Week 3
Invertible Matrices
Determinants

> **Theorem.**
>
> Let A be an $n \times n$ matrix. The following statements are equivalent:
>
> 1. A is invertible.
> 2. The matrix equation $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as its only solution.
> 3. A is row equivalent to I.

Proof of 1 implying 2

If A^{-1} exists, then:

$$\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x}) = A^{-1}\vec{0} = \vec{0}.$$

So, $\vec{x} = \vec{0}$ is the only solution.

Proof of 2 implying 3

This follows from our previous discussion of Gaussian elimination on augmented matrices, and the fact that a matrix in strict triangular form is row equivalent to the identity matrix.

Proof of 3 implying 1

If A is row equivalent to I, then there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that:

$$E_k \cdots E_2 E_1 A = I.$$

It is easy to see that every elementary matrix is invertible (**Exercise.**). Multiplying from the left both sides of the above equation with the inverses of the E_i's, we obtain:

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1},$$

which is a product of invertible matrices. Hence, A is invertible, with inverse $A^{-1} = E_k \cdots E_2 E_1$.

Corollary.

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that either BA or AB is equal I, then A is invertible, with $A^{-1} = B$.

Proof.

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If $BA = I$, then for any vector $\vec{x}$ satisfying $A\vec{x} = \vec{0}$, we have:

$$\vec{x} = (BA)\vec{x} = B(A\vec{x}) = B\vec{0} = \vec{0}.$$

Hence, $\vec{x} = \vec{0}$ is the only solution to $A\vec{x} = \vec{0}$, which by the previous theorem implies that A is invertible. Moreover, $BA = I$ now implies that:

$$A^{-1} = IA^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI = B.$$

If $AB = I$, then by the same argument as before B is invertible. We have $A = A(BB^{-1}) = (AB)B^{-1} = B^{-1}$. Hence, A is invertible, with $A^{-1} = (B^{-1})^{-1} = B$.

Finding the Inverse of a Matrix

Suppose an $n \times n$ matrix A is invertible, then we know from the theorem that A is row equivalent to I_n. In other words, there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I$. Moreover, we have $A^{-1} = E_k \cdots E_2 E_1$. Consider the augmented matrix:

$$\left(\begin{array}{c|c} A & I_n \end{array} \right)$$

Performing a row operation corresponding to E_1 on the augmented matrix, we obtain: >

$$E_1 \left(\begin{array}{c|c} A & I_n \end{array} \right) = \left(\begin{array}{c|c} E_1 A & E_1 \end{array} \right)$$

Performing a row operation corresponding E_2 on this new augmented matrix, we obtain: >

$$\left(\begin{array}{c|c} E_2 E_1 A & E_2 E_1 \end{array} \right)$$

After performing successive row operations corresponding to the E_i's, we obtain: >

$$\left(\begin{array}{c|c} E_k \cdots E_2 E_1 A & E_k \cdots E_2 E_1 \end{array}\right) = \left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$$

In other words... >
to find A^{-1}, we perform Gaussian elimination on $(A|I)$ until the left half is reduced to I. Once the left half is reduced to I, the right half is precisely A^{-1}. If an $n \times n$ matrix A cannot be row reduced to I (i.e. A is row equivalent to a matrix which has a row whose entries are all zero), then it is not invertible.

Example.

Find the inverse of

$$A = \begin{pmatrix} -1 & 3 & -3 \\ 0 & -6 & 5 \\ -5 & -3 & 1 \end{pmatrix},$$

if it exists.
We perform Gaussian elimination on the following augmented matrix:

$$\left(\begin{array}{ccc|ccc} -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & -6 & 5 & 0 & 1 & 0 \\ -5 & -3 & 1 & 0 & 0 & 1 \end{array}\right)$$

We get: >

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1 & -1/2 \\ 0 & 1 & 0 & -25/6 & -8/3 & 5/6 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right)$$

Hence, A is row equivalent to I, which implies that it is invertible. Moreover,

$$A^{-1} = \begin{pmatrix} 3/2 & 1 & -1/2 \\ -25/6 & -8/3 & 5/6 \\ -5 & -3 & 1 \end{pmatrix}$$

Useful Fact:

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $ad - bc \neq 0$, then A is invertible, with:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Determinants

For a 1×1 matrix $A = (a)$. We define the **determinant** of A to be:

$$\det A = a.$$

Now, let $A = (a_{ij})$ be an $n \times n$ matrix, with $n > 1$. For $i, j \in \{1, 2, \ldots, n\}$, let M_{ij} denote the $(n-1) \times (n-1)$ **minor** matrix obtained from A by removing the i-th row and j-th column of A. The **determinant** of A is defined recursively as follows:

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$$\det A = |A| := \sum_{k=1}^{n} \underbrace{a_{1k}(-1)^{1+k} \det M_{1k}}_{1k-\text{th cofactor}}$$
$$= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| - \cdots + a_{1n}(-1)^{1+n}|M_{1n}|.$$

$$A = \left(\begin{array}{cccc|c|ccc} * & * & \cdots & * & a_{1k} & * & \cdots & * \\ \hline a_{21} & a_{22} & \cdots & a_{2(k-1)} & * & a_{2(k+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & a_{3(k-1)} & * & a_{3(k+1)} & \vdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & * & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{n(k-1)} & * & a_{n(k+1)} & \cdots & a_{nn} \end{array}\right),$$

$$M_{1k} = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2(k-1)} & a_{2(k+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & a_{3(k-1)} & a_{3(k+1)} & \vdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{n(k-1)} & a_{n(k+1)} & \cdots & a_{nn} \end{pmatrix}$$

The sum defining $\det A$ written above is called the **cofactor expansion** along the first row.

Theorem.

The determinant $\det A$ can be computed using a cofactor expansion along any row or any column:

$$\det A = \sum_{k=1}^{n} a_{ik}(-1)^{i+k} \det M_{ik} = \sum_{l=1}^{n} a_{lj}(-1)^{l+j} \det M_{lj}$$

The resulting values will be the same.

Exercise.

For a 2×2 matrix, we have:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Example.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \\ 7 & 1 & -2 \end{pmatrix}$$

$$M_{11} = \begin{pmatrix} 0 & 5 \\ 1 & -2 \end{pmatrix}, M_{12} = \begin{pmatrix} -1 & 5 \\ 7 & -2 \end{pmatrix}, M_{13} = \begin{pmatrix} -1 & 0 \\ 7 & 1 \end{pmatrix}$$

$$\det A = 1 \cdot \begin{vmatrix} 0 & 5 \\ 1 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} -1 & 5 \\ 7 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 0 \\ 7 & 1 \end{vmatrix} = 58.$$

Properties of the Determinant

Let A be an $n \times n$ matrix.

- $$\det A = \det A^\top,$$

  where $A^\top$ is the transpose of A, defined by $A^\top_{ij} = A_{ji}$. This follows from the fact that $\det A$ may be computed from the cofactor expansion along any row or column.

- If A is an upper or lower triangular matrix, then $\det A$ is equal to the product of its diagonal entries:

  $$\det \begin{pmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} = a_{11}a_{22}\cdots a_{nn}$$

- If one row or one column of A consists entirely of zeroes, then $\det A = 0$.

  $$\det \begin{pmatrix} 1 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & -7 & 9 \\ 0 & 3 & 1 & 8 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 1 & 2 & 0 & -5 \\ 6 & -7 & 0 & 3 \\ 3 & 4 & 0 & 9 \\ 0 & 3 & 0 & 8 \end{pmatrix} = 0$$

- If one row (column) of A is equal to a scalar multiple of another row (column), then $\det A = 0$.

- If a matrix B is obtained from a square matrix A by switching two rows, then $\det B = -\det A$.

- The determinant of an elementary matrix is nonzero.

- If E is an $n \times n$ elementary matrix, then $\det(EA) = (\det E)(\det A)$.