THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 5

Differentiability A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **differentiable at** $a \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, in which case this limit is called the **derivative** of f **at** a and is denoted by $f'(a)$. $f : \mathbb{R} \to \mathbb{R}$ is said to be **differentiable** if it is differentiable at every point in $\mathbb{R}$.

- If we write $y = f(x)$ and $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, we may also denote $f'(a)$ by another notation $\frac{dy}{dx}|_{x=a}$. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, by $\frac{dy}{dx}$ we mean the function $a \mapsto \frac{dy}{dx}|_{x=a}$.
- If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a, then it is continuous at a.
- Higher derivatives: for $n \in \mathbb{Z}^+$, the n-**derivative** of f **at** a, denoted by $f^{(n)}(a)$ (or $\frac{d^n y}{dx^n}|_{x=a}$ if we write $y = f(x)$), is defined as the derivative of $f^{(n-1)}(x)$ at a if they exist, where by convention, $f^{(0)}(a) = f(a)$.

Basic rules of differentiation Sum rule, product rule, chain rule and Leibniz rule.

Implicit function Let $F(x, y) = 0$ be an implicit function. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Mean value theorem Suppose a, b are real numbers and $a < b$.

1. *Lagrange's.* Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

2. *Cauchy's.* Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remark. Rolle's theorem is used to prove mean value theorems. But once we have proved the latter, the former would become a special case.

Relationship between derivatives and monotonicity Suppose $a, b \in \mathbb{R}$ and $a < b$.

1. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and attains a local extremum at a, then $f'(a) = 0$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on (a, b). Then $f : \mathbb{R} \to \mathbb{R}$ is monotonic increasing (decreasing) on (a, b) if and only if $f'(x) \geq 0$ ($f'(x) \leq 0$) for any $x \in (a, b)$.
3. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on (a, b) and $f'(x) > 0$ ($f'(x) < 0$) for any $x \in (a, b)$, then it is strictly increasing (decreasing) on (a, b).

L'Hopital's rule Let $a \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. If

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L_0$, where either $L_0 = 0$ or $L_0 = \pm\infty$;
2. There exists $\delta > 0$ such that $g'(x) \neq 0$ for any $x \in (a - \delta, a + \delta)$ with $x \neq a$;
3. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, where $L \in [-\infty, \infty]$.

then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

L'Hopital's rule can be applied to indeterminate forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, ∞^0, 1^∞.

Problems that may be demonstrated in class :

Q1. Determine whether the following functions are differentiable at the specified points. If yes, find the derivatives at those points.
 (a) $\frac{\pi}{8}(1 + x^2)^{\frac{8}{\pi}\arctan x}$ at $x = 1$;
 (b) $|\tan \pi x \arcsin x|$ at $x = 0$;
 (c) $\max\{e^x \sin x, -x^3\}$ at $x = 0$.

Q2. Use L'Hopital's rule to evaluate the following limits.
 (a) $\lim_{x\to+\infty} \frac{x^2-6x+2}{e^x}$; (b) $\lim_{x\to 0}(\cosh x)^{\cot x}$; (c) $\lim_{x\to-\infty}(1 + x^2)^{\pi/2+\arctan x}$.

Q3. Find $\frac{dy}{dx}$ for the implicit function $x^2 + y^2 = e^{x^2-y^2}$.

Q4. Suppose a differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) = f(x+1)$ for any $x \in \mathbb{R}$.
 (a) Prove that there exist $\alpha, \beta \in \mathbb{R}$ such that
$$f(\alpha) \leq f(x) \leq f(\beta) \quad \text{for any } x \in \mathbb{R}.$$
 (b) Prove that $f'(x + 1) = f'(x)$ for any $x \in \mathbb{R}$.
 (c) Let $\alpha, \beta \in \mathbb{R}$ and $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in \mathbb{R}$. Prove that there exists $\xi \in \mathbb{R}$ such that $f(\beta) - f(\alpha) \leq f'(\xi) \leq \xi$.

Q5. Suppose $n \in \mathbb{Z}^+$ and $a_1, ..., a_n$ are positive real numbers. Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} \left(\frac{a_1^x+\cdots+a_n^x}{n}\right)^{1/x}, & \text{if } x \neq 0; \\ \sqrt[n]{a_1 \cdots a_n}, & \text{if } x = 0. \end{cases}$$
 (a) Show that f is a continuous at 0.
 (b) Show that $\lim_{x\to+\infty} f(x) = \max\{a_1, ..., a_n\}$ and $\lim_{x\to-\infty} f(x) = \min\{a_1, ..., a_n\}$.

Q6. Let $f : \mathbb{R} \to \mathbb{R}$ be an injective continuous function, $a, b \in \mathbb{R}$, $a < b$ and $f(b) \leq f(a)$.
 (a) Show that $f(b) < f(x) < f(a)$ for any $x \in (a, b)$ (Hint: use intermediate value theorem).
 (b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Prove that $f'(x) \leq 0$ for any $x \in (a, b)$.

Q7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, $f(f(x)) = x$ for any $x \in \mathbb{R}$ but $f(x) \not\equiv x$.
 (a) Verify that $f : \mathbb{R} \to \mathbb{R}$ is injective.
 (b) Prove that f has a fixed point $\xi \in \mathbb{R}$, i.e. $f(\xi) = \xi$, such that $f'(\xi) = -1$.