THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 12 Solutions

Problems that may be demonstrated in class :

Q1. For each of the following sequence, compute the limit if it exists.

(a) $\lim_{n\to\infty} \frac{3^n + (-2)^{n+1}}{3^{n-2} - 2^{2n-1}}$ (b) $\lim_{n\to\infty} \frac{\ln^2(n+1)}{(n-1)^2}$ (c) $\lim_{n\to\infty} \frac{n^2 + n\sin n}{n^2}$

Q2. Compute the following limits if exist.

(a) $\lim_{x\to 0} x^2 \cos\frac{1}{x}$ (b) $\lim_{x\to\infty} \frac{2e^{3x} + 2e^x + 1}{3e^{3x} + 3}$ (c) $\lim_{x\to\infty} \frac{x^5 + 2x + 3}{x^4 + 3}$ (d) $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$

Q3. Let

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x < 1 \\ Ax - 4 & \text{if } x \geq 1 \end{cases}$$

Find the value of A if f is continuous.

Q4. Determine whether the following functions are differentiable.

(a) $f(x) = |x+2|$ (b) $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2\sqrt{2x-1} & \text{if } x > 1 \end{cases}$

Q5. Compute $f'(x)$.

(a) $f(x) = x^2 e^x$ (b) $f(x) = \frac{x^2+1}{x^3+2}$ (c) $f(x) = \int_0^x (3t^2 + 3)dt$

(d) $f(x) = \int_{-x^3}^{e^{2x}} t\,dt$

Q6. Prove that the equation $x^5 + 7x - 2 = 0$ has exactly one real root.

Q7. Find the Taylor series of the following functions at $x = 0$.

(a) $f(x) = x^4 e^{-x}$ (b) $f(x) = \frac{2x}{(1+x)^2}$

Q8. (a) Let $f(x) = \sin x$, using Taylor theorem to show that

$$\frac{599}{6000} - \frac{0.1^4}{4!} \leq f(0.1) \leq \frac{599}{6000} + \frac{0.1^4}{4!}$$

(b) If we use the Taylor polynomial of f of degree n to approximate f, find one n such that the absolute error is less than 10^{-7}.

Q9. Compute the following integral.

(a) (1.1.14) $\int \frac{dx}{1+e^x}$ (b) (1.2.20) $\int e^{2x} \cos 3x dx$ (c) (1.3.18) $\int \sin^2 x \cos^4 x dx$

(d) (1.5.3) $\int \sqrt{\frac{1+x}{1-x}} dx$ (e) (1.6.10) $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx$ (f) (1.7.3) $\int \frac{dx}{\sin x \cos^4 x}$

(g) (1.8.36) $\int_0^1 \frac{dx}{1+\sqrt{x}}$

Q10. (1.3.4) Prove the following reduction formula:

$$I_n = \int \frac{1}{\sin^n x} dx;\ I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2},\ n \geq 2$$

Solution

Q1. (a) $\lim_{n\to\infty} \frac{3^n + (-2)^{n+1}}{3^{n-2} - 2^{2n-1}} = \lim_{n\to\infty} \frac{\frac{3^n}{2^{2n-1}} + \frac{(-2)^{n+1}}{2^{2n-1}}}{\frac{3^{n-2}}{2^{2n-1}} + 1} = \lim_{n\to\infty} \frac{2\left(\frac{3}{4}\right)^n - 4\left(\frac{-1}{2}\right)^n}{\frac{2}{9}\left(\frac{3}{4}\right)^n + 1} = 0$

(b)

$$\lim_{n\to\infty} \frac{\ln^2(n+1)}{(n-1)^2} = \lim_{n\to\infty} \frac{\frac{\ln(n+1)}{n+1}}{n-1} = \lim_{n\to\infty} \frac{\ln(n+1)}{n^2-1} = \frac{1}{2n(n+1)} = 0$$

(c) Since $\frac{n^2-n}{n^2} \leq \frac{n^2 + n\sin n}{n^2} \leq \frac{n^2+n}{n^2}$ and $\lim_{n\to\infty} \frac{n^2 \pm n}{n^2} = 1$. By Sandwich theorem,

$$\lim_{n\to\infty} \frac{n^2 + n\sin n}{n^2} = 1$$

Q2. (a) Since $0 \leq |x^2 \cos \frac{1}{x}| \leq x^2$ and $\lim_{x\to 0} x^2 = 0$. By Sandwich theorem,

$$\lim_{x\to 0} x^2 \cos \frac{1}{x} = 0$$

(b) $\lim_{x\to\infty} \frac{2e^{3x} + 2e^x + 1}{3e^{3x} + 3} = \lim_{x\to\infty} \frac{2 + 2e^{-2} + e^{-3x}}{3 + 3e^{-3x}} = \frac{2}{3}$

(c) $\lim_{x\to\infty} \frac{x^5 + 2x + 3}{x^4 + 3} = \lim_{x\to\infty} \frac{x + 2x^{-3} + 3x^{-4}}{1 + 3x^{-4}} = \infty$, therefore limit does not exist.

(d) $\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} x + 3 = 6$

Q3. By continuity, we have $-1 = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = A - 4$. Hence $A = 3$.

Q4. (a) f is clearly differentiable at $x \neq -2$. Next,

$$\lim_{x\to -2^-} \frac{f(x) - f(-2)}{x+2} = \lim_{x\to -2^-} \frac{-(x+2) - 0}{x+2} = -1$$

But

$$\lim_{x\to -2^+} \frac{f(x) - f(-2)}{x+2} = \lim_{x\to -2^+} \frac{(x+2) - 0}{x+2} = 1$$

Since $\lim_{x\to -2^-} \frac{f(x) - f(-2)}{x+2} \neq \lim_{x\to -2^+} \frac{f(x) - f(-2)}{x+2}$, f is not differentiable at $x = -2$

(b) Again, f is clearly differentiable at $x \neq 1$. But f is not continuous at $x = 1$ (Think about why). Therefore f is not differentiable at $x = 1$. (What theorem did I use here?)

Q5. (a) $f'(x) = x^2 \dfrac{d}{dx}(e^x) + \dfrac{d}{dx}(x^2)e^x = e^x(x^2 + 2x)$

(b) $f'(x) = \dfrac{(x^3+2)\frac{d}{dx}(x^2+1) - (x^2+1)\dfrac{d}{dx}(x^3+2)}{(x^3+2)^2} = \dfrac{-x^4 - 3x^2 + 4x}{(x^3+2)^2}$

(c) $f'(x) = (3x^2+3)\dfrac{d}{dx}(x) = 3(x^2+1)$

(d) $f'(x) = e^{2x}\dfrac{d}{dx}(e^{2x}) - (-x^3)\dfrac{d}{dx}(-x^3) = 2e^{4x} - 3x^5$

Q6. Let $f(x) = x^5 + 7x - 2$. Note that f is continuous. Moreover, $f(0) = -2 < 0$ and $f(1) = 6 > 0$. By intermediate value theorem, there exists $c \in (0,1)$ such that $f(c) = 0$.

Next we want to show that such c is unique.

Assume there exists $a \neq b$ such that $f(a) = 0 = f(b)$. By Mean value theorem (or Rolle's theorem since $f(a) = f(b) = 0$), there exists $d \in (a,b) \subset (0,1)$ such that $f'(d) = 0$. But this is impossible since $f'(x) = 5x^4 + 7 > 0$ for all $x \in (0,1)$. Hence there exists exactly one root.

Q7. (a) Let $g(x) = e^{-x}$. We then have $g^{(n)}(0) = (-1)^n$. Therefore the Taylor series of f is

$$T(x) = x^4 \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+4}$$

(b) Let $g(x) = \dfrac{1}{(1+x)^2}$. Observe that $g'(x) = \dfrac{-2}{(1+x)^3}$, $g''(x) = \dfrac{(-2)(-3)}{(1+x)^4}$. One can easily see that $g^{(n)}(0) = (-1)^n(n+1)!$. Hence the Taylor series of f is

$$T(x) = 2x \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} 2(-1)^n(n+1)x^{n+1}$$

Q8. (a) Consider the Taylor polynomial of degree 3,

$$\sin x = T_3(x) + \frac{f^{(4)}(c)}{4!}x^4 = x - \frac{x^3}{3!} + \frac{f^{(4)}(c)}{4!}x^4$$

where c lies between 0 and x. Hence

$$|\sin(0.1) - (0.1 - \frac{0.1^3}{3!})| \leq \left| \frac{f^{(4)}(c)}{4!}(0.1^4) \right| \leq \frac{0.1^4}{4!}$$

Therefore

$$\frac{599}{6000} - \frac{0.1^4}{4!} \leq f(0.1) \leq \frac{599}{6000} + \frac{0.1^4}{4!}$$

(b) When using the Taylor polynomial of degree n, the absolute error, $E_n(x) = \left| \dfrac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right|$

For $f(x) = \sin x$, we have

$$E_n(0.1) \leq \frac{0.1^{n+1}}{(n+1)!}$$

Hence if we can find n such that $\dfrac{0.1^{n+1}}{(n+1)!} \le 10^{-7}$, then $E_n(0.1) \le 10^{-7}$.

Note that $\dfrac{0.1^5}{(5)!} \le 10^{-7}$, therefore $E_4(0.1) \le 10^{-7}$. Therefore we can set $n = 4$.

Q9. (a) $\displaystyle\int \frac{dx}{1+e^x} = \int \frac{(1+e^x)-e^x}{1+e^x}dx = \int dx - \int \frac{e^x dx}{1+e^x} = \int dx - \int \frac{d(1+e^x)}{1+e^x}$

$= x - \ln|1+e^x| + C$

(b) Let $I = \displaystyle\int e^{2x}\cos 3x dx$. By integration by parts,

$$\begin{aligned} I &= \frac{1}{2}\int \cos 3x \, d(e^{2x}) \\ &= \frac{1}{2}\left(e^{2x}\cos 3x - \int e^{2x}(-3\sin 3x)dx\right) \\ &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}\int \sin 3x \, d(e^{2x}) \\ &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}\left(e^{2x}\sin 3x - \int e^{2x}(3\cos 3x)dx\right) \\ &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C' \end{aligned}$$

Hence $\dfrac{13}{4}I = \dfrac{1}{2}e^{2x}\cos 3x + \dfrac{3}{4}e^{2x}\sin 3x + C'$ and $I = \dfrac{2}{13}e^{2x}\cos 3x + \dfrac{3}{13}e^{2x}\sin 3x + C$

(c)

$$\begin{aligned} \int \sin^2 x \cos^4 x dx &= \frac{1}{4}\int 4\sin^2 x\cos^2 x \cdot \cos^2 x dx \\ &= \frac{1}{8}\int \sin^2 2x \cdot (1+\cos 2x)dx \\ &= \frac{1}{16}\int 1 - \cos 4x \, dx + \frac{1}{16}\int \sin^2 2x \, d(\sin 2x) \\ &= \frac{1}{16}\left(x - \frac{1}{4}\sin 4x\right) + \frac{\sin^3 2x}{48} + C \\ &= \frac{\sin^3 2x}{48} + \frac{x}{16} - \frac{\sin 4x}{64} + C \end{aligned}$$

Remark: One may verify that this is equivalent to the answer in the exercise.

(d) For $x \in (0,1)$,

$$\begin{aligned} \int \sqrt{\frac{1+x}{1-x}}dx &= \int \frac{1+x}{\sqrt{1-x^2}}dx \\ &= \int \frac{1}{\sqrt{1-x^2}}dx + \int \frac{xdx}{\sqrt{1-x^2}} \\ &= \int \frac{\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} + \frac{-1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} \\ &= \arcsin x - \sqrt{1-x^2} + C \end{aligned}$$

(e) By long division,

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}=\frac{2x^3-4x^2-x-3}{(x+1)(x-3)}=2x+\frac{5x-3}{(x+1)(x-3)}$$

Hence by the partial fraction decomposition,

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3}=2x+\frac{3}{x-3}+\frac{2}{x+1}$$

and thus

$$\int\frac{2x^3-4x^2-x-3}{x^2-2x-3}dx=2\int xdx+3\int\frac{dx}{x-3}+2\int\frac{dx}{x+1}=x^2+3\ln|x-3|+2\ln|x+1|+C$$

(f)

$$\begin{aligned}\int\frac{dx}{\sin x\cos^4 x}&=\int\frac{\sin x dx}{(1-\cos^2 x)\cos^4 x}\\&=-\int\frac{d(\cos x)}{(1-\cos^2 x)\cos^4 x}\\&=-\int\frac{dt}{t^4(1-t^2)}\qquad(\text{Let } t=\cos x)\\&=-\int\frac{1}{t^4}+\frac{1}{t^2}+\frac{1}{2(t+1)}-\frac{1}{2(t-1)}dt\\&=\frac{1}{3t^3}+\frac{1}{t}-\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|+C\\&=\frac{1}{3\cos^3 x}+\frac{1}{\cos x}+\frac{1}{2}\ln(\tan^2\frac{x}{2})+C\\&=\frac{1}{3\cos^3 x}+\frac{1}{\cos x}+\ln\left|\tan\frac{x}{2}\right|+C\end{aligned}$$

(g) Let $t=1+\sqrt{x}$, then $dx=2(t-1)dt$. Moreover, when $x=0, t=1$, when $x=1, t=2$. Then

$$\int_0^1\frac{dx}{1+\sqrt{x}}=\int_1^2\frac{2(t-1)}{t}dt=\int_1^2 2-\frac{2}{t}dt=[\,2t-2\ln|t|\,]_1^2=(4-2\ln 2-2+2\ln 1)=2(1-\ln 2)$$

Q10. For $n\geq 2$,

$$\begin{aligned}I_n&=\int\csc^n x dx\\&=\int\csc^{n-2}x\csc^2 x dx\\&=-\int\csc^{n-2}d(\cot x)\\&=-\int\cot x(n-2)\csc^{n-3}x\csc x\cot x dx-\cot x\csc^{n-2}x\\&=-(n-2)\int\cot^2 x\csc^{n-2}x dx-\cot x\csc^{n-2}x\\&=-(n-2)\int(\csc^2-1)\csc^{n-2}x dx-\cot x\csc^{n-2}x\\&=(n-2)I_{n-2}-(n-2)I_n-\cot x\csc^{n-2}x\end{aligned}$$

Therefore

$$I_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} I_{n-2}$$