THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 5 Solutions

Problems that may be demonstrated in class :

Q1. Determine whether the following functions are differentiable at the specified points. If yes, find the derivatives at those points.

(a) $\frac{\pi}{8}(1+x^2)^{\frac{8}{\pi}\arctan x}$ at $x=1$;

(b) $|\tan \pi x \arcsin x|$ at $x=0$;

(c) $\max\{e^x \sin x, -x^3\}$ at $x=0$.

Q2. Use L'Hopital's rule to evaluate the following limits.
(a) $\lim_{x\to+\infty} \frac{x^2-6x+2}{e^x}$; (b) $\lim_{x\to 0}(\cosh x)^{\cot x}$; (c) $\lim_{x\to-\infty}(1+x^2)^{\pi/2+\arctan x}$.

Q3. Find $\frac{dy}{dx}$ for the implicit function $x^2+y^2=e^{x^2-y^2}$.

Q4. Suppose a differentiable function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(x)=f(x+1)$ for any $x\in\mathbb{R}$.

(a) Prove that there exist $\alpha,\beta\in\mathbb{R}$ such that

$$f(\alpha)\le f(x)\le f(\beta) \quad \text{for any } x\in\mathbb{R}.$$

(b) Prove that $f'(x+1)=f'(x)$ for any $x\in\mathbb{R}$.

(c) Let $\alpha,\beta\in\mathbb{R}$ and $f(\alpha)\le f(x)\le f(\beta)$ for any $x\in\mathbb{R}$. Prove that there exists $\xi\in\mathbb{R}$ such that $f(\beta)-f(\alpha)\le f'(\xi)\le\xi$.

Q5. Suppose $n\in\mathbb{Z}^+$ and $a_1,...,a_n$ are positive real numbers. Define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x)=\begin{cases}\left(\frac{a_1^x+\cdots+a_n^x}{n}\right)^{1/x}, & \text{if } x\neq 0;\\ \sqrt[n]{a_1\cdots a_n}, & \text{if } x=0.\end{cases}$$

(a) Show that f is a continuous at 0.

(b) Show that $\lim_{x\to+\infty} f(x)=\max\{a_1,...,a_n\}$ and $\lim_{x\to-\infty} f(x)=\min\{a_1,...,a_n\}$.

Q6. Let $f:\mathbb{R}\to\mathbb{R}$ be an injective continuous function, $a,b\in\mathbb{R}$, $a<b$ and $f(b)\le f(a)$.

(a) Show that $f(b)<f(x)<f(a)$ for any $x\in(a,b)$ (Hint: use intermediate value theorem).

(b) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable. Prove that $f'(x)\le 0$ for any $x\in(a,b)$.

Q7. Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable, $f(f(x))=x$ for any $x\in\mathbb{R}$ but $f(x)\not\equiv x$.

(a) Verify that $f:\mathbb{R}\to\mathbb{R}$ is injective.

(b) Prove that f has a fixed point $\xi\in\mathbb{R}$, i.e. $f(\xi)=\xi$, such that $f'(\xi)=-1$.

Solutions :

Q1. (a) Let $f(x) = \frac{\pi}{8}(1+x^2)^{\frac{8}{\pi}\arctan x}$. Then

$$\begin{aligned}
f(1) &= \frac{\pi}{8}(1+1^2)^{\frac{8}{\pi}\arctan 1} = \frac{\pi}{8}\cdot 2^{\frac{8}{\pi}\frac{\pi}{4}} = \frac{\pi}{2},\\
\frac{f'(x)}{f(x)} &= \frac{d}{dx}(\ln f(x)) = \frac{d}{dx}\left(\frac{8}{\pi}\arctan x\ln(1+x^2) + \ln\pi - \ln 8\right)\\
&= \frac{8}{\pi}\left(\ln(1+x^2)\frac{d}{dx}\arctan x + \arctan x\frac{d}{dx}\ln(1+x^2)\right)\\
&= \frac{8}{\pi}\left(\frac{1}{1+x^2}\ln(1+x^2) + \frac{2x}{1+x^2}\arctan x\right),\\
\therefore f'(x) &= \frac{8f(x)}{\pi}\left(\frac{1}{1+x^2}\ln(1+x^2) + \frac{2x}{1+x^2}\arctan x\right),\\
f'(1) &= \frac{8}{\pi}\cdot\frac{\pi}{2}\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right) = 4\left(\frac{1}{2}\ln 2 + \frac{\pi}{4}\right) = \pi + \ln 4.
\end{aligned}$$

(b) Let $f(x) = |\tan\pi x \arcsin x|$. When $0 \le x < 1/2$, $\tan\pi x \ge 0$ and $\arcsin x \ge 0$, thus $f(x) = \tan\pi x\arcsin x$. When $-1/2 < x < 0$, $\tan\pi x < 0$ and $\arcsin x < 0$, thus $f(x) = \tan\pi x\arcsin x$. Therefore,

$$\begin{aligned}
f'(0) &= \frac{d}{dx}(\tan\pi x\arcsin x)\Big|_{x=0} = \left(\pi\sec^2\pi x\arcsin x + \frac{\tan\pi x}{\sqrt{1-x^2}}\right)\Big|_{x=0}\\
&= \pi\cdot 1\cdot 0 + \frac{0}{\sqrt{1-0^2}} = 0.
\end{aligned}$$

(c) Let $f(x) = \max\{e^x\sin x, -x^3\}$. When $0 \le x \le \pi$, $e^x\sin x \ge 0 \ge -x^3$ and thus $f(x) = e^x\sin x$. When $-\pi < x < 0$, $e^x\sin x < 0 < -x^3$ and thus $f(x) = -x^3$.

$$\begin{aligned}
\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0}\frac{e^h\sin h - 0}{h} = \frac{d}{dx}e^x\sin x\Big|_{x=0}\\
&= (e^x\sin x + e^x\cos x)|_{x=0} = 1,\\
\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0}\frac{-h^3-0}{h} = \frac{d}{dx}x^3\Big|_{x=0} = 3x^2\big|_{x=0} = 0 \ne 1.
\end{aligned}$$

Therefore, $f(x)$ is not differentiable at $x = 0$.

Q2. (a)

$$\begin{aligned}
\lim_{x\to+\infty}\frac{x^2-6x+2}{e^x} &= \lim_{x\to+\infty}\frac{\frac{d}{dx}(x^2-6x+2)}{\frac{d}{dx}e^x} = \lim_{x\to+\infty}\frac{2x-6}{e^x}\\
&= \lim_{x\to+\infty}\frac{\frac{d}{dx}(2x-6)}{\frac{d}{dx}e^x} = \lim_{x\to+\infty}\frac{2}{e^x} = 0.
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\cot x\ln\cosh x &= \lim_{x\to 0}\frac{\ln\cosh x}{\tan x} = \lim_{x\to 0}\frac{\frac{d}{dx}\ln\cosh x}{\frac{d}{dx}\tan x} = \lim_{x\to 0}\frac{1}{\sec^2 x}\cdot\frac{\sinh x}{\cosh x} = 0,\\
\lim_{x\to 0}(\cosh x)^{\cot x} &= \lim_{x\to 0}e^{\cot x\ln\cosh x} = e^0 = 1.
\end{aligned}$$

(c)

$$
\begin{aligned}
&\lim_{x\to-\infty}\left(\frac{\pi}{2}+\arctan x\right)\ln(1+x^2)\\
&=\lim_{x\to-\infty}\frac{\frac{\pi}{2}+\arctan x}{\frac{1}{\ln(1+x^2)}}=\lim_{x\to-\infty}\frac{\frac{1}{1+x^2}}{\frac{-1}{(\ln(1+x^2))^2}\cdot\frac{2x}{1+x^2}}=\lim_{x\to-\infty}\frac{-(\ln(1+x^2))^2}{2x}\\
&=\lim_{x\to-\infty}\frac{-2\ln(1+x^2)\cdot\frac{2x}{1+x^2}}{2}=\lim_{x\to-\infty}\frac{-2\ln(1+x^2)}{x+\frac{1}{x}}=\lim_{x\to-\infty}\frac{\frac{4x}{1+x^2}}{\frac{1}{x^2}-1}\\
&=\lim_{x\to-\infty}\frac{4x^3}{1-x^4}=\lim_{x\to-\infty}\frac{\frac{4}{x}}{\frac{1}{x^4}-1}=0.
\end{aligned}
$$

Thus, $\lim_{x\to-\infty}(1+x^2)^{\frac{\pi}{2}+\arctan x}=\lim_{x\to-\infty}e^{(\frac{\pi}{2}+\arctan x)\ln(1+x^2)}=e^0=1.$

Q3. Differentiating on both sides,

$$
\begin{aligned}
2x+2y\frac{dy}{dx}&=e^{x^2-y^2}\left(2x-2y\frac{dy}{dx}\right),\\
x+y\frac{dy}{dx}&=x(x^2+y^2)-y(x^2+y^2)\frac{dy}{dx},\\
y(x^2+y^2+1)\frac{dy}{dx}&=x(x^2+y^2-1),\\
\therefore\frac{dy}{dx}&=\frac{x(x^2+y^2-1)}{y(x^2+y^2+1)}.
\end{aligned}
$$

Q4. (a) f is in particular continuous on the closed and bounded interval $[0,1]$. By extreme value theorem, there exist $\alpha,\beta\in[0,1]$ such that $f(\alpha)\le f(x)\le f(\beta)$ for any $x\in[0,1]$. Consider any $x\in\mathbb{R}$. Let $n=\lfloor x\rfloor$, i.e. n be the integral part of x. Then $n\le x<n+1$ and hence $0\le x-n<1$. We have

$$f(\alpha)\le f(x-n)=f(x)\le f(\beta).$$

(b) Fix $x\in\mathbb{R}$. By differentiability of f,

$$f'(x+1)=\lim_{h\to0}\frac{f(x+1+h)-f(x+1)}{h}=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=f'(x).$$

(c) Let $n=\lfloor\beta-\alpha\rfloor$ and $\gamma=\beta-n$. Then $n\le\beta-\alpha<n+1$ and thus $0\le\gamma-\alpha<1$. By Lagrange's mean value theorem, there exists $\eta\in(\alpha,\gamma)$ such that

$$f'(\eta)=\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}=\frac{f(\beta)-f(\alpha)}{\gamma-\alpha}\ge f(\beta)-f(\alpha).$$

Define $m=\lfloor\eta-f'(\eta)\rfloor$ and $\xi=\eta-m$. Then $m\le\eta-f'(\eta)$ and therefore $f(\beta)-f(\alpha)\le f'(\xi)=f'(\xi-m)=f'(\eta)\le\eta-m=\xi$.

Q5. (a) Let $g(x)=\ln f(x)$ and $h(x)=x$ for any $x\in\mathbb{R}$. Note that $h'(x)=1\ne0$ for any $x\in\mathbb{R}$, $\lim_{x\to0}[\ln(a_1^x+\cdots+a_n^x)-\ln n]=\lim_{x\to0}h(x)=0$, and

$$
\begin{aligned}
\lim_{x\to0}\frac{\frac{d}{dx}(\ln(a_1^x+\cdots+a_n^x)-\ln n)}{\frac{dx}{dx}}&=\lim_{x\to0}\frac{a_1^x\ln a_1+\cdots+a_n^x\ln a_n}{a_1^x+\cdots+a_n^x}\\
&=\frac{\ln a_1+\cdots+\ln a_n}{n}=\ln\sqrt[n]{a_1\cdots a_n}=\ln f(0).
\end{aligned}
$$

By L'Hopital's rule,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{\ln(a_1^x + \cdots + a_n^x) - \ln n}{x} = \ln f(0).$$

We know that the exponential function $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto e^x$ is continuous. Therefore, $f : \mathbb{R} \to \mathbb{R}$ is continuous at 0 since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{g(x)} = e^{\ln f(0)} = f(0).$$

(b) Without loss of generality, assume $a_1 \le \cdots \le a_n$. For any $x > 0$,

$$\frac{a_n^x}{n} < \frac{a_1^x + \cdots + a_n^x}{n} \le \frac{na_n^x}{n} = a_n^x \quad \text{and thus} \quad \frac{a_n}{\sqrt[x]{n}} < f(x) \le a_n.$$

Because $\lim_{x\to+\infty} \frac{a_n}{\sqrt[x]{n}} = a_n$, by Sandwich theorem, $\lim_{x\to+\infty} f(x) = a_n$. On the other hand, for any $x < 0$,

$$\frac{a_1^x}{n} < \frac{a_1^x + \cdots + a_n^x}{n} \le \frac{na_1^x}{n} = a_1^x \quad \text{and thus} \quad a_1 \le f(x) < \frac{a_1}{\sqrt[x]{n}}.$$

Because $\lim_{x\to-\infty} \frac{a_1}{\sqrt[x]{n}} = a_1$, by Sandwich theorem, $\lim_{x\to-\infty} f(x) = a_1$.

Q6. (a) Fix $x \in (a, b)$. We claim $f(x) \le f(a)$. Assume the contrary that $f(x) > f(a)$. Define $c = \frac{1}{2}(f(a) + f(x))$. Then $f(b) \le f(a) < c < f(x)$ and by intermediate value theorem, $\exists \xi_0 \in (a, x)$ and $\xi_1 \in (x, b)$ such that $f(\xi_0) = f(\xi_1) = c$. But this violates injectivity of f. Thus, $f(x) \le f(a)$. By injectivity of f, $f(x) < f(a)$. Define $g(y) = -f(-y)$ for any $y \in \mathbb{R}$. Suppose $y, z \in \mathbb{R}$ and $g(y) = g(z)$. Then $f(-y) = -g(y) = -g(z) = f(-z)$. By injectivity of f, $-y = -z$, whence $y = z$. Then $g : \mathbb{R} \to \mathbb{R}$ is injective. Now we have $-b < -x < -a$ and $g(-a) = -f(a) < -f(b) = g(-b)$. Applying the previous argument, we have $g(-x) < g(-b)$ and hence $f(b) = -g(-b) < -g(-x) < f(x)$.

(b) Consider any $x, y \in (a, b)$ with $x < y$. $f(b) < f(x)$ by (a). Since $x < y < b$, applying (a) again, $f(y) < f(x)$. Therefore, f is strictly decreasing on (a, b). We can conclude that $f'(x) \le 0$ for any $x \in (a, b)$.

Q7. (a) If $x, y \in \mathbb{R}$ and $f(x) = f(y)$, then $x = f(f(x)) = f(f(y)) = y$. f is injective.

(b) Because f is not the identity function, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \ne x_0$. Let $a = \min\{x_0, f(x_0)\}$ and $b = \max\{x_0, f(x_0)\}$. Then $f(b) = a < b = f(a)$. Define a continuous function $g(x) = f(x) - x$ for any $x \in \mathbb{R}$. We check that $g(a) = f(a) - a = b - a > 0$ and $g(b) = f(b) - b = a - b < 0$. By intermediate value theorem, $\exists \xi \in (a, b)$ such that $f(\xi) - \xi = g(\xi) = 0$, whence $f(\xi) = \xi$. ξ is a fixed point of f. By (a) and Q6(b), $f'(\xi) \le 0$. By chain rule,

$$[f'(\xi)]^2 = f'(f(\xi))f'(\xi) = (f \circ f)'(\xi) = \left.\frac{dx}{dx}\right|_{x=\xi} = 1.$$

Therefore, $f'(\xi) = -1$.