Week 6
Cross Product
Function in n variables

Cross Product

Given a plane containing the origin which is the linear span of $\vec{v}$, $\vec{w}$, we want to find a normal vector $\vec{n}$ perpendicular to it. To do so, observe that:

$$\begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0,$$

for both expressions are determinants of matrices with repeated rows. By the definition of the cofactor expansion along the first row, we have:

$$\langle v_1, v_2, v_3 \rangle \cdot \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle = 0$$

$$\langle w_1, w_2, w_3 \rangle \cdot \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle = 0$$

Definition.

The **cross product** of $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ is the *vector*:

$$\vec{v} \times \vec{w} = \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$$

For nonzero vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$, their cross product $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

Properties of the Cross Product

1. $$\vec{v} \times \vec{0} = \vec{0}$$

2. $$\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0$$

3.

$$\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$$

4.

$$(s\vec{v}) \times (r\vec{w}) = (rs)(\vec{v} \times \vec{w}), \quad r, s \in \mathbb{R}$$

5.

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$$

6.

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

7.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

8.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

9.
Given $\vec{v}, \vec{w} \neq \vec{0}$ in $\mathbb{R}^3$, there are exactly two *unit* vectors in $\mathbb{R}^3$ which are perpendicular to both $\vec{v}$ and $\vec{w}$. Namely:

$$\pm \frac{\vec{v} \times \vec{w}}{|\vec{v} \times \vec{w}|}$$

10.
For $\vec{v}, \vec{w} \neq \vec{0}$ in $\mathbb{R}^3$,

$$|\vec{v} \times \vec{w}| = |\vec{v}|\,|\vec{w}| \sin\theta,$$

where θ is the angle ($0 \leq \theta \leq \pi$) between $\vec{v}$ and $\vec{w}$.

11.
Two nonzero vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$ are parallel to each other if and only if $\vec{v} \times \vec{w} = \vec{0}$. In particular,

$$\vec{v} \times \vec{v} = \vec{0}.$$

Example.

Let two planes in $\mathbb{R}^3$ be given:

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

where $\vec{n}_i := \langle a_i, b_i, c_i \rangle \neq \vec{0}$ $(i = 1, 2)$. Suppose $\vec{n}_1$ and $\vec{n}_2$ are not parallel to each other. Then, the two planes are non-parallel, and the intersection of the two planes is a line parallel to the vector $\vec{v} = \vec{n}_1 \times \vec{n}_2$. Note that the vector $\vec{v}$ is nonzero, since $\vec{n}_1$ and $\vec{n}_2$ are by assumption non-parallel.

Distance Between a Point and a Plane

Given a plane in $\mathbb{R}^3$ corresponding to:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

The (minimal) distance between a point $P \in \mathbb{R}^3$ and the plane is:

$$\boxed{d = \left| \text{Proj}_{\vec{n}} \overrightarrow{P_0P} \right| = \left| \overrightarrow{P_0P} \cdot \frac{\vec{n}}{|\vec{n}|} \right|}$$

where $P_0 = (x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c \rangle$.