

# MATH1010 University Mathematics

Department of Mathematics  
The Chinese University of Hong Kong

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# Limits of sequences

## Definition (Infinite sequence of real numbers)

An **infinite sequence of real numbers** is defined by a function from the set of positive integers  $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$  to the set of real numbers  $\mathbb{R}$ .

## Example (Sequences)

- Arithmetic sequence:  $a_n = 3n + 4$ ; 7, 10, 13, 16...
- Geometric sequence:  $a_n = 3 \cdot 2^n$ ; 6, 12, 24, 48...
- Fibonacci's sequence:

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right);$$

1, 1, 2, 3, 5, 8, 13, ...

## Definition (Limit of sequence)

- 1 Suppose there exists real number  $L$  such that for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for any  $n > N$ , we have  $|a_n - L| < \epsilon$ . Then we say that  $a_n$  is **convergent**, or  $a_n$  **converges to**  $L$ , and write

$$\lim_{n \rightarrow \infty} a_n = L.$$

Otherwise we say that  $a_n$  is **divergent**.

- 2 Suppose for any  $M > 0$ , there exists  $N \in \mathbb{N}$  such that for any  $n > N$ , we have  $a_n > M$ . Then we say that  $a_n$  **tends to**  $+\infty$  as  $n$  tends to infinity, and write

$$\lim_{n \rightarrow \infty} a_n = +\infty.$$

We define  $a_n$  **tends to**  $-\infty$  in a similar way. Note that  $a_n$  is divergent if it tends to  $\pm\infty$ .

## Example (Intuitive meaning of limits of infinite sequences)

$a_n$	First few terms	Limit
$\frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$	0
$\frac{n}{n+1}$	$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$	1
$(-1)^{n+1}$	$1, -1, 1, -1, \dots$	does not exist
$2n$	$2, 4, 6, 8, \dots$	does not exist/ $+\infty$
$\left(1 + \frac{1}{n}\right)^n$	$2, \frac{9}{4}, \frac{64}{27}, \frac{625}{256}, \dots$	$e \approx 2.71828$
$\frac{F_{n+1}}{F_n}$	$1, 2, \frac{3}{2}, \frac{5}{3}, \dots$	$\frac{1 + \sqrt{5}}{2} \approx 1.61803$

## Definition (Monotonic sequence)

- 1 We say that  $a_n$  is **monotonic increasing (decreasing)** if for any  $m < n$ , we have  $a_m \leq a_n$  ( $a_m \geq a_n$ ). We say that  $a_n$  is **monotonic** if  $a_n$  is either monotonic increasing or monotonic decreasing.
- 2 We say that  $a_n$  is **strictly increasing (decreasing)** if for any  $m < n$ , we have  $a_m < a_n$  ( $a_m > a_n$ ).

## Definition (Bounded sequence)

We say that  $a_n$  is **bounded** if there exists real number  $M$  such that  $|a_n| < M$  for any  $n \in \mathbb{N}$ .

## Example (Bounded and monotonic sequence)

$a_n$	Terms	Bounded	Monotonic	Convergent (Limit)
$\frac{1}{n^2}$	$1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$	✓	✓	✓ (0)
$1 - \frac{(-1)^n}{n}$	$2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \dots$	✓	×	✓ (1)
$n^2$	$1, 4, 9, 16, \dots$	×	✓	×
$1 - (-1)^n$	$2, 0, 2, 0, \dots$	✓	×	×
$(-1)^n n$	$-1, 2, -3, 4, \dots$	×	×	×



## Theorem

*If  $a_n$  is convergent, then  $a_n$  is bounded.*

**Convergent  $\Rightarrow$  Bounded**

Note that the converse of the above statement is not correct.

**Bounded  $\not\Rightarrow$  Convergent**

The following theorem is very important and we will discuss it in details later.

## Theorem (Monotone convergence theorem)

*If  $a_n$  is bounded and monotonic, then  $a_n$  is convergent.*

**Bounded and Monotonic  $\Rightarrow$  Convergent**

## Exercise (True or False)

Suppose  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = a \pm b.$$

**Answer: T**

## Exercise (True or False)

Suppose  $\lim_{n \rightarrow \infty} a_n = a$  and  $c$  is a real number. Then

$$\lim_{n \rightarrow \infty} ca_n = ca.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , then

$$\lim_{n \rightarrow \infty} a_n b_n = ab.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

**Answer: F**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b \neq 0$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = 0$ , then

$$\lim_{n \rightarrow \infty} a_n b_n = 0.$$

**Answer: F**

## Example

For  $a_n = \frac{1}{n}$  and  $b_n = n$ , we have  $\lim_{n \rightarrow \infty} a_n = 0$  but

$$\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} \frac{1}{n} \cdot n = \lim_{n \rightarrow \infty} 1 = 1 \neq 0.$$

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = 0$  and  $b_n$  is **convergent**, then

$$\lim_{n \rightarrow \infty} a_n b_n = 0.$$

**Answer: T**

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n b_n &= \lim_{n \rightarrow \infty} a_n \lim_{n \rightarrow \infty} b_n \\ &= 0 \end{aligned}$$





## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = 0$  and  $b_n$  is **bounded**, then

$$\lim_{n \rightarrow \infty} a_n b_n = 0.$$

**Answer: T**

Caution! The previous proof does not work.

## Exercise (True or False)

*If  $a_n^2$  is convergent, then  $a_n$  is convergent.*

**Answer: F**

## Example

For  $a_n = (-1)^n$ ,  $a_n^2$  converges to 1 but  $a_n$  is divergent.

## Exercise (True or False)

*If  $a_n$  is convergent, then  $|a_n|$  is convergent.*

**Answer: T**

## Exercise (True or False)

*If  $|a_n|$  is convergent, then  $a_n$  is convergent.*

**Answer: F**

## Exercise (True or False)

*If  $a_n$  and  $b_n$  are divergent, then  $a_n + b_n$  is divergent.*

**Answer: F**

## Example

The sequences  $a_n = n$  and  $b_n = -n$  are divergent but  $a_n + b_n = 0$  converges to 0.

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} b_n = +\infty$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$

**Answer: F**

## Example

For  $a_n = n^2$  and  $b_n = n$ , we have  $\lim_{n \rightarrow \infty} b_n = +\infty$  but

$$\frac{a_n}{b_n} = \frac{n^2}{n} = n \text{ is divergent.}$$

## Exercise (True or False)

If  $a_n$  is convergent and  $\lim_{n \rightarrow \infty} b_n = \pm\infty$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$

**Answer: T**

## Exercise (True or False)

If  $a_n$  is **bounded** and  $\lim_{n \rightarrow \infty} b_n = \pm\infty$ , then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$

**Answer: T**



## Exercise (True or False)

*Suppose  $a_n$  is bounded. Suppose  $b_n$  is a sequence and there exists  $N$  such that  $b_n = a_n$  for any  $n > N$ . Then  $b_n$  is bounded.*

**Answer: T**

## Exercise (True or False)

Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . Suppose  $b_n$  is a sequence and there exists  $N$  such that  $b_n = a_n$  for any  $n > N$ . Then

$$\lim_{n \rightarrow \infty} b_n = a.$$

**Answer: T**

## Exercise (True or False)

Suppose  $a_n$  and  $b_n$  are convergent sequences such that  $a_n < b_n$  for any  $n$ . Then

$$\lim_{n \rightarrow \infty} a_n < \lim_{n \rightarrow \infty} b_n.$$

**Answer: F**

## Example

The sequences  $a_n = 0$  and  $b_n = \frac{1}{n}$  satisfy  $a_n < b_n$  for any  $n$ .  
However

$$\lim_{n \rightarrow \infty} a_n \not< \lim_{n \rightarrow \infty} b_n$$

because both of them are 0.

## Exercise (True or False)

Suppose  $a_n$  and  $b_n$  are convergent sequences such that  $a_n \leq b_n$  for any  $n$ . Then

$$\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_n = a$ , then

$$\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{2n+1} = a.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} a_{2n} = \lim_{n \rightarrow \infty} a_{2n+1} = a$ , then

$$\lim_{n \rightarrow \infty} a_n = a.$$

**Answer: T**

## Exercise (True or False)

If  $a_n$  is convergent, then

$$\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0.$$

**Answer: T**

## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$ , then  $a_n$  is convergent.

**Answer: F**

## Example

Let  $a_n = \sqrt{n}$ . Then  $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$  and  $a_n$  is divergent.



## Exercise (True or False)

If  $\lim_{n \rightarrow \infty} (a_{n+1} - a_n) = 0$  and  $a_n$  is bounded, then  $a_n$  is convergent.

**Answer: F**

## Example

$$0, \frac{1}{2}, 1, \frac{2}{3}, \frac{1}{3}, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, 0, \frac{1}{6}, \frac{2}{6}, \dots$$

## Example

Let  $a > 0$  be a positive real number.

$$\lim_{n \rightarrow \infty} a^n = \begin{cases} +\infty, & \text{if } a > 1 \\ 1, & \text{if } a = 1 \\ 0, & \text{if } 0 < a < 1 \end{cases} .$$

## Example

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2n - 5}{3n + 1} &= \lim_{n \rightarrow \infty} \frac{2 - \frac{5}{n}}{3 + \frac{1}{n}} \\ &= \frac{2 - 0}{3 + 0} \\ &= \frac{2}{3}\end{aligned}$$

## Example

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n^3 - 2n + 7}{4n^3 + 5n^2 - 3} &= \lim_{n \rightarrow \infty} \frac{1 - \frac{2}{n^2} + \frac{7}{n^3}}{4 + \frac{5}{n} - \frac{3}{n^3}} \\ &= \frac{1}{4}\end{aligned}$$

## Example

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{3n - \sqrt{4n^2 + 1}}{3n + \sqrt{9n^2 + 1}} &= \lim_{n \rightarrow \infty} \frac{3 - \frac{\sqrt{4n^2 + 1}}{n}}{3 + \frac{\sqrt{9n^2 + 1}}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{3 - \sqrt{4 + \frac{1}{n^2}}}{3 + \sqrt{9 + \frac{1}{n^2}}} \\ &= \frac{1}{6}\end{aligned}$$

## Example

$$\begin{aligned} & \lim_{n \rightarrow \infty} (n - \sqrt{n^2 - 4n + 1}) \\ = & \lim_{n \rightarrow \infty} \frac{(n - \sqrt{n^2 - 4n + 1})(n + \sqrt{n^2 - 4n + 1})}{n + \sqrt{n^2 - 4n + 1}} \\ = & \lim_{n \rightarrow \infty} \frac{n^2 - (n^2 - 4n + 1)}{n + \sqrt{n^2 - 4n + 1}} \\ = & \lim_{n \rightarrow \infty} \frac{4n - 1}{n + \sqrt{n^2 - 4n + 1}} \\ = & \lim_{n \rightarrow \infty} \frac{4 - \frac{1}{n}}{1 + \sqrt{1 - \frac{4}{n} + \frac{1}{n^2}}} \\ = & 2 \end{aligned}$$

## Example

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{\ln(n^4 + 1)}{\ln(n^3 + 1)} &= \lim_{n \rightarrow \infty} \frac{\ln(n^4(1 + \frac{1}{n^4}))}{\ln(n^3(1 + \frac{1}{n^3}))} \\ &= \lim_{n \rightarrow \infty} \frac{\ln n^4 + \ln(1 + \frac{1}{n^4})}{\ln n^3 + \ln(1 + \frac{1}{n^3})} \\ &= \lim_{n \rightarrow \infty} \frac{4 \ln n + \ln(1 + \frac{1}{n^4})}{3 \ln n + \ln(1 + \frac{1}{n^3})} \\ &= \lim_{n \rightarrow \infty} \frac{4 + \frac{\ln(1 + \frac{1}{n^4})}{\ln n}}{3 + \frac{\ln(1 + \frac{1}{n^3})}{\ln n}} \\ &= \frac{4}{3}\end{aligned}$$

# Squeeze theorem

## Theorem (Squeeze theorem)

Suppose  $a_n, b_n, c_n$  are sequences such that  $a_n \leq b_n \leq c_n$  for any  $n$  and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$ . Then  $b_n$  is convergent and

$$\lim_{n \rightarrow \infty} b_n = L.$$



## Theorem

If  $a_n$  is bounded and  $\lim_{n \rightarrow \infty} b_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .

## Proof.

Since  $a_n$  is bounded, there exists  $M$  such that  $-M < a_n < M$  for any  $n$ .  
Thus

$$-M|b_n| < a_n b_n < M|b_n|$$

for any  $n$ . Now

$$\lim_{n \rightarrow \infty} (-M|b_n|) = \lim_{n \rightarrow \infty} M|b_n| = 0.$$

Therefore by squeeze theorem, we have

$$\lim_{n \rightarrow \infty} a_n b_n = 0.$$



## Example

Find  $\lim_{n \rightarrow \infty} \frac{\sqrt{n} + (-1)^n}{\sqrt{n} - (-1)^n}$ .

## Solution

Since  $(-1)^n$  is bounded and  $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{\sqrt{n}} = 0 \text{ and therefore}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\sqrt{n} + (-1)^n}{\sqrt{n} - (-1)^n} &= \lim_{n \rightarrow \infty} \frac{1 + \frac{(-1)^n}{\sqrt{n}}}{1 - \frac{(-1)^n}{\sqrt{n}}} \\ &= 1 \end{aligned}$$

## Example

Show that  $\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0$ .

## Proof.

Observe that for any  $n \geq 3$ ,

$$0 < \frac{2^n}{n!} = 2 \left( \frac{2}{2} \cdot \frac{2}{3} \cdot \frac{2}{4} \cdots \frac{2}{n-1} \right) \frac{2}{n} \leq 2 \cdot \frac{2}{n} = \frac{4}{n}$$

and  $\lim_{n \rightarrow \infty} \frac{4}{n} = 0$ . By squeeze theorem, we have

$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0.$$



# Monotone convergence theorem

Theorem (Monotone convergence theorem)

If  $a_n$  is **bounded and monotonic**, then  $a_n$  is **convergent**.

**Bounded and Monotonic  $\Rightarrow$  Convergent**

## Example

Let  $a_n$  be the sequence defined by the recursive relation

$$\begin{cases} a_{n+1} = \sqrt{a_n + 1} \text{ for } n \geq 1 \\ a_1 = 1 \end{cases}$$

Find  $\lim_{n \rightarrow \infty} a_n$ .

$n$	$a_n$
1	1
2	1.414213562
3	1.553773974
4	1.598053182
5	1.611847754
10	1.618016542
15	1.618033940

## Solution

Suppose  $\lim_{n \rightarrow \infty} a_n = a$ . Then  $\lim_{n \rightarrow \infty} a_{n+1} = a$  and thus

$$a = \sqrt{a+1}$$

$$a^2 = a+1$$

$$a^2 - a - 1 = 0$$

By solving the quadratic equation, we have

$$a = \frac{1 + \sqrt{5}}{2} \text{ or } \frac{1 - \sqrt{5}}{2}.$$

It is obvious that  $a > 0$ . Therefore

$$a = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887$$

## Solution

The above solution is not complete. The solution is valid only after we have proved that  $\lim_{n \rightarrow \infty} a_n$  exists and is positive. This can be done by using **monotone convergent theorem**. We are going to show that  $a_n$  is **bounded and monotonic**.

### Boundedness

We prove that  $1 \leq a_n < 2$  for all  $n \geq 1$  by induction.

(Base case) When  $n = 1$ , we have  $a_1 = 1$  and  $1 \leq a_1 < 2$ .

(Induction step) Assume that  $1 \leq a_k < 2$ . Then

$$a_{k+1} = \sqrt{a_k + 1} \geq \sqrt{1 + 1} > 1$$

$$a_{k+1} = \sqrt{a_k + 1} < \sqrt{2 + 1} < 2$$

Thus  $1 \leq a_n < 2$  for any  $n \geq 1$  which implies that  $a_n$  is bounded.

## Solution

**Monotonicity**

We prove that  $a_{n+1} > a_n$  for any  $n \geq 1$  by induction.

(Base case) When  $n = 1$ ,  $a_1 = 1$ ,  $a_2 = \sqrt{2}$  and thus  $a_2 > a_1$ .

(Induction step) Assume that

$$a_{k+1} > a_k \text{ (Induction hypothesis).}$$

Then

$$\begin{aligned} a_{k+2} &= \sqrt{a_{k+1} + 1} > \sqrt{a_k + 1} \text{ (by induction hypothesis)} \\ &= a_{k+1} \end{aligned}$$

This completes the induction step and thus  $a_n$  is strictly increasing.

We have proved that  $a_n$  is bounded and strictly increasing. Therefore  $a_n$  is convergent by monotone convergence theorem. Since  $a_n \geq 1$  for any  $n$ , we have  $\lim_{n \rightarrow \infty} a_n \geq 1$  is positive.



## Example

Let  $a_n = \frac{F_{n+1}}{F_n}$  where  $F_n$  is the Fibonacci's sequence defined by

$$\begin{cases} F_{n+2} = F_{n+1} + F_n \\ F_1 = F_2 = 1 \end{cases}$$

Find  $\lim_{n \rightarrow \infty} a_n$ .

$n$	$a_n$
1	1
2	2
3	1.5
4	1.666666666
5	1.6
10	1.618181818
15	1.618032787
20	1.618033999

## Theorem

For any  $n \geq 1$ ,

- 1  $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$
- 2  $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$

## Proof

- 1 When  $n = 1$ , we have  $F_3F_1 - F_2^2 = 2 \cdot 1 - 1^2 = 1 = (-1)^2$ . Assume

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}.$$

Then

$$\begin{aligned} F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 \\ &= F_{k+2}(F_{k+1} - F_{k+2}) + F_{k+1}^2 \\ &= -F_{k+2}F_k + F_{k+1}^2 \\ &= (-1)^{k+2} \text{ (by induction hypothesis)} \end{aligned}$$

Therefore  $F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$  for any  $n \geq 1$ .

## Proof.

The proof for the second statement is basically the same. When  $n = 1$ , we have  $F_4F_1 - F_3F_2 = 3 \cdot 1 - 2 \cdot 1 = 1 = (-1)^2$ . Assume

$$F_{k+3}F_k - F_{k+2}F_{k+1} = (-1)^{k+1}.$$

Then

$$\begin{aligned} F_{k+4}F_{k+1} - F_{k+3}F_{k+2} &= (F_{k+3} + F_{k+2})F_{k+1} - F_{k+3}F_{k+2} \\ &= F_{k+3}(F_{k+1} - F_{k+2}) + F_{k+2}F_{k+1} \\ &= -F_{k+3}F_k + F_{k+2}F_{k+1} \\ &= -(-1)^{k+1} \text{ (by induction hypothesis)} \\ &= (-1)^{k+2} \end{aligned}$$

Therefore  $F_{n+3}F_n - F_{n+2}F_{n+1} = (-1)^{n+1}$  for any  $n \geq 1$ . □

## Theorem

$$\text{Let } a_n = \frac{F_{n+1}}{F_n}.$$

- 1 The sequence  $a_1, a_3, a_5, a_7, \dots$ , is strictly increasing.
- 2 The sequence  $a_2, a_4, a_6, a_8, \dots$ , is strictly decreasing.

## Proof.

For any  $k \geq 1$ , we have

$$\begin{aligned} a_{2k+1} - a_{2k-1} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k}}{F_{2k-1}} = \frac{F_{2k+2}F_{2k-1} - F_{2k+1}F_{2k}}{F_{2k+1}F_{2k-1}} \\ &= \frac{(-1)^{2k}}{F_{2k+1}F_{2k-1}} = \frac{1}{F_{2k+1}F_{2k-1}} > 0 \end{aligned}$$

Therefore  $a_1, a_3, a_5, a_7, \dots$ , is strictly increasing. The second statement can be proved in a similar way.  $\square$

## Theorem

$$\lim_{k \rightarrow \infty} (a_{2k+1} - a_{2k}) = 0$$

## Proof.

For any  $k \geq 1$ ,

$$\begin{aligned} a_{2k+1} - a_{2k} &= \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{2k+1}}{F_{2k}} \\ &= \frac{F_{2k+2}F_{2k} - F_{2k+1}^2}{F_{2k+1}F_{2k}} = \frac{1}{F_{2k+1}F_{2k}} \end{aligned}$$

Therefore

$$\lim_{k \rightarrow \infty} (a_{2k+1} - a_{2k}) = \lim_{k \rightarrow \infty} \frac{1}{F_{2k+1}F_{2k}} = 0.$$



## Theorem

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2}$$

## Proof

First we prove that  $a_n = \frac{F_{n+1}}{F_n}$  is convergent.

$a_n$  is bounded. ( $1 \leq a_n \leq 2$  for any  $n$ .)

$a_{2k+1}$  and  $a_{2k}$  are convergent. (They are bounded and monotonic.)

$$\lim_{k \rightarrow \infty} (a_{2k+1} - a_{2k}) = 0 \Rightarrow \lim_{k \rightarrow \infty} a_{2k+1} = \lim_{k \rightarrow \infty} a_{2k}$$

It follows that  $a_n$  is convergent and

$$\lim_{n \rightarrow \infty} a_n = \lim_{k \rightarrow \infty} a_{2k+1} = \lim_{k \rightarrow \infty} a_{2k}.$$

Proof.

To evaluate the limit, suppose  $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = L$ . Then

$$L = \lim_{n \rightarrow \infty} \frac{F_{n+2}}{F_{n+1}} = \lim_{n \rightarrow \infty} \frac{F_{n+1} + F_n}{F_{n+1}} = \lim_{n \rightarrow \infty} \left( 1 + \frac{F_n}{F_{n+1}} \right) = 1 + \frac{1}{L}$$
$$L^2 - L - 1 = 0$$

By solving the quadratic equation, we have

$$L = \frac{1 + \sqrt{5}}{2} \text{ or } \frac{1 - \sqrt{5}}{2}.$$

We must have  $L \geq 1$  since  $a_n \geq 1$  for any  $n$ . Therefore

$$L = \frac{1 + \sqrt{5}}{2}.$$



## Remarks

The limit can be calculate directly using the formula

$$\begin{aligned}F_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)\end{aligned}$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}$$

are the roots of the quadratic equation

$$x^2 - x - 1 = 0.$$



## Theorem

Let

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^n \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Then

- 1  $a_n < b_n$  for any  $n > 1$ .
- 2  $a_n$  and  $b_n$  are convergent and

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

$$b_n = \sum_{k=0}^n \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

n	$a_n$	$b_n$
1	2	2
5	2.48832	2.71666666666666
10	2.593742	2.718281801146
100	2.704813	2.718281828459
100000	2.718268	2.718281828459

The limit of the two sequences is the important Euler's number

$$e \approx 2.71828\ 18284\ 59045\ 23536\ \dots$$

which is also known as the Napier's constant.

## Definition (Convergence of infinite series)

We say that an **infinite series**

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is **convergent** if the sequence of partial sums

$s_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_n$  is convergent. If the infinite series is convergent, then we define

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

# Limits of functions

## Definition (Function)

A real valued **function** on a subset  $D \subset \mathbb{R}$  is a real value  $f(x)$  assigned to each of the values  $x \in D$ . The set  $D$  is called the **domain** of the function.

Given an expression  $f(x)$  in  $x$ , the domain  $D$  is understood to be taken as the set of all real numbers  $x$  such that  $f(x)$  is defined. This is called the maximum domain of definition of  $f(x)$ .

## Definition (Graph of function)

Let  $f(x)$  is a real valued function. The graph of  $f(x)$  is the set

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

## Definition

Let  $f(x)$  be a real valued function and  $D$  be its domain. We say that  $f(x)$  is

- 1 **injective** if for any  $x_1, x_2 \in D$  with  $x_1 \neq x_2$ , we have  $f(x_1) \neq f(x_2)$ .
- 2 **surjective** if for any real number  $y \in \mathbb{R}$ , there exists  $x \in D$  such that  $f(x) = y$ .
- 3 **bijective** if  $f(x)$  is both injective and surjective.

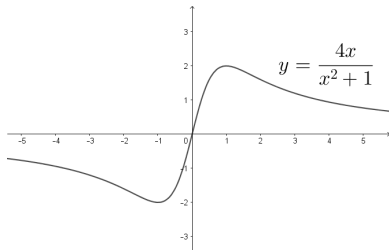
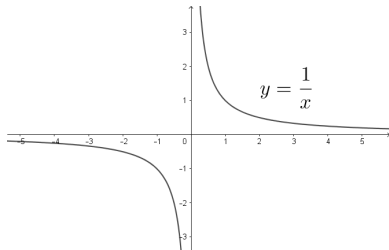
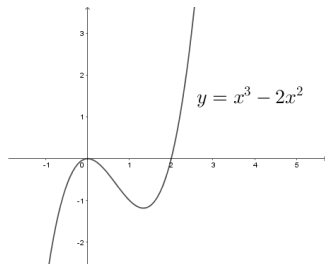
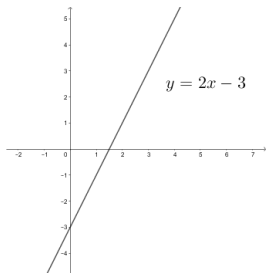
## Definition

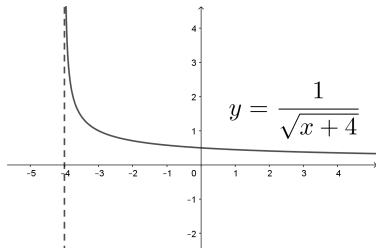
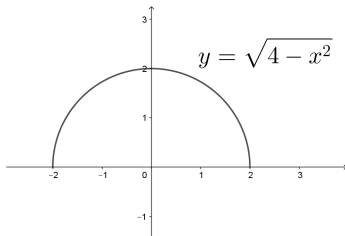
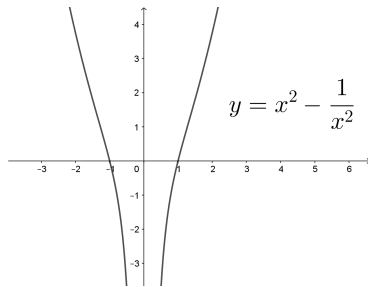
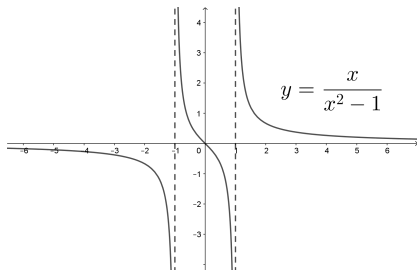
Let  $f(x)$  be a real valued function. We say that  $f(x)$  is

- 1 **even** if  $f(-x) = f(x)$  for any  $x$ .
- 2 **odd** if  $f(-x) = -f(x)$  for any  $x$ .

## Example

$f(x)$	Domain	Injective	Surjective	Bijjective	Even	Odd
$2x - 3$	$\mathbb{R}$	✓	✓	✓	×	×
$x^3 - 2x^2$	$\mathbb{R}$	×	✓	×	×	×
$\frac{1}{x}$	$x \neq 0$	✓	×	×	×	✓
$\frac{4x}{x^2 + 1}$	$\mathbb{R}$	×	×	×	×	✓
$\frac{x}{x^2 - 1}$	$x \neq \pm 1$	×	✓	×	×	✓
$x^2 - \frac{1}{x^2}$	$x \neq 0$	×	✓	×	✓	×
$\sqrt{4 - x^2}$	$-2 \leq x \leq 2$	×	×	×	✓	×
$\frac{1}{\sqrt{x + 4}}$	$x > -4$	✓	×	×	×	×







## Definition (Limit of function)

Let  $f(x)$  be a real valued function.

- ① We say that a real number  $l$  is a limit of  $f(x)$  at  $x = a$  if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - l| < \epsilon$$

and write

$$\lim_{x \rightarrow a} f(x) = l.$$

- ② We say that a real number  $l$  is a limit of  $f(x)$  at  $+\infty$  if for any  $\epsilon > 0$ , there exists  $R > 0$  such that

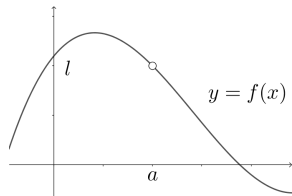
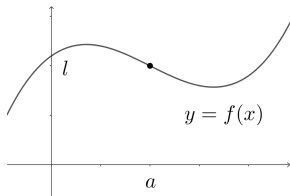
$$\text{if } x > R, \text{ then } |f(x) - l| < \epsilon$$

and write

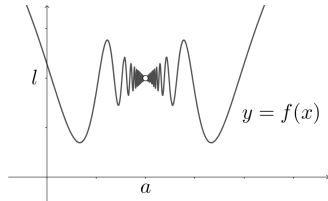
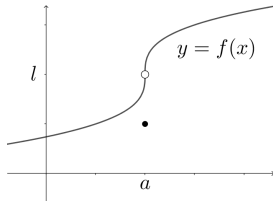
$$\lim_{x \rightarrow +\infty} f(x) = l.$$

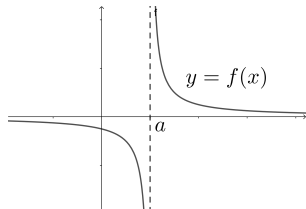
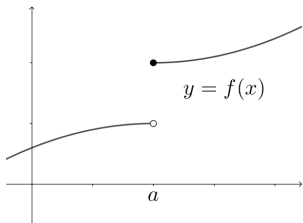
The limit of  $f(x)$  at  $-\infty$  is defined similarly.

- 1 Note that for the limit of  $f(x)$  at  $x = a$  to exist,  $f(x)$  may not be defined at  $x = a$  and even if  $f(a)$  is defined, the value of  $f(a)$  does not affect the value of  $\lim_{x \rightarrow a} f(x)$ .
- 2 The limit of  $f(x)$  at  $x = a$  may not exist. However the limit is unique if it exists.

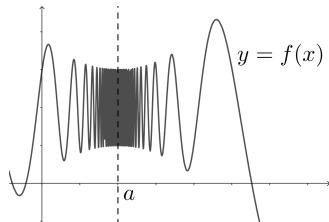
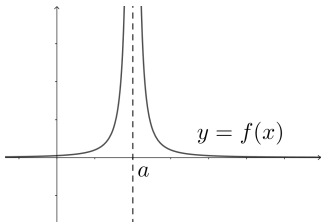


$$\lim_{x \rightarrow a} f(x) = l$$





$\lim_{x \rightarrow a} f(x)$   
does not exist



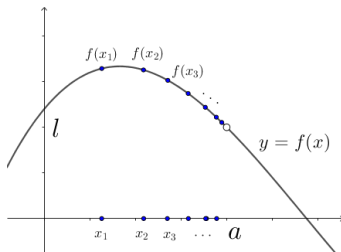
## Theorem (Sequential criterion for limits of functions)

Let  $f(x)$  be a real valued function. Then

$$\lim_{x \rightarrow a} f(x) = l$$

if and only if for any sequence  $x_n$  of real numbers with  $\lim_{n \rightarrow \infty} x_n = a$ , we have

$$\lim_{n \rightarrow \infty} f(x_n) = l.$$



## Theorem

Let  $f(x)$ ,  $g(x)$  be functions such that  $\lim_{x \rightarrow a} f(x)$ ,  $\lim_{x \rightarrow a} g(x)$  exist and  $c$  be a real number. Then

$$\textcircled{1} \quad \lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

$$\textcircled{3} \quad \lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

$$\textcircled{4} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

## Theorem

Let  $g(u)$  be a function of  $u$  and  $u = f(x)$  be a function of  $x$ .  
Suppose

- 1  $\lim_{x \rightarrow a} f(x) = b \in [-\infty, +\infty]$
- 2  $\lim_{u \rightarrow b} g(u) = l$
- 3  $f(x) \neq b$  when  $x \neq a$  or  $g(b) = l$ .

Then

$$\lim_{x \rightarrow a} (g \circ f)(x) = l.$$

$$x \xrightarrow{f} u = f(x) \xrightarrow{g} (g \circ f)(x) = g(u) = g(f(x))$$

## Example

$$\begin{aligned} 1. \quad \lim_{x \rightarrow +\infty} \frac{6x^3 + 2x^2 - 5}{2x^3 - 3x + 1} &= \lim_{x \rightarrow +\infty} \frac{6 + \frac{2}{x} - \frac{5}{x^3}}{2 - \frac{3}{x^2} + \frac{1}{x^3}} \\ &= \lim_{y \rightarrow 0} \frac{6 + 2y - 5y^3}{2 - 3y + y^3} \\ &= 3 \end{aligned}$$

$$\begin{aligned} 2. \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^2}\right)^{n^2} &= \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m \\ &= e \end{aligned}$$



### Theorem (Squeeze theorem)

Let  $f(x), g(x), h(x)$  be real valued functions. Suppose

- 1  $f(x) \leq g(x) \leq h(x)$  for any  $x \neq a$  on a neighborhood of  $a$ , and
- 2  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = l$ .

Then the limit of  $g(x)$  at  $x = a$  exists and  $\lim_{x \rightarrow a} g(x) = l$ .

### Theorem

Suppose

- 1  $f(x)$  is bounded, and
- 2  $\lim_{x \rightarrow a} g(x) = 0$

Then  $\lim_{x \rightarrow a} f(x)g(x) = 0$ .

# Exponential, logarithmic and trigonometric functions

## Definition (Exponential function)

The **exponential function** is defined for real number  $x \in \mathbb{R}$  by

$$\begin{aligned}e^x &= \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\end{aligned}$$

- 1 It can be proved that the two limits in the definition exist and converge to the same value for any real number  $x$ .
- 2  $e^x$  is just a notation for the exponential function. One should not interpret it as 'e to the power  $x$ '.

## Theorem

*For any  $x, y \in \mathbb{R}$ , we have*

$$e^{x+y} = e^x e^y.$$

Caution! One cannot use law of indices to prove the above identity. It is because  $e^x$  is just a notation for the exponential function and it does not mean 'e to the power  $x$ '. In fact we have not defined what  $a^x$  means when  $x$  is a real number which is not rational.

## Theorem

- 1  $e^x > 0$  for any real number  $x$ .
- 2  $e^x$  is strictly increasing.

## Proof.

- 1 For any  $x > 0$ , we have  $e^x > 1 + x > 1$ . If  $x < 0$ , then

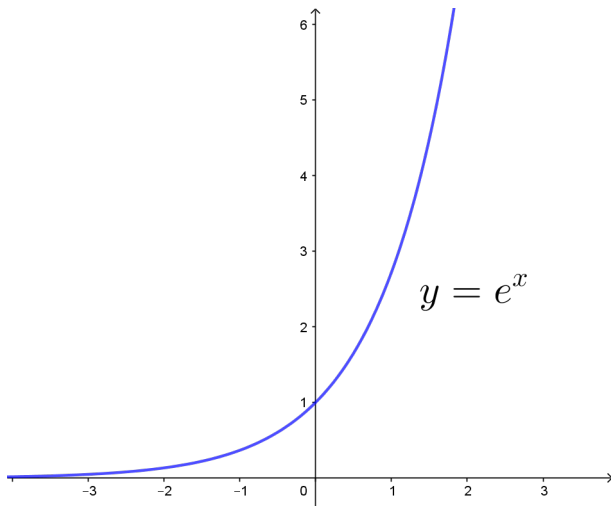
$$\begin{aligned}e^x e^{-x} &= e^{x+(-x)} = e^0 = 1 \\e^x &= \frac{1}{e^{-x}} > 0\end{aligned}$$

since  $e^{-x} > 1$ . Therefore  $e^x > 0$  for any  $x \in \mathbb{R}$ .

- 2 Let  $x, y$  be real numbers with  $x < y$ . Then  $y - x > 0$  which implies  $e^{y-x} > 1$ . Therefore

$$e^y = e^{x+(y-x)} = e^x e^{y-x} > e^x.$$





### Definition (Logarithmic function)

The **logarithmic function** is the function  $\ln : \mathbb{R}^+ \rightarrow \mathbb{R}$  defined for  $x > 0$  by

$$y = \ln x \text{ if } e^y = x.$$

In other words,  $\ln x$  is the inverse function of  $e^x$ .

It can be proved that for any  $x > 0$ , there exists unique real number  $y$  such that  $e^y = x$ .

## Theorem

- 1  $\ln xy = \ln x + \ln y$
- 2  $\ln \frac{x}{y} = \ln x - \ln y$
- 3  $\ln x^n = n \ln x$  for any integer  $n \in \mathbb{Z}$ .

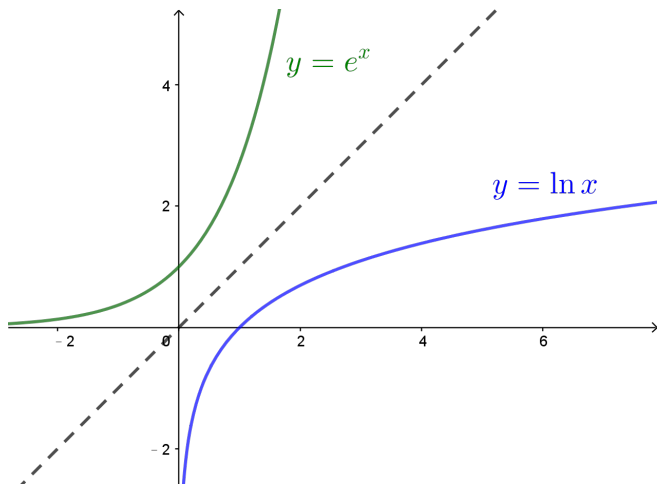
## Proof.

- 1 Let  $u = \ln x$  and  $v = \ln y$ . Then  $x = e^u$ ,  $y = e^v$  and we have

$$xy = e^u e^v = e^{u+v} = e^{\ln x + \ln y}$$

which means  $\ln xy = \ln x + \ln y$ .

Other parts can be proved similarly. □



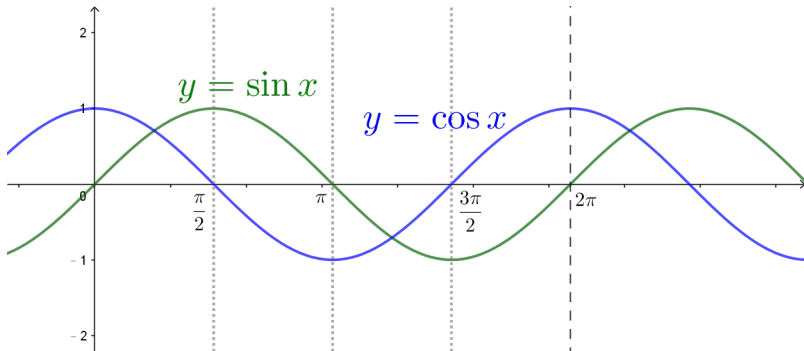


## Definition (Cosine and sine functions)

The **cosine** and **sine** functions are defined for real number  $x \in \mathbb{R}$  by the infinite series

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\end{aligned}$$

- 1 When the sine and cosine are interpreted as trigonometric ratios, the angles are measured in radian. ( $180^\circ = \pi$ )
- 2 The series for cosine and sine are convergent for any real number  $x \in \mathbb{R}$ .



There are four more trigonometric functions namely tangent, cotangent, secant and cosecant functions. All of them are defined in terms of sine and cosine.

### Definition (Trigonometric functions)

$$\tan x = \frac{\sin x}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, k \in \mathbb{Z}$$

$$\cot x = \frac{\cos x}{\sin x}, \text{ for } x \neq k\pi, k \in \mathbb{Z}$$

$$\sec x = \frac{1}{\cos x}, \text{ for } x \neq \frac{2k+1}{2}\pi, k \in \mathbb{Z}$$

$$\csc x = \frac{1}{\sin x}, \text{ for } x \neq k\pi, k \in \mathbb{Z}$$

## Theorem (Trigonometric identities)

$$\textcircled{1} \quad \cos^2 x + \sin^2 x = 1; \quad \sec^2 x - \tan^2 x = 1; \quad \csc^2 x - \cot^2 x = 1$$

$$\textcircled{2} \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y;$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y;$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\textcircled{3} \quad \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x;$$

$$\sin 2x = 2 \sin x \cos x;$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\textcircled{4} \quad 2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$2 \cos x \sin y = \sin(x + y) - \sin(x - y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$\textcircled{5} \quad \cos x + \cos y = 2 \cos \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right)$$

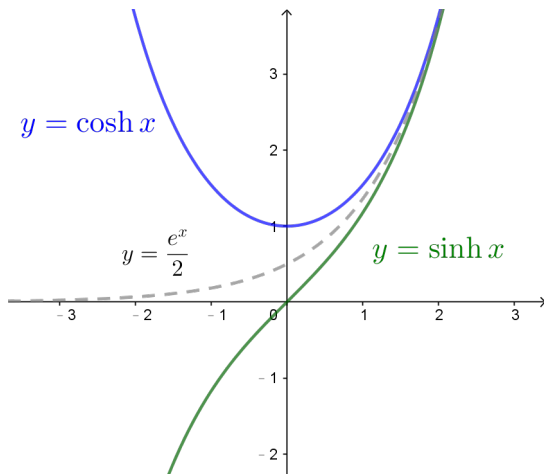
$$\sin x + \sin y = 2 \sin \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)$$

$$\sin x - \sin y = 2 \cos \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right)$$

## Definition (Hyperbolic function)

The **hyperbolic functions** are defined for  $x \in \mathbb{R}$  by

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \sinh x &= \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\end{aligned}$$



## Theorem (Hyperbolic identities)

- 1  $\cosh^2 x - \sinh^2 x = 1$
- 2  $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$   
 $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- 3  $\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x;$   
 $\sinh 2x = 2 \sinh x \cosh x$

## Theorem

$$\textcircled{1} \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\textcircled{2} \quad \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



Proof.  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$

For any  $-1 < x < 1$  with  $x \neq 0$ , we have

$$\begin{aligned} \frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \dots \\ &\leq 1 + \frac{x}{2} + \left( \frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \dots \right) = 1 + \frac{x}{2} + \frac{x^2}{2} \end{aligned}$$

$$\begin{aligned} \frac{e^x - 1}{x} &= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots \\ &\geq 1 + \frac{x}{2} - \left( \frac{x^2}{4} + \frac{x^2}{8} + \frac{x^2}{16} + \dots \right) = 1 + \frac{x}{2} - \frac{x^2}{2} \end{aligned}$$

and  $\lim_{x \rightarrow 0} \left(1 + \frac{x}{2} + \frac{x^2}{2}\right) = \lim_{x \rightarrow 0} \left(1 + \frac{x}{2} - \frac{x^2}{2}\right) = 1.$  Therefore  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1.$   $\square$

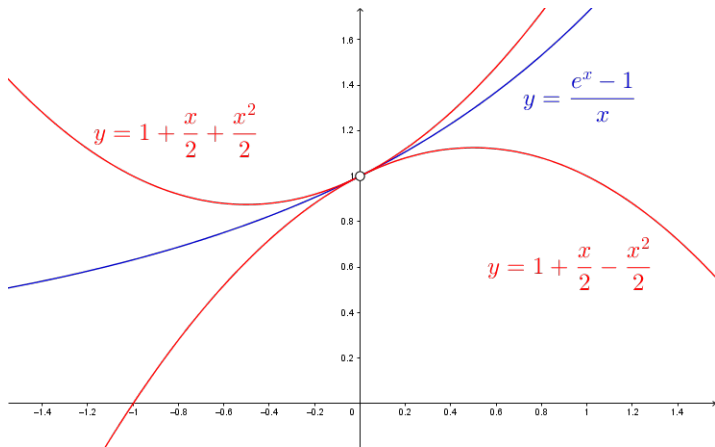


Figure:  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

Proof.  $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1.$

Let  $y = \ln(1+x)$ . Then

$$\begin{aligned}e^y &= 1+x \\x &= e^y - 1\end{aligned}$$

and  $x \rightarrow 0$  as  $y \rightarrow 0$ . We have

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} &= \lim_{y \rightarrow 0} \frac{y}{e^y - 1} \\&= 1\end{aligned}$$



Note that the first part implies  $\lim_{y \rightarrow 0} (e^y - 1) = 0.$

Proof.  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$

Note that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \dots$$

For any  $-1 < x < 1$  with  $x \neq 0$ , we have

$$\frac{\sin x}{x} = 1 - \left( \frac{x^2}{3!} - \frac{x^4}{5!} \right) - \left( \frac{x^6}{7!} - \frac{x^8}{9!} \right) - \dots \leq 1$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \left( \frac{x^4}{5!} - \frac{x^6}{7!} \right) + \left( \frac{x^8}{9!} - \frac{x^{10}}{11!} \right) + \dots \geq 1 - \frac{x^2}{6}$$

and  $\lim_{x \rightarrow 0} 1 = \lim_{x \rightarrow 0} \left( 1 - \frac{x^2}{6} \right) = 1.$  Therefore

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$



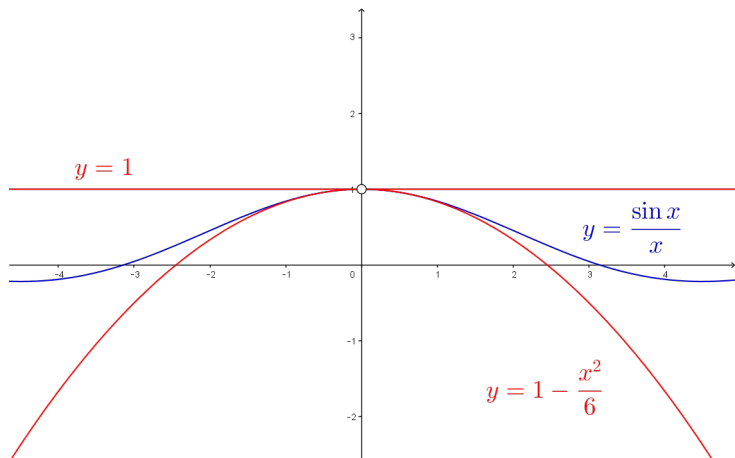


Figure:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

## Theorem

Let  $k$  be a positive integer.

$$\textcircled{1} \quad \lim_{x \rightarrow +\infty} \frac{x^k}{e^x} = 0$$

$$\textcircled{2} \quad \lim_{x \rightarrow +\infty} \frac{(\ln x)^k}{x} = 0$$

## Proof.

- 1 For any  $x > 0$ ,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots > \frac{x^{k+1}}{(k+1)!}$$

and thus

$$0 < \frac{x^k}{e^x} < \frac{(k+1)!}{x}.$$

Moreover  $\lim_{x \rightarrow +\infty} \frac{(k+1)!}{x} = 0$ . Therefore

$$\lim_{x \rightarrow +\infty} \frac{x^k}{e^x} = 0.$$

- 2 Let  $x = e^y$ . Then  $x \rightarrow +\infty$  as  $y \rightarrow +\infty$  and  $\ln x = y$ . We have

$$\lim_{x \rightarrow +\infty} \frac{(\ln x)^k}{x} = \lim_{y \rightarrow +\infty} \frac{y^k}{e^y} = 0.$$



## Example

$$\begin{aligned} 1. \lim_{x \rightarrow 4} \frac{x^2 - 16}{\sqrt{x} - 2} &= \lim_{x \rightarrow 4} \frac{(x-4)(x+4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4} \frac{(x-4)(x+4)(\sqrt{x}+2)}{x-4} \\ &= \lim_{x \rightarrow 4} (x+4)(\sqrt{x}+2) = 32 \\ 2. \lim_{x \rightarrow +\infty} \frac{3e^{2x} + e^x - x^4}{4e^{2x} - 5e^x + 2x^4} &= \lim_{x \rightarrow +\infty} \frac{3 + e^{-x} - x^4 e^{-2x}}{4 - 5e^{-x} + 2x^4 e^{-2x}} = \frac{3}{4} \\ 3. \lim_{x \rightarrow +\infty} \frac{\ln(2e^{4x} + x^3)}{\ln(3e^{2x} + 4x^5)} &= \lim_{x \rightarrow +\infty} \frac{4x + \ln(2 + x^3 e^{-4x})}{2x + \ln(3 + 4x^5 e^{-2x})} \\ &= \lim_{x \rightarrow +\infty} \frac{4 + \frac{\ln(2 + x^3 e^{-4x})}{x}}{2 + \frac{\ln(3 + 4x^5 e^{-2x})}{x}} = 2 \\ 4. \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 - 2x}) &= \lim_{x \rightarrow -\infty} \frac{(x + \sqrt{x^2 - 2x})(x - \sqrt{x^2 - 2x})}{x - \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow -\infty} \frac{2x}{x - \sqrt{x^2 - 2x}} \\ &= \lim_{x \rightarrow -\infty} \frac{2}{1 + \sqrt{1 - \frac{2}{x}}} = 1 \end{aligned}$$



## Example

$$5. \lim_{x \rightarrow 0} \frac{\sin 6x - \sin x}{\sin 4x - \sin 3x} = \lim_{x \rightarrow 0} \frac{\frac{6 \sin 6x}{6x} - \frac{\sin x}{x}}{\frac{4 \sin 4x}{4x} - \frac{3 \sin 3x}{3x}} = \frac{6 - 1}{4 - 3} = 5$$

$$6. \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \tan x} = \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x)}{x \frac{\sin x}{\cos x} (1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{(1 - \cos^2 x) \cos x}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right) \frac{\cos x}{1 + \cos x} = \frac{1}{2}$$

$$7. \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\ln(1 + 3x)} = \lim_{x \rightarrow 0} \frac{2}{3} \cdot \frac{e^{2x} - 1}{2x} \cdot \frac{3x}{\ln(1 + 3x)} = \frac{2}{3}$$

$$8. \lim_{x \rightarrow 0} \frac{x \ln(1 + \sin x)}{1 - \sqrt{\cos x}} = \lim_{x \rightarrow 0} \frac{x(1 + \sqrt{\cos x})(1 + \cos x) \ln(1 + \sin x)}{1 - \cos^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \frac{\ln(1 + \sin x)}{\sin x} (1 + \sqrt{\cos x})(1 + \cos x)$$

$$= 4$$

# Continuity of functions

## Definition (Continuity)

Let  $f(x)$  be a real valued function. We say that  $f(x)$  is **continuous** at  $x = a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

In other words,  $f(x)$  is continuous at  $x = a$  if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$\text{if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \epsilon.$$

We say that  $f(x)$  is continuous on an interval in  $\mathbb{R}$  if  $f(x)$  is continuous at every point on the interval.

## Theorem

Let  $g(u)$  be a function in  $u$  and  $u = f(x)$  be a function in  $x$ .  
Suppose  $g(u)$  is continuous and the limit of  $f(x)$  at  $x = a$  exists.  
Then

$$\lim_{x \rightarrow a} (g \circ f)(x) = \lim_{x \rightarrow a} g(f(x)) = g\left(\lim_{x \rightarrow a} f(x)\right).$$

$$x \xrightarrow{f} u = f(x) \xrightarrow{g} (g \circ f)(x) = g(u) = g(f(x))$$

## Theorem

- 1 For any non-negative integer  $n$ ,  $f(x) = x^n$  is continuous on  $\mathbb{R}$ .
- 2 The functions  $e^x$ ,  $\cos x$ ,  $\sin x$  are continuous on  $\mathbb{R}$ .
- 3 The logarithmic function  $\ln x$  is continuous on  $\mathbb{R}^+$ .

## Theorem

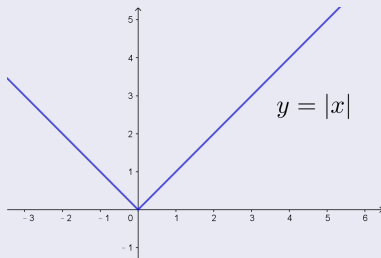
Suppose  $f(x)$ ,  $g(x)$  are continuous functions and  $c$  is a real number. Then the following functions are continuous.

- 1  $f(x) + g(x)$
- 2  $cf(x)$
- 3  $f(x)g(x)$
- 4  $\frac{f(x)}{g(x)}$  at the points where  $g(x) \neq 0$ .
- 5  $(f \circ g)(x)$

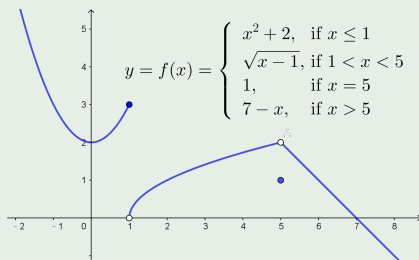
## Definition

The **absolute value** of  $x \in \mathbb{R}$  is defined by

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

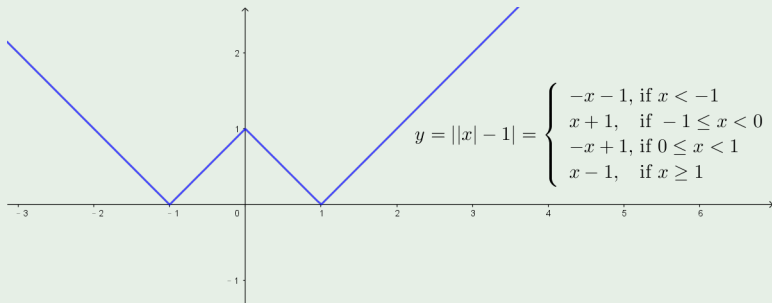


## Example (Piecewise defined function)



$a$	1	5
$\lim_{x \rightarrow a^-} f(x)$	3	2
$\lim_{x \rightarrow a^+} f(x)$	0	2
$\lim_{x \rightarrow a} f(x)$	does not exist	2

## Example





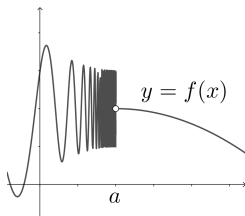
## Theorem

A function  $f(x)$  is continuous at  $x = a$  if

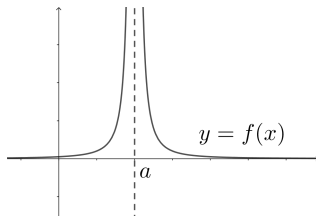
$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = f(a).$$

The theorem is usually used to check whether a piecewise defined function is continuous.

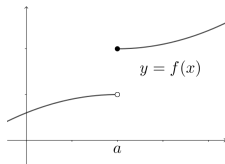
The following functions are not continuous at  $x = a$ .



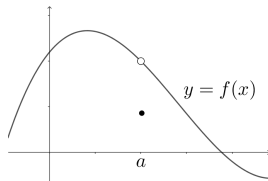
$\lim_{x \rightarrow a^-} f(x)$  does not exist



$\lim_{x \rightarrow a} f(x)$  does not exist



$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$



$\lim_{x \rightarrow a} f(x) \neq f(a)$

## Example

Given that the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 2 \\ a & \text{if } x = 2 \\ x^2 + b & \text{if } x > 2 \end{cases}$$

is continuous at  $x = 2$ . Find the value of  $a$  and  $b$ .

## Solution

*Note that*

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (2x - 1) = 3$$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x^2 + b) = 4 + b$$

$$f(2) = a$$

Since  $f(x)$  is continuous at  $x = 2$ , we have  $3 = 4 + b = a$  which implies  $a = 3$  and  $b = -1$ .

## Example

Prove that the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

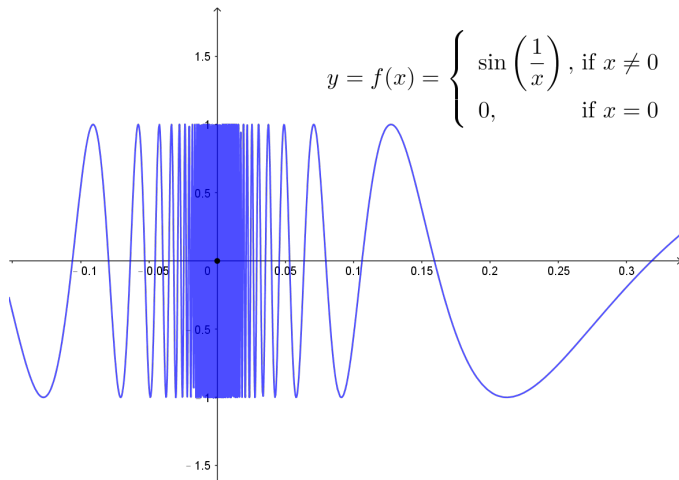
is not continuous at  $x = 0$ .

## Proof.

Let  $x_n = \frac{2}{(2n+1)\pi}$  for  $n = 1, 2, 3, \dots$ . Then  $\lim_{n \rightarrow \infty} x_n = 0$  and

$$f(x_n) = \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n.$$

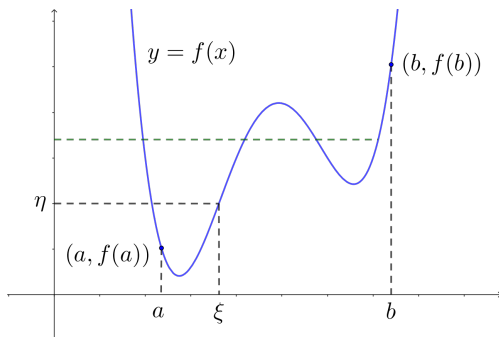
Thus  $\lim_{n \rightarrow \infty} f(x_n)$  does not exist. Therefore  $f(x)$  is not continuous at  $x = 0$ . □



$f(x)$  is not continuous at  $x = 0$ .

## Theorem (Intermediate value theorem)

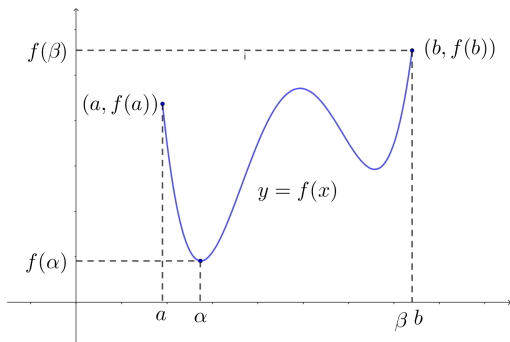
Suppose  $f(x)$  is a function which is **continuous** on  $[a, b]$ . Then for any real number  $\eta$  between  $f(a)$  and  $f(b)$ , there exists  $\xi \in (a, b)$  such that  $f(\xi) = \eta$ .



## Theorem (Extreme value theorem)

Suppose  $f(x)$  is a function which is **continuous** on a **closed and bounded** interval  $[a, b]$ . Then there exists  $\alpha, \beta \in [a, b]$  such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$



# Differentiable functions

## Definition (Differentiable function)

Let  $f(x)$  be a function. Denote

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

and we say that  $f(x)$  is **differentiable** at  $x = a$  if the above limit exists. We say that  $f(x)$  is differentiable on  $(a, b)$  if  $f(x)$  is differentiable at every point in  $(a, b)$ .

The above limit can also be written as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$



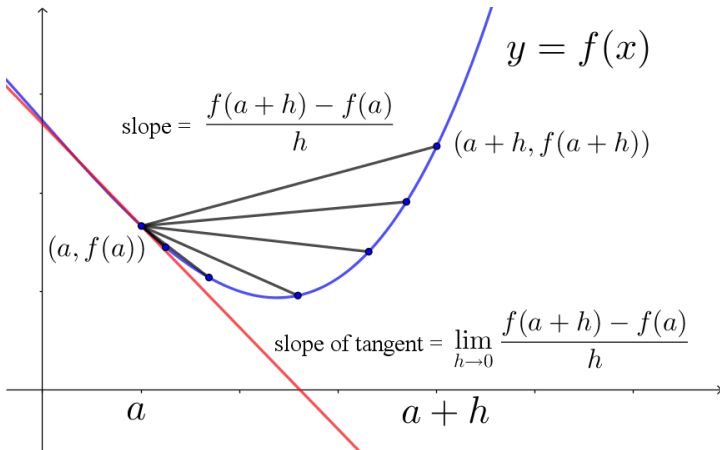
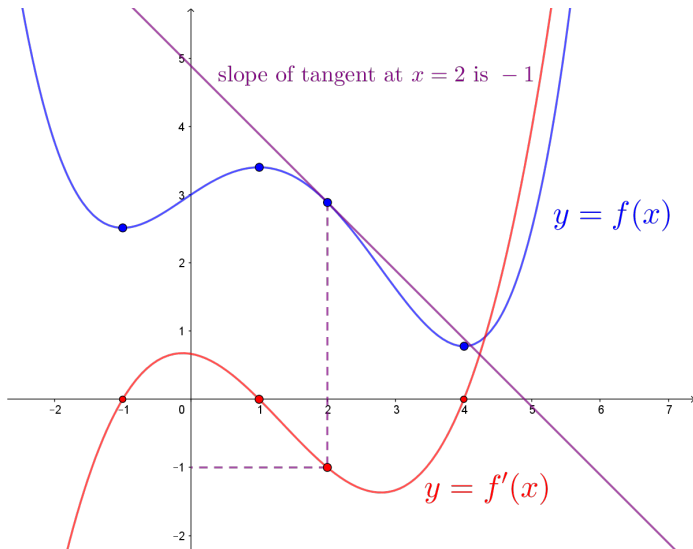


Figure: Definition of derivative



## Theorem

If  $f(x)$  differentiable at  $x = a$ , then  $f(x)$  is continuous at  $x = a$ .

**Differentiable at  $x = a \Rightarrow$  Continuous at  $x = a$**

## Proof.

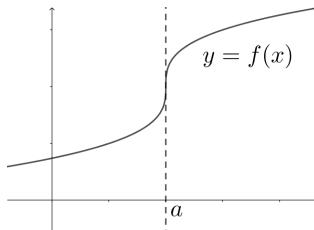
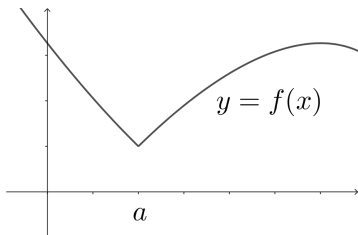
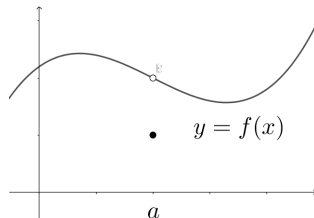
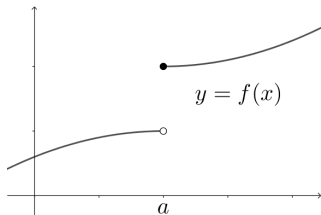
Suppose  $f(x)$  is differentiable at  $x = a$ . Then

$$\begin{aligned}\lim_{x \rightarrow a} (f(x) - f(a)) &= \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} \right) (x - a) \\ &= \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x - a} \right) \lim_{x \rightarrow a} (x - a) \\ &= f'(a) \cdot 0 = 0\end{aligned}$$

Therefore  $f(x)$  is continuous at  $x = a$ . □

Note that the converse of the above theorem does not hold. The function  $f(x) = |x|$  is continuous but not differentiable at 0.

The following functions are not differentiable at  $x = a$ .



## Example

$$\textcircled{1} \quad f(x) = e^x: f'(0) = \lim_{h \rightarrow 0} \frac{e^h - e^0}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

$$\textcircled{2} \quad f(x) = \ln x: f'(1) = \lim_{h \rightarrow 0} \frac{\ln(1+h) - \ln 1}{h} = \lim_{h \rightarrow 0} \frac{\ln(1+h)}{h} = 1.$$

$$\textcircled{3} \quad f(x) = \sin x: f'(0) = \lim_{h \rightarrow 0} \frac{\sin h - \sin 0}{h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

## Example

Find the values of  $a, b$  if  $f(x) = \begin{cases} 4x - 1, & \text{if } x \leq 1 \\ ax^2 + bx, & \text{if } x > 1 \end{cases}$  is differentiable at  $x = 1$ .

**Solution:** Since  $f(x)$  is differentiable at  $x = 1$ ,  $f(x)$  is continuous at  $x = 1$  and we have

$$\lim_{x \rightarrow 1^+} f(x) = f(1) \Rightarrow \lim_{x \rightarrow 1^+} (ax^2 + bx) = a + b = 3.$$

Moreover,  $f(x)$  is differentiable at  $x = 1$  and we have

$$\lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{(4(1+h) - 1) - 3}{h} = 4$$

$$\lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{a(1+h)^2 - b(1+h) - 3}{h} = 2a + b$$

$$\text{Therefore } \begin{cases} a + b = 3 \\ 2a + b = 4 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases} .$$

## Definition (First derivative)

Let  $y = f(x)$  be a differentiable function on  $(a, b)$ . The **first derivative** of  $f(x)$  is the function on  $(a, b)$  defined by

$$\frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

## Theorem

*Let  $f(x)$  and  $g(x)$  be differentiable functions and  $c$  be a real number. Then*

- 1  $(f + g)'(x) = f'(x) + g'(x)$
- 2  $(cf)'(x) = cf'(x)$

## Theorem

$$\textcircled{1} \quad \frac{d}{dx} x^n = nx^{n-1}, \quad n \in \mathbb{Z}^+, \text{ for } x \in \mathbb{R}$$

$$\textcircled{2} \quad \frac{d}{dx} e^x = e^x \text{ for } x \in \mathbb{R}$$

$$\textcircled{3} \quad \frac{d}{dx} \ln x = \frac{1}{x} \text{ for } x > 0$$

$$\textcircled{4} \quad \frac{d}{dx} \cos x = -\sin x \text{ for } x \in \mathbb{R}$$

$$\textcircled{5} \quad \frac{d}{dx} \sin x = \cos x \text{ for } x \in \mathbb{R}$$



Proof ( $\frac{d}{dx}x^n = nx^{n-1}$ )

Let  $y = x^n$ . For any  $x \in \mathbb{R}$ , we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h-x)((x+h)^{n-1} + (x+h)^{n-2}x + \dots + x^{n-1})}{h} \\ &= \lim_{h \rightarrow 0} ((x+h)^{n-1} + (x+h)^{n-2}x + \dots + x^{n-1}) \\ &= nx^{n-1}\end{aligned}$$

Note that the above proof is valid only when  $n \in \mathbb{Z}^+$  is a positive integer.

Proof ( $\frac{d}{dx}e^x = e^x$ )

Let  $y = e^x$ . For any  $x \in \mathbb{R}$ , we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h} = e^x.$$

(Alternative proof)

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ &= e^x \end{aligned}$$

In general, differentiation cannot be applied term by term to infinite series. The second proof is valid only after we prove that this can be done to **power series**.

## Proof

$\left(\frac{d}{dx} \ln x = \frac{1}{x}\right)$  Let  $f(x) = \ln x$ . For any  $x > 0$ , we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h} = \frac{1}{x}.$$

$\left(\frac{d}{dx} \cos x = -\sin x\right)$  Let  $f(x) = \cos x$ . For any  $x \in \mathbb{R}$ , we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{-2 \sin\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)}{h} = -\sin x.$$

$\left(\frac{d}{dx} \sin x = \cos x\right)$  Let  $f(x) = \sin x$ . For any  $x \in \mathbb{R}$ , we have

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{2 \cos\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)}{h} = \cos x.$$

## Definition

Let  $a > 0$  be a positive real number. For  $x \in \mathbb{R}$ , we define

$$a^x = e^{x \ln a}.$$

## Theorem

Let  $a > 0$  be a positive real number. We have

- 1  $a^{x+y} = a^x a^y$  for any  $x, y \in \mathbb{R}$
- 2  $\frac{d}{dx} a^x = a^x \ln a.$

## Proof.

- 1  $a^{x+y} = e^{(x+y) \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$
- 2  $\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$



## Example

Let  $f(x) = |x|$  for  $x \in \mathbb{R}$ . Show that  $f(x)$  is not differentiable at  $x = 0$ .

## Proof.

Observe that

$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \\ \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1\end{aligned}$$

Thus the limit

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

does not exist. Therefore  $f(x)$  is not differentiable at  $x = 0$ . □

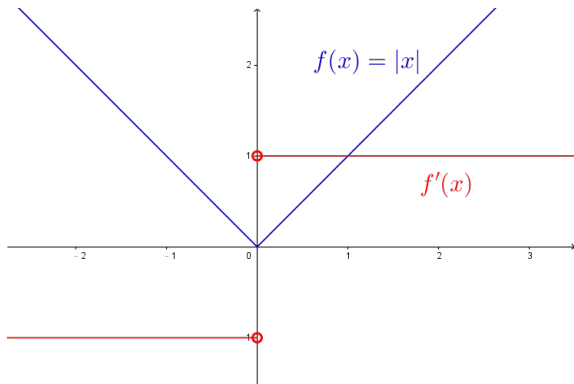


Figure:  $f(x) = |x|$  is not differentiable at  $x = 0$

## Exercise (True or False)

Suppose  $f(x)$  is bounded and is differentiable on  $(a, b)$ . Then

- ①  $f'(x)$  is differentiable on  $(a, b)$ .

**Answer: F**

- ②  $f'(x)$  is continuous on  $(a, b)$ .

**Answer: F**

- ③  $f'(x)$  is bounded on  $(a, b)$ .

**Answer: F**

## Example

Let  $f(x) = |x|x$  for  $x \in \mathbb{R}$ . Find  $f'(x)$ .

**Solution:** When  $x < 0$ ,  $f(x) = -x^2$  and  $f'(x) = -2x$ . When  $x > 0$ ,  $f(x) = x^2$  and  $f'(x) = 2x$ . When  $x = 0$ , we have

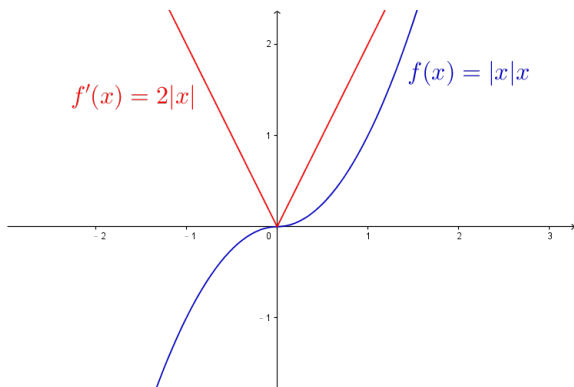
$$\begin{aligned}\lim_{h \rightarrow 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{-h^2 - 0}{h} = 0 \\ \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{h^2 - 0}{h} = 0\end{aligned}$$

Thus  $f'(0) = 0$ . Therefore

$$\begin{aligned}f'(x) &= \begin{cases} -2x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 2x, & \text{if } x > 0 \end{cases} \\ &= 2|x|.\end{aligned}$$

Note that  $f'(x) = 2|x|$  is continuous at  $x = 0$ .





- $f(x)$  is differentiable at  $x = 0$ . ( $f(x)$  is differentiable on  $\mathbb{R}$ .)
- $f'(x)$  is continuous on  $\mathbb{R}$ .
- $f'(x)$  is not differentiable at  $x = 0$ .

## Example

Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

- 1 Find  $f'(x)$  for  $x \neq 0$ .
- 2 Determine whether  $f(x)$  is differentiable at  $x = 0$ .

## Solution

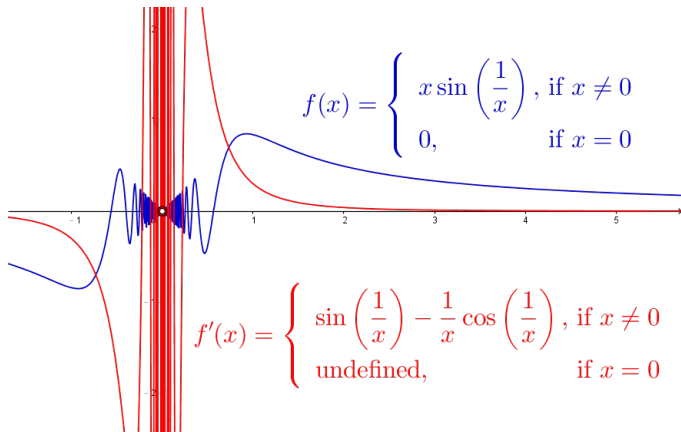
1. When  $x \neq 0$ ,

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}.$$

2. We have

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h}$$

does not exist. Therefore  $f(x)$  is not differentiable at  $x = 0$ .



- $f(x)$  is not differentiable at  $x = 0$ .

## Example

Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

- 1 Find  $f'(x)$ .
- 2 Determine whether  $f'(x)$  is continuous at  $x = 0$ .

## Solution

1. When  $x \neq 0$ , we have

$$f'(x) = 2x \sin \frac{1}{x} + x^2 \left( -\frac{1}{x^2} \cos \frac{1}{x} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

## Solution

2. When  $x = 0$ , we have

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h}.$$

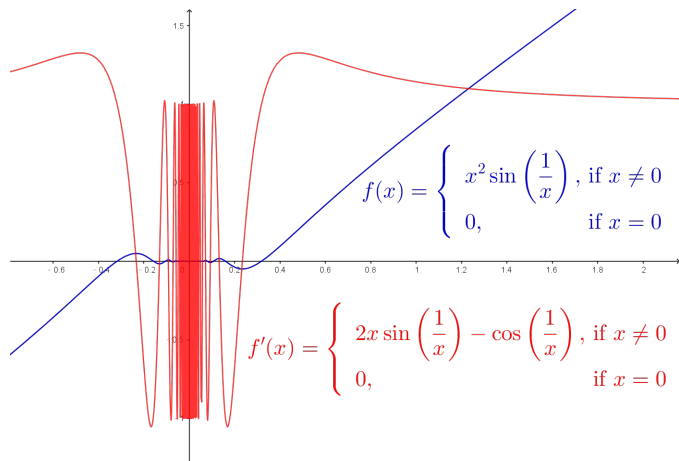
Since  $\lim_{h \rightarrow 0} h = 0$  and  $|\sin \frac{1}{h}| \leq 1$  is bounded, we have  $f'(0) = 0$ . Therefore

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

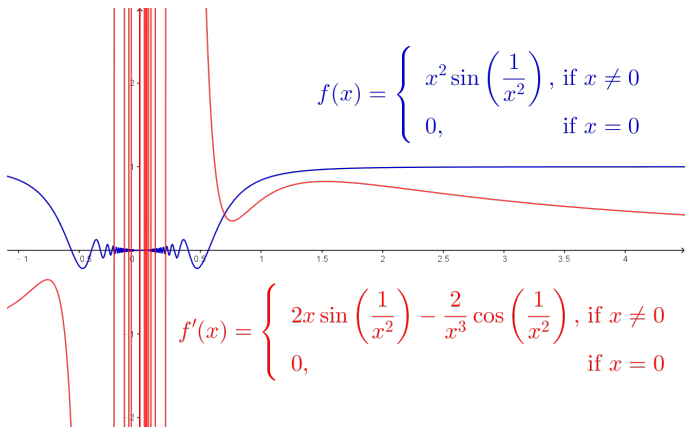
Observe that

$$\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \left( 2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist. We conclude that  $f'(x)$  is not continuous at  $x = 0$ .



- $f'(0) = 0$  ( $f(x)$  is differentiable on  $\mathbb{R}$ )
- $f'(x)$  is not continuous at  $x = 0$



- $f'(0) = 0$  ( $f(x)$  is differentiable on  $\mathbb{R}$ )
- $f'(x)$  is not continuous at  $x = 0$
- $f'(x)$  is not bounded near  $x = 0$

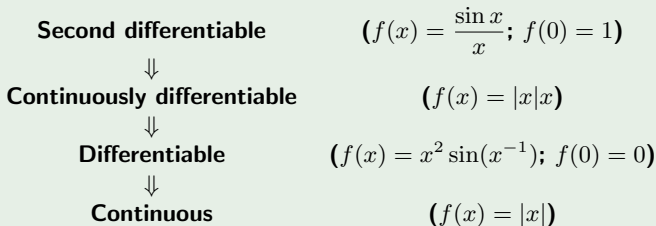
## Example

$f(x)$	$f(x)$ is continuous at $x = 0$	$f(x)$ is differentiable at $x = 0$	$f'(x)$ is continuous at $x = 0$
$ x $	Yes	No	Not applicable
$ x x$	Yes	Yes	Yes
$x \sin\left(\frac{1}{x}\right); f(0) = 0$	Yes	No	Not applicable
$x^2 \sin\left(\frac{1}{x}\right); f(0) = 0$	Yes	Yes	No



## Example

The following diagram shows the relations between the existence of limit, continuity and differentiability of a function at a point  $a$ . (Examples in the bracket is for  $a = 0$ .)



# Rules of differentiation

## Theorem (Basic formulas for differentiation)

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \sinh x = \cosh x$$

## Theorem (Product rule and quotient rule)

Let  $u$  and  $v$  be differentiable functions of  $x$ . Then

$$\frac{d}{dx} uv = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

## Proof

Let  $u = f(x)$  and  $v = g(x)$ .

$$\begin{aligned} \frac{d}{dx} uv &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left( f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \frac{f(x+h) - f(x)}{h} \right) \\ &= u \frac{dv}{dx} + v \frac{du}{dx} \end{aligned}$$

Proof.

$$\begin{aligned}\frac{d}{dx} \frac{u}{v} &= \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(x+h)g(x) - f(x)g(x)}{hg(x)g(x+h)} - \frac{f(x)g(x+h) - f(x)g(x)}{hg(x)g(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \left( g(x) \cdot \frac{f(x+h) - f(x)}{hg(x)g(x+h)} - f(x) \cdot \frac{g(x+h) - g(x)}{hg(x)g(x+h)} \right) \\ &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}\end{aligned}$$

□

### Theorem (Chain rule)

Let  $y = f(u)$  be a function of  $u$  and  $u = g(x)$  be a function of  $x$ . Suppose  $g(x)$  is differentiable at  $x = a$  and  $f(u)$  is differentiable at  $u = g(a)$ . Then  $f \circ g(x) = f(g(x))$  is differentiable at  $x = a$  and

$$(f \circ g)'(a) = f'(g(a))g'(a).$$

In other words,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

## Proof

$$\begin{aligned} & (f \circ g)'(a) \\ = & \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{h} \\ = & \lim_{h \rightarrow 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} \\ = & \lim_{k \rightarrow 0} \frac{f(g(a)+k) - f(g(a))}{k} \lim_{h \rightarrow 0} \frac{g(a+h) - g(a)}{h} \\ & \text{(Note that } g(a+h) - g(a) = k \rightarrow 0 \text{ as } h \rightarrow 0 \text{ because } g(x) \text{ is continuous.)} \\ = & f'(g(a))g'(a) \end{aligned}$$

The above proof is valid only if  $g(a+h) - g(a) \neq 0$  whenever  $h$  is sufficiently close to 0. This is true when  $g'(a) \neq 0$  because of the following proposition.

## Proposition

Suppose  $g(x)$  is a function such that  $g'(a) \neq 0$ . Then there exists  $\delta > 0$  such that if  $0 < |h| < \delta$ , then

$$g(a+h) - g(a) \neq 0.$$

When  $g'(a) = 0$ , we need another proposition.

### Proposition

*Suppose  $f(u)$  is a function which is differentiable at  $u = b$ . Then there exists  $\delta > 0$  and  $M > 0$  such that*

$$|f(b+h) - f(b)| < M|h| \text{ for any } |h| < \delta.$$

The proof of chain rule when  $g'(a) = 0$  goes as follows. There exists  $\delta > 0$  such that

$$|f(g(a+h)) - f(g(a))| < M|g(a+h) - g(a)| \text{ for any } |h| < \delta.$$

Therefore

$$\lim_{h \rightarrow 0} \left| \frac{f(g(a+h)) - f(g(a))}{h} \right| \leq \lim_{h \rightarrow 0} M \left| \frac{g(a+h) - g(a)}{h} \right| = 0$$

which implies  $(f \circ g)'(a) = 0$ .

## Example

The chain rule is used in the following way. Suppose  $u$  is a differentiable function of  $x$ . Then

$$\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

$$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$$



## Example

$$1. \frac{d}{dx} \sin^3 x = 3 \sin^2 x \frac{d}{dx} \sin x = 3 \sin^2 x \cos x$$

$$2. \frac{d}{dx} e^{\sqrt{x}} = e^{\sqrt{x}} \frac{d}{dx} \sqrt{x} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

$$3. \frac{d}{dx} \frac{1}{(\ln x)^2} = -\frac{2}{(\ln x)^3} \frac{d}{dx} \ln x = -\frac{2}{x(\ln x)^3}$$

$$4. \frac{d}{dx} \ln \cos 2x = \frac{1}{\cos 2x} (-\sin 2x) \cdot 2 = -\frac{2 \sin 2x}{\cos 2x} = -2 \tan 2x$$

$$5. \frac{d}{dx} \tan \sqrt{1+x^2} = \sec^2 \sqrt{1+x^2} \cdot \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x \sec^2 \sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$6. \frac{d}{dx} \sec^3 \sqrt{\sin x} = 3 \sec^2 \sqrt{\sin x} (\sec \sqrt{\sin x} \tan \sqrt{\sin x}) \frac{1}{2\sqrt{\sin x}} \cdot \cos x$$
$$= \frac{3 \sec^3 \sqrt{\sin x} \tan \sqrt{\sin x} \cos x}{2\sqrt{\sin x}}$$

## Example

$$\begin{aligned} 7. \frac{d}{dx} \cos^4 x \sin x &= \cos^4 x \cos x + 4 \cos^3 x (-\sin x) \sin x \\ &= \cos^5 x - 4 \cos^3 x \sin^2 x \end{aligned}$$

$$\begin{aligned} 8. \frac{d}{dx} \frac{\sec 2x}{\ln x} &= \frac{\ln x (2 \sec 2x \tan 2x) - \sec 2x (\frac{1}{x})}{(\ln x)^2} \\ &= \frac{\sec 2x (2x \tan 2x \ln x - 1)}{x (\ln x)^2} \end{aligned}$$

$$9. e^{\frac{\tan x}{x}} = e^{\frac{\tan x}{x}} \left( \frac{x \sec^2 x - \tan x}{x^2} \right)$$

$$\begin{aligned} 10. \sin \left( \frac{\ln x}{\sqrt{1+x^2}} \right) &= \cos \left( \frac{\ln x}{\sqrt{1+x^2}} \right) \left( \frac{\sqrt{1+x^2} (\frac{1}{x}) - \ln x (\frac{2x}{2\sqrt{1+x^2}})}{1+x^2} \right) \\ &= \left( \frac{1+x^2 - x^2 \ln x}{x(1+x^2)^{\frac{3}{2}}} \right) \cos \left( \frac{\ln x}{\sqrt{1+x^2}} \right) \end{aligned}$$

## Definition (Implicit functions)

An **implicit function** is an equation of the form  $F(x, y) = 0$ . An implicit function may not define a function. Sometimes it defines a function when the domain and range are specified.

## Theorem

Let  $F(x, y) = 0$  be an implicit function. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

and we have

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

Here  $\frac{\partial F}{\partial x}$  is called the partial derivative of  $F$  with respect to  $x$  which is the derivative of  $F$  with respect to  $x$  while considering  $y$  as constant. Similarly the partial derivative  $\frac{\partial F}{\partial y}$  is the derivative of  $F$  with respect to  $y$  while considering  $x$  as constant.

## Example

Find  $\frac{dy}{dx}$  for the following implicit functions.

- 1  $x^2 - xy - xy^2 = 0$
- 2  $\cos(xe^y) + x^2 \tan y = 1$

## Solution

$$\begin{aligned} 1. \quad 2x - (y + xy') - (y^2 + 2xyy') &= 0 \\ xy' + 2xyy' &= 2x - y - y^2 \\ y' &= \frac{2x - y - y^2}{x + 2xy} \end{aligned}$$

$$\begin{aligned} 2. \quad -\sin(xe^y)(e^y + xe^y y') + 2x \tan y + x^2 \sec^2 y y' &= 0 \\ x^2 \sec^2 y y' - xe^y \sin(xe^y) y' &= e^y \sin(xe^y) - 2x \tan y \\ y' &= \frac{e^y \sin(xe^y) - 2x \tan y}{x^2 \sec^2 y - xe^y \sin(xe^y)} \end{aligned}$$

## Theorem

Suppose  $f(y)$  is a differentiable function with  $f'(y) \neq 0$  for any  $y$ . Then the inverse function  $y = f^{-1}(x)$  of  $f(y)$  is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In other words,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

## Proof.

$$\begin{aligned} f(f^{-1}(x)) &= x \\ f'(f^{-1}(x))(f^{-1})'(x) &= 1 \\ (f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$



## Theorem

① For  $\sin^{-1} : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ ,

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

② For  $\cos^{-1} : [-1, 1] \rightarrow [0, \pi]$ ,

$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

③ For  $\tan^{-1} : \mathbb{R} \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ ,

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}.$$

Proof.

1

$$\begin{aligned}y &= \sin^{-1} x \\ \sin y &= x \\ \cos y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1 - \sin^2 y}} \quad (\text{Note: } \cos y \geq 0 \text{ for } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}) \\ &= \frac{1}{\sqrt{1 - x^2}}\end{aligned}$$

The other parts can be proved similarly. □

## Example

Find  $\frac{dy}{dx}$  if  $y = x^x$ .

## Solution

There are 2 methods.

Method 1. Note that  $y = x^x = e^{x \ln x}$ . Thus

$$\frac{dy}{dx} = e^{x \ln x} (1 + \ln x) = x^x (1 + \ln x).$$

Method 2. Taking logarithm on both sides, we have

$$\begin{aligned} \ln y &= x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= 1 + \ln x \\ \frac{dy}{dx} &= y(1 + \ln x) \\ &= x^x (1 + \ln x) \end{aligned}$$



## Example

Let  $u$  and  $v$  be functions of  $x$ . Show that

$$\frac{d}{dx}u^v = u^v v' \ln u + u^{v-1} v u'.$$

## Proof.

We have

$$\begin{aligned}\frac{d}{dx}u^v &= \frac{d}{dx}e^{v \ln u} \\ &= e^{v \ln u} \left( v' \ln u + v \cdot \frac{u'}{u} \right) \\ &= u^v v' \left( \ln u + \frac{v u'}{u} \right) \\ &= u^v v' \ln u + u^{v-1} v u'\end{aligned}$$



## Second and higher derivatives

### Definition (Second and higher derivatives)

Let  $y = f(x)$  be a function. The **second derivative** of  $f(x)$  is the function

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right).$$

The second derivative of  $y = f(x)$  is also denoted as  $f''(x)$  or  $y''$ . Let  $n$  be a non-negative integer. The  **$n$ -th derivative** of  $y = f(x)$  is defined inductively by

$$\frac{d^n y}{dx^n} = \frac{d}{dx} \left( \frac{d^{n-1} y}{dx^{n-1}} \right) \text{ for } n \geq 1$$

$$\frac{d^0 y}{dx^0} = y$$

The  $n$ -th derivative is also denoted as  $f^{(n)}(x)$  or  $y^{(n)}$ . Note that  $f^{(0)}(x) = f(x)$ .

## Example

Find  $\frac{d^2y}{dx^2}$  for the following functions.

①  $y = \ln(\sec x + \tan x)$

②  $x^2 - y^2 = 1$

## Solution

$$\begin{aligned} 1. \quad y' &= \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \\ &= \sec x \end{aligned}$$

$$y'' = \sec x \tan x$$

$$\begin{aligned} 2. \quad 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \\ y'' &= \frac{y - xy'}{y^2} \\ &= \frac{y - x(\frac{x}{y})}{y^2} \\ &= \frac{y^2 - x^2}{y^3} \end{aligned}$$

## Theorem (Leinbiz's rule)

Let  $u$  and  $v$  be differentiable function of  $x$ . Then

$$(uv)^{(n)} = \sum_{k=0}^n \binom{n}{k} u^{(n-k)} v^{(k)}$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  is the binomial coefficient.

## Example

$$(uv)^{(0)} = u^{(0)}v^{(0)}$$

$$(uv)^{(1)} = u^{(1)}v^{(0)} + u^{(0)}v^{(1)}$$

$$(uv)^{(2)} = u^{(2)}v^{(0)} + 2u^{(1)}v^{(1)} + u^{(0)}v^{(2)}$$

$$(uv)^{(3)} = u^{(3)}v^{(0)} + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + u^{(0)}v^{(3)}$$

$$(uv)^{(4)} = u^{(4)}v^{(0)} + 4u^{(3)}v^{(1)} + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + u^{(0)}v^{(4)}$$

## Proof

We prove the Leibniz's rule by induction on  $n$ . When  $n = 0$ ,  $(uv)^{(0)} = uv = u^{(0)}v^{(0)}$ . Assume that for some nonnegative  $m$ ,

$$(uv)^{(m)} = \sum_{k=0}^m \binom{m}{k} u^{(m-k)} v^{(k)}.$$

Then

$$\begin{aligned} & (uv)^{(m+1)} \\ = & \frac{d}{dx} (uv)^{(m)} \\ = & \frac{d}{dx} \sum_{k=0}^m \binom{m}{k} u^{(m-k)} v^{(k)} \\ = & \sum_{k=0}^m \binom{m}{k} (u^{(m-k+1)} v^{(k)} + u^{(m-k)} v^{(k+1)}) \end{aligned}$$

## Proof.

$$\begin{aligned}
&= \sum_{k=0}^m \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=0}^m \binom{m}{k} u^{(m-k)} v^{(k+1)} \\
&= \sum_{k=0}^m \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-(k-1))} v^{(k)} \\
&= \sum_{k=0}^m \binom{m}{k} u^{(m-k+1)} v^{(k)} + \sum_{k=1}^{m+1} \binom{m}{k-1} u^{(m-k+1)} v^{(k)} \\
&= \sum_{k=0}^{m+1} \left( \binom{m}{k} + \binom{m}{k-1} \right) u^{(m-k+1)} v^{(k)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} u^{(m+1-k)} v^{(k)}
\end{aligned}$$

Here we use the convention  $\binom{m}{-1} = \binom{m}{m+1} = 0$  in the second last equality. This completes the induction step and the proof of the Leibniz's rule.  $\square$

## Example

Let  $y = x^2 e^{3x}$ . Find  $y^{(n)}$  where  $n$  is a nonnegative integer.

## Solution

Let  $u = x^2$  and  $v = e^{3x}$ . Then  $u^{(0)} = x^2$ ,  $u^{(1)} = 2x$ ,  $u^{(2)} = 2$  and  $u^{(k)} = 0$  for  $k \geq 3$ . On the other hand,  $v^{(k)} = 3^k e^{3x}$  for any  $k \geq 0$ . Therefore by Leibniz's rule, we have

$$\begin{aligned}y^{(n)} &= \binom{n}{0} u^{(0)} v^{(n)} + \binom{n}{1} u^{(1)} v^{(n-1)} + \binom{n}{2} u^{(2)} v^{(n-2)} \\&= x^2 (3^n e^{3x}) + n(2x)(3^{n-1} e^{3x}) + \frac{n(n-1)}{2!} (2)(3^{n-2} e^{3x}) \\&= (3^n x^2 + 2 \cdot 3^{n-1} n x + 3^{n-2} (n^2 - n)) e^{3x} \\&= 3^{n-2} (9x^2 + 6nx + n^2 - n) e^{3x}\end{aligned}$$

# Mean value theorem

Suppose  $f(x)$  is a function which is differentiable on  $(a, b)$ .

- ① If  $f(x)$  attains its maximum or minimum at  $x = c \in (a, b)$ , then  $f'(c) = 0$ .

**Answer: T**

- ② If  $f'(c) = 0$ , then  $f(x)$  attains its maximum or minimum at  $x = c \in (a, b)$ .

**Answer: F**

- ③ If  $f'(x) = 0$  for any  $x \in (a, b)$ , then  $f(x)$  is constant on  $(a, b)$ .

**Answer: T**

- ④ If  $f(x)$  is strictly increasing on  $(a, b)$ , then  $f'(x) > 0$  for any  $x \in (a, b)$ .

**Answer: F**

- ⑤ If  $f'(x) > 0$  for any  $(a, b)$ , then  $f(x)$  is strictly increasing on  $(a, b)$ .

**Answer: T**

- ⑥ If  $f(x)$  is monotonic increasing on  $(a, b)$ , then  $f'(x) \geq 0$  for any  $x \in (a, b)$ .

**Answer: T**



## Theorem

Let  $f$  be a function on  $(a, b)$  and  $c \in (a, b)$  such that

- 1  $f$  is differentiable at  $x = c$ , and
- 2 either  $f(x) \leq f(c)$  for any  $x \in (a, b)$ , or  $f(x) \geq f(c)$  for any  $x \in (a, b)$ .

Then  $f'(c) = 0$ .

## Proof.

Suppose  $f(x) \leq f(c)$  for any  $x \in (a, b)$ . The proof for the other case is essentially the same. For any  $h < 0$  with  $a < c + h < c$ , we have  $f(c + h) - f(c) \leq 0$  and  $h$  is negative. Thus

$$f'(c) = \lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \geq 0$$

On the other hand, for any  $h > 0$  with  $c < c + h < b$ , we have  $f(c + h) - f(c) \leq 0$  and  $h$  is positive. Thus we have

$$f'(c) = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \leq 0$$

Therefore  $f'(c) = 0$ .



## Example

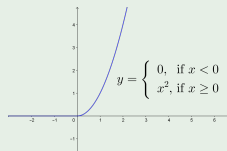
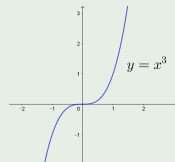
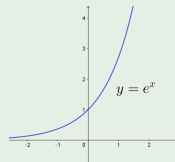
$$f'(x) > 0 \text{ for any } x$$



Strictly increasing



$$\text{Monotonic increasing} \Leftrightarrow f'(x) \geq 0 \text{ for any } x$$

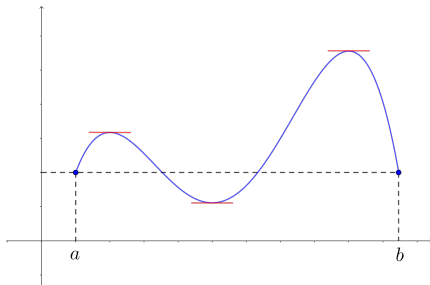


## Theorem (Rolle's theorem)

Suppose  $f(x)$  is a function which satisfies the following conditions.

- 1  $f(x)$  is continuous on  $[a, b]$ .
- 2  $f(x)$  is differentiable on  $(a, b)$ .
- 3  $f(a) = f(b)$

Then there exists  $\xi \in (a, b)$  such that  $f'(\xi) = 0$ .



## Proof.

By extreme value theorem, there exist  $a \leq \alpha, \beta \leq b$  such that

$$f(\alpha) \leq f(x) \leq f(\beta) \text{ for any } x \in [a, b].$$

Since  $f(a) = f(b)$ , at least one of  $\alpha, \beta$  can be chosen in  $(a, b)$  and we let it be  $\xi$ . Then we have  $f'(\xi) = 0$  since  $f(x)$  attains its maximum or minimum at  $\xi$ . □

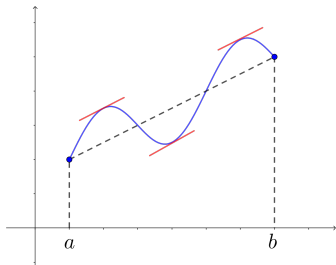
## Theorem (Lagrange's mean value theorem)

Suppose  $f(x)$  is a function which satisfies the following conditions.

- 1  $f(x)$  is continuous on  $[a, b]$ .
- 2  $f(x)$  is differentiable on  $(a, b)$ .

Then there exists  $\xi \in (a, b)$  such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$



## Proof.

Let  $g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$ . Since  $g(a) = g(b) = f(a)$ ,  
by Rolle's theorem, there exists  $\xi \in (a, b)$  such that

$$g'(\xi) = 0$$

$$f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0$$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$



## Theorem

Let  $f(x)$  be a function which is differentiable on  $(a, b)$ . Then  $f(x)$  is monotonic increasing if and only if  $f'(x) \geq 0$  for any  $x \in (a, b)$ .

**Proof.** Suppose  $f(x)$  is monotonic increasing on  $(a, b)$ . Then for any  $x \in (a, b)$ , we have  $f(x+h) - f(x) \geq 0$  for any  $h > 0$  and thus

$$f'(x) = \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \geq 0.$$

On the other hand, suppose  $f'(x) \geq 0$  for any  $x \in (a, b)$ . Then for any  $\alpha, \beta \in (a, b)$  with  $\alpha < \beta$ , by Lagrange's mean value theorem, there exists  $\xi \in (\alpha, \beta)$  such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) \geq 0.$$

Therefore  $f(x)$  is monotonic increasing on  $(a, b)$ . □

## Corollary

$f(x)$  is constant on  $(a, b)$  if and only if  $f'(x) = 0$  for any  $x \in (a, b)$ .

## Theorem

If  $f(x)$  is a differentiable function such that  $f'(x) > 0$  for any  $x \in (a, b)$ , then  $f(x)$  is strictly increasing.

## Proof.

Suppose  $f'(x) > 0$  for any  $x \in (a, b)$ . Then for any  $\alpha, \beta \in (a, b)$  with  $\alpha < \beta$ , by Lagrange's mean value theorem, there exists  $\xi \in (\alpha, \beta)$  such that

$$f(\beta) - f(\alpha) = f'(\xi)(\beta - \alpha) > 0.$$

Therefore  $f(x)$  is strictly increasing on  $(a, b)$ . □

The converse of the above theorem is false.

## Example

$f(x) = x^3$  is strictly increasing on  $\mathbb{R}$  but  $f'(0) = 0$  is not positive.



## Example

Prove that  $1 - \frac{1}{x} \leq \ln x \leq x - 1$  for any  $x > 0$ .

**Solution.** Let  $f(x) = \ln x - \left(1 - \frac{1}{x}\right)$ . Then  $f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}$ . Now  $f'(1) = 0$  and

	$0 < x < 1$	$x > 1$
$f'(x)$	-	+

Therefore  $f(x)$  attains its minimum at  $x = 1$  and we have

$f(x) = \ln x - \frac{x-1}{x} \geq f(1) = 0$  for any  $x > 0$ . On the other hand, let

$g(x) = x - 1 - \ln x$ . Then  $g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$ . Now  $g'(1) = 0$  and

	$0 < x < 1$	$x > 1$
$f'(x)$	-	+

Therefore  $g(x)$  attains its minimum at  $x = 1$  and we have

$g(x) = x - 1 - \ln x \geq g(1) = 0$  for any  $x > 0$ .

## Example

Let  $0 < \alpha < 1$ . Prove that

$$1 + \alpha x - \frac{\alpha(1 - \alpha)x^2}{2} < (1 + x)^\alpha < 1 + \alpha x, \text{ for any } x > 0.$$

**Solution.** Let  $f(x) = 1 + \alpha x - (1 + x)^\alpha$ . Then  $f(0) = 0$  and for any  $x > 0$ ,

$$f'(x) = \alpha - \frac{\alpha}{(1 + x)^{1-\alpha}} > \alpha - \alpha = 0.$$

Therefore  $f(x) > 0$  for any  $x > 0$ . On the other hand, let

$g(x) = (1 + x)^\alpha - \left(1 + \alpha x - \frac{\alpha(1 - \alpha)x^2}{2}\right)$ . Then  $g(0) = 0$  and for any  $x > 0$ ,

$$\begin{aligned} g'(x) &= \frac{\alpha}{(1 + x)^{1-\alpha}} - \alpha + \alpha(1 - \alpha)x \\ &> \frac{\alpha}{1 + (1 - \alpha)x} - \alpha(1 - (1 - \alpha)x) \\ &= \frac{\alpha(1 - \alpha)^2 x^2}{1 + (1 - \alpha)x} > 0 \end{aligned}$$

Therefore  $g(x) > 0$  for any  $x > 0$ .

## Theorem (Cauchy's mean value theorem)

Suppose  $f(x)$  and  $g(x)$  are functions which satisfies the following conditions.

- 1  $f(x), g(x)$  is continuous on  $[a, b]$ .
- 2  $f(x), g(x)$  is differentiable on  $(a, b)$ .
- 3  $g'(x) \neq 0$  for any  $x \in (a, b)$ .

Then there exists  $\xi \in (a, b)$  such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

**Proof.** Let  $h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$ .

Since  $h(a) = h(b) = f(a)$ , by Rolle's theorem, there exists  $\xi \in (a, b)$  such that

$$\begin{aligned} f'(\xi) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(\xi) &= 0 \\ \frac{f'(\xi)}{g'(\xi)} &= \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

# L'Hopital's rule

## Theorem (L'Hopital's rule)

Let  $a \in [-\infty, +\infty]$ . Suppose  $f$  and  $g$  are differentiable functions such that

- 1  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$  (or  $\pm\infty$ ).
- 2  $g'(x) \neq 0$  for any  $x \neq a$  (on a neighborhood of  $a$ ).
- 3  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ .

Then the limit of  $\frac{f(x)}{g(x)}$  at  $x = a$  exists and  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$ .

## Proof.

We give here the proof for  $a \in (-\infty, +\infty)$ . For any  $x \neq a$ , by applying Cauchy's mean value theorem to  $f(x)$ ,  $g(x)$  on  $[a, x]$  or  $[x, a]$ , there exists  $\xi$  between  $a$  and  $x$  such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Here we redefine  $f(a) = g(a) = 0$ , if necessary, so that  $f$  and  $g$  are continuous at  $a$ . Note that  $\xi \rightarrow a$  as  $x \rightarrow a$ . We have

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(\xi)}{g'(\xi)} = L.$$



Example (Indeterminate form of types  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$ )

$$1. \lim_{x \rightarrow 0} \frac{\sin x - x \cos x}{x^3} = \lim_{x \rightarrow 0} \frac{x \sin x}{3x^2} = \frac{1}{3}$$

$$2. \lim_{x \rightarrow 0} \frac{x^2}{\ln \sec x} = \lim_{x \rightarrow 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = \lim_{x \rightarrow 0} \frac{2x}{\tan x} = \lim_{x \rightarrow 0} \frac{2}{\sec^2 x} = 2$$

$$3. \lim_{x \rightarrow 0} \frac{\ln(1+x^3)}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{1+x^3}}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{1}{1+x^3} \lim_{x \rightarrow 0} \frac{x^2}{1 - \cos x}$$
$$= \lim_{x \rightarrow 0} \frac{2x}{\sin x} = 2$$

$$4. \lim_{x \rightarrow +\infty} \frac{\ln(1+x^4)}{\ln(1+x^2)} = \lim_{x \rightarrow +\infty} \frac{\frac{4x^3}{1+x^4}}{\frac{2x}{1+x^2}} = \lim_{x \rightarrow +\infty} \frac{4x^3(1+x^2)}{2x(1+x^4)} = 2$$

Example (Indeterminate form of types  $\infty - \infty$  and  $0 \cdot \infty$ )

$$\begin{aligned} 5. \lim_{x \rightarrow 1} \left( \frac{1}{\ln x} - \frac{1}{x-1} \right) &= \lim_{x \rightarrow 1} \frac{x-1 - \ln x}{(x-1)\ln x} = \lim_{x \rightarrow 1} \frac{1 - \frac{1}{x}}{\frac{x-1}{x} + \ln x} \\ &= \lim_{x \rightarrow 1} \frac{x-1}{x-1+x\ln x} = \lim_{x \rightarrow 1} \frac{1}{2 + \ln x} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 6. \lim_{x \rightarrow 0} \cot 3x \tan^{-1} x &= \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{\tan 3x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{3 \sec^2 3x} \\ &= \lim_{x \rightarrow 0} \frac{1}{3(1+x^2)\sec^2 3x} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} 7. \lim_{x \rightarrow 0^+} x \ln \sin x &= \lim_{x \rightarrow 0^+} \frac{\ln \sin x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^2 \cos x}{\sin x} = 0 \end{aligned}$$

$$\begin{aligned} 8. \lim_{x \rightarrow +\infty} x \ln \left( \frac{x+1}{x-1} \right) &= \lim_{x \rightarrow +\infty} \frac{\ln(x+1) - \ln(x-1)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow +\infty} \frac{\frac{1}{x+1} - \frac{1}{x-1}}{-\frac{1}{x^2}} = \lim_{x \rightarrow +\infty} \frac{2x^2}{(x+1)(x-1)} = 2 \end{aligned}$$

Example (Indeterminate form of types  $0^0$ ,  $1^\infty$  and  $\infty^0$ )

Evaluate the following limits.

$$1 \quad \lim_{x \rightarrow 0^+} x^{\sin x}$$

$$2 \quad \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}}$$

$$3 \quad \lim_{x \rightarrow +\infty} (1 + 2x)^{\frac{1}{3 \ln x}}$$



## Solution

$$\begin{aligned} \textcircled{1} \quad \ln \left( \lim_{x \rightarrow 0^+} x^{\sin x} \right) &= \lim_{x \rightarrow 0^+} \ln(x^{\sin x}) = \lim_{x \rightarrow 0^+} \sin x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} = \lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x \cos x} = 0. \end{aligned}$$

$$\text{Thus } \lim_{x \rightarrow 0^+} x^{\sin x} = e^0 = 1.$$

$$\begin{aligned} \textcircled{2} \quad \ln \left( \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}} \right) &= \lim_{x \rightarrow 0} \ln(\cos x)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \frac{\ln \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{-\tan x}{2x} \\ &= \lim_{x \rightarrow 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}. \end{aligned}$$

$$\text{Thus } \lim_{x \rightarrow 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}.$$

$$\textcircled{3} \quad \ln \left( \lim_{x \rightarrow +\infty} (1+2x)^{\frac{3}{\ln x}} \right) = \lim_{x \rightarrow +\infty} \frac{3 \ln(1+2x)}{\ln x} = \lim_{x \rightarrow +\infty} \frac{\frac{6}{1+2x}}{\frac{1}{x}} = 3.$$

$$\text{Thus } \lim_{x \rightarrow +\infty} (1+2x)^{\frac{3}{\ln x}} = e^3.$$

## Example

The following shows some wrong use of L'Hopital rule.

1.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sec x - 1}{e^{2x} - 1} &= \lim_{x \rightarrow 0} \frac{\sec x \tan x}{2e^{2x}} \\ &= \lim_{x \rightarrow 0} \frac{\sec^2 x \tan x + \sec^3 x}{4e^{2x}} \\ &= \frac{1}{4}\end{aligned}$$

This is wrong because  $\lim_{x \rightarrow 0} e^{2x} \neq 0, \pm\infty$ . One cannot apply L'Hopital rule to  $\lim_{x \rightarrow 0} \frac{\sec x \tan x}{2e^{2x}}$ . The correct solution is

$$\lim_{x \rightarrow 0} \frac{\sec x - 1}{e^{2x} - 1} = \lim_{x \rightarrow 0} \frac{\sec x \tan x}{2e^{2x}} = 0.$$

## Example

2.

$$\begin{aligned}\lim_{x \rightarrow +\infty} \frac{5x - 2 \cos^2 x}{3x + \sin^2 x} &= \lim_{x \rightarrow +\infty} \frac{5 + 2 \cos x \sin x}{3 + \sin x \cos x} \\ &= \lim_{x \rightarrow +\infty} \frac{2(\cos^2 x - \sin^2 x)}{\cos^2 x - \sin^2 x} \\ &= 2\end{aligned}$$

This is wrong because  $\lim_{x \rightarrow +\infty} (5 + 2 \cos x \sin x)$  and  $\lim_{x \rightarrow +\infty} (3 + \sin x \cos x)$  do not exist. One cannot apply L'Hopital rule to  $\lim_{x \rightarrow +\infty} \frac{5 + 2 \cos x \sin x}{3 + \sin x \cos x}$ . The correct solution is

$$\lim_{x \rightarrow +\infty} \frac{5x - 2 \cos^2 x}{3x + \sin^2 x} = \lim_{x \rightarrow +\infty} \frac{5 - \frac{2 \cos^2 x}{x}}{3 + \frac{\sin^2 x}{x}} = \frac{5}{3}.$$

# Taylor series

## Definition (Taylor polynomial)

Let  $f(x)$  be a function such that the  $n$ -th derivative exists at  $x = a$ . The **Taylor polynomial** of degree  $n$  of  $f(x)$  at  $x = a$  is the polynomial

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

## Theorem

*The Taylor polynomial  $p_n(x)$  of degree  $n$  of  $f(x)$  at  $x = a$  is the unique polynomial such that*

$$p_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, 2, \dots, n.$$

## Example

Find the Taylor polynomial  $p_3(x)$  of degree 3 of  $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$  at  $x = 0$ .

**Solution.** The derivatives  $f^{(k)}(x)$  up to order 3 are

$k$	0	1	2	3
$f^{(k)}(x)$	$(1+x)^{\frac{1}{2}}$	$\frac{1}{2}(1+x)^{-\frac{1}{2}}$	$-\frac{1}{4}(1+x)^{-\frac{3}{2}}$	$\frac{3}{8}(1+x)^{-\frac{5}{2}}$
$f^{(k)}(0)$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$

Therefore the Taylor polynomial of  $f(x)$  of degree 3 at  $x = 0$  is

$$\begin{aligned} p_3(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} \\ &= 1 + \left(\frac{1}{2}\right)x + \left(-\frac{1}{4}\right)\frac{x^2}{2!} + \left(\frac{3}{8}\right)\frac{x^3}{3!} \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} \end{aligned}$$

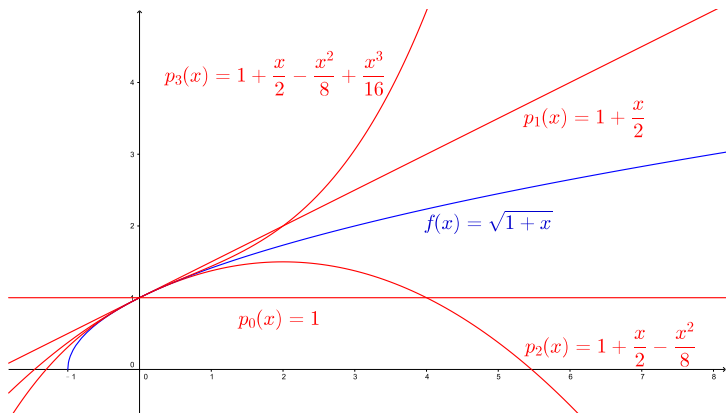


Figure: Taylor polynomials for  $f(x) = \sqrt{1+x}$  at  $x=0$

## Example

Let  $f(x) = \cos x$ . The first few derivatives are

$k$	0	1	2	3	4
$f^{(k)}(x)$	$\cos x$	$-\sin x$	$-\cos x$	$-\sin x$	$\cos x$
$f^{(k)}(0)$	1	0	-1	0	1

We see that

$$f^{(n)}(x) = \begin{cases} (-1)^k \cos x, & \text{if } n = 2k \\ (-1)^k \sin x, & \text{if } n = 2k - 1 \end{cases} \quad \text{and} \quad f^{(n)}(0) = \begin{cases} (-1)^k, & \text{if } n = 2k \\ 0, & \text{if } n = 2k - 1 \end{cases}$$

Therefore the Taylor polynomial of  $f(x)$  of degree  $n = 2k$  at  $x = 0$  is

$$\begin{aligned} p_{2k}(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(2k)}(0)x^{2k}}{(2k)!} \\ &= 1 + (0)x + \frac{(-1)x^2}{2!} + \frac{(0)x^3}{3!} + \frac{(1)x^4}{4!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!} \end{aligned}$$

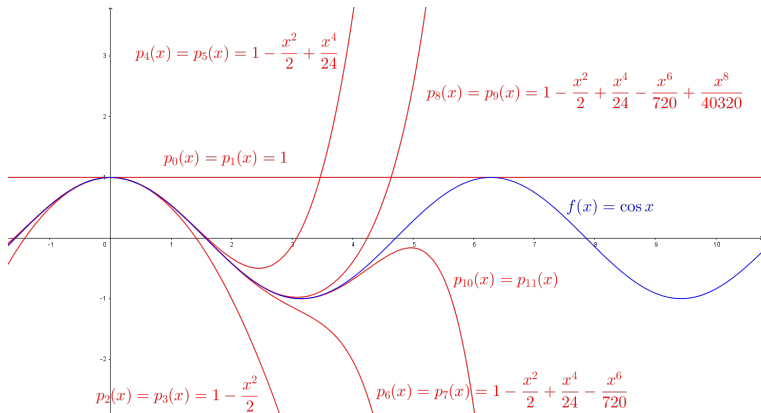


Figure: Taylor polynomials for  $f(x) = \cos x$  at  $x = 0$



## Example

Find the Taylor polynomial of degree  $n$  of  $f(x) = \frac{1}{x}$  at  $x = 1$ .

**Solution.** The derivatives  $f^{(k)}(x)$  are

$k$	0	1	2	3	$\dots$	$n$
$f^{(k)}(x)$	$x^{-1}$	$-x^{-2}$	$2x^{-3}$	$-6x^{-4}$	$\dots$	$(-1)^n n! x^{-(n+1)}$
$f^{(k)}(1)$	1	-1	2	-6	$\dots$	$(-1)^n n!$

Therefore the Taylor polynomial of  $f(x)$  of degree  $n$  at  $x = 1$  is

$$\begin{aligned} p_n(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \dots + \frac{f^{(n)}(1)(x-1)^n}{(n)!} \\ &= 1 - (x-1) + \frac{2(x-1)^2}{2!} + \frac{(-6)(x-1)^3}{3!} + \dots + \frac{(-1)^n n!(x-1)^n}{n!} \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots + (-1)^n (x-1)^n \end{aligned}$$

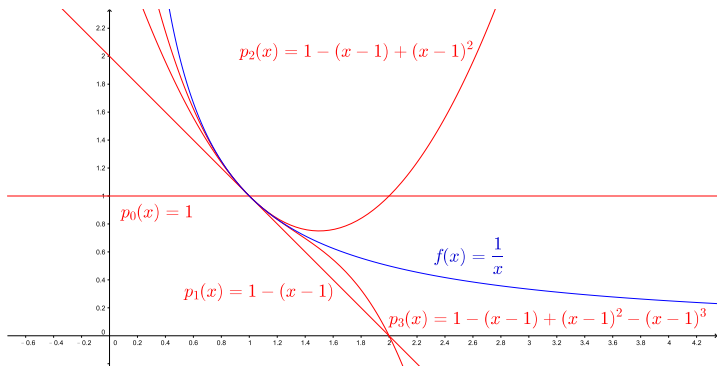


Figure: Taylor polynomials for  $f(x) = \frac{1}{x}$  at  $x = 1$

## Example

Find the Taylor polynomial of  $f(x) = (1+x)^\alpha$  at  $x=0$ , where  $\alpha \in \mathbb{R}$ .

**Solution.** The derivatives are

$$f(x) = (1+x)^\alpha$$

$$f'(x) = \alpha(1+x)^{\alpha-1}$$

$$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$$

$$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$

$$\vdots$$

$$f^{(k)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$$

## Example

Thus we have

$$\begin{aligned}f(0) &= 1 \\f'(0) &= \alpha \\f''(0) &= \alpha(\alpha - 1)\end{aligned}$$

$$\vdots$$

$$f^{(k)}(0) = \alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - k + 1)$$

Therefore the Taylor polynomial of  $f(x) = (1 + x)^\alpha$  of degree  $n$  at  $x = 0$  is

$$\begin{aligned}p_n(x) &= f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{(n)!} \\&= 1 + \alpha x + \frac{\alpha(\alpha - 1)x^2}{2!} + \cdots + \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)x^n}{n!} \\&= \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n\end{aligned}$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n + 1)}{n!}.$$

## Example

The Taylor polynomials of degree  $n$  for  $f(x)$  at  $x = 0$ .

$f(x)$	Taylor polynomial
$e^x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^k x^{2k}}{(2k)!}, n = 2k$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!}, n = 2k+1$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1} x^n}{n}$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots + x^n$
$\sqrt{1+x}$	$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots + \frac{(-1)^{n+1} (2n-3)!! x^n}{2^n n!}$
$(1+x)^\alpha$	$1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \cdots + \binom{\alpha}{n} x^n$

## Example

The Taylor polynomials of degree  $n$  for  $f(x)$  at  $x = a$ .

$f(x)$	Taylor polynomial
$\cos x; a = \pi$	$-1 + \frac{(x - \pi)^2}{2!} - \frac{(x - \pi)^4}{4!} + \cdots + \frac{(-1)^{k+1}(x - \pi)^{2k}}{(2k)!}$
$e^x; a = 2$	$e^2 + e^2(x - 2) + \frac{e^2(x - 2)^2}{2!} + \cdots + \frac{e^2(x - 2)^n}{n!}$
$\frac{1}{x}; x = 1$	$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots + (-1)^n(x - 1)^n$
$\frac{1}{2 + x}; a = 0$	$\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \cdots + \frac{(-1)^n x^n}{2^{n+1}}$
$\frac{1}{3 - 2x}; x = 1$	$1 + 2(x - 1) + 4(x - 1)^2 + 8(x - 1)^3 + \cdots + 2^n(x - 1)^n$
$\sqrt{100 - 2x}; a = 0$	$10 - \frac{x}{10} - \frac{x^2}{2000} - \frac{x^3}{200000} - \cdots - \frac{(2n - 3)!!x^n}{10^{2n-1}n!}$

### Definition (Taylor series)

Let  $f(x)$  be an infinitely differentiable function. The **Taylor series** of  $f(x)$  at  $x = a$  is the infinite power series

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

## Example

The following table shows the Taylor series for  $f(x)$  at  $x = a$ .

$f(x)$	Taylor series
$e^x; a = 0$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
$\cos x; a = 0$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
$\sin x; a = \pi$	$-(x - \pi) + \frac{(x - \pi)^3}{3!} - \frac{(x - \pi)^5}{5!} + \dots$
$\ln x; a = 1$	$(x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots$
$\sqrt{1+x}; a = 0$	$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots$
$\frac{1}{\sqrt{1+x}}; a = 0$	$1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \frac{63x^5}{256} + \dots$
$(1+x)^\alpha; a = 0$	$1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \dots$



$$e^x; \quad \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\cos x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\ln(1+x); \quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\frac{1}{1-x}; \quad \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^\alpha; \quad \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \frac{\alpha(\alpha-1)(\alpha-2)x^3}{3!} + \dots$$

$$\tan^{-1} x; \quad \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\sin^{-1} x; \quad \sum_{k=0}^{\infty} \frac{(2k)! x^{2k+1}}{4^k (k!)^2 (2k+1)} = x + \left(\frac{1}{2}\right) \frac{x^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) \frac{x^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right) \frac{x^7}{7} + \dots$$

## Theorem

Suppose  $T(x)$  is the Taylor series of  $f(x)$  at  $x = 0$ . Then for any positive integer  $k$ , the Taylor series for  $f(x^k)$  at  $x = 0$  is  $T(x^k)$ .

## Example

$f(x)$	Taylor series at $x = 0$
$\frac{1}{1+x^2}$	$1 - x^2 + x^4 - x^6 + \dots$
$\frac{1}{\sqrt{1-x^2}}$	$1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \frac{35x^8}{128} + \dots$
$\frac{\sin x^2}{x^2}$	$1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \dots$

## Theorem

Suppose the Taylor series for  $f(x)$  at  $x = 0$  is

$$T(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then the Taylor series for  $f'(x)$  is

$$T'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

## Example

Find the Taylor series of the following functions.

①  $\frac{1}{(1+x)^2}$

②  $\tan^{-1} x$

## Solution

① Let  $F(x) = -\frac{1}{1+x}$  so that  $F'(x) = \frac{1}{(1+x)^2}$ . The Taylor series for  $F(x)$  at  $x = 0$  is

$$T(x) = -1 + x - x^2 + x^3 - x^4 + \dots$$

Therefore the Taylor series for  $F'(x) = \frac{1}{(1+x)^2}$  is

$$T'(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$$

## Solution

2. Suppose the Taylor series for  $f(x) = \tan^{-1} x$  at  $x = 0$  is

$$T(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \dots$$

Now comparing  $T'(x)$  with the Taylor series for  $f'(x) = \frac{1}{1+x^2}$  which takes the form

$$1 - x^2 + x^4 - x^6 + \dots,$$

we obtain the values of  $a_1, a_2, a_3, \dots$  and get

$$T(x) = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Since  $a_0 = f(0) = 0$ , we have

$$T(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

## Theorem

Suppose the Taylor series for  $f(x)$  and  $g(x)$  at  $x = 0$  are

$$S(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,$$

$$T(x) = \sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots,$$

respectively. Then the Taylor series for  $f(x)g(x)$  at  $x = 0$  is

$$\begin{aligned} & \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots \end{aligned}$$

## Proof.

The coefficient of  $x^n$  of the Taylor series of  $f(x)g(x)$  at  $x = 0$  is

$$\begin{aligned}\frac{(fg)^{(n)}(0)}{n!} &= \sum_{k=0}^n \binom{n}{k} \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \quad (\text{Leibniz's formula}) \\ &= \sum_{k=0}^n \frac{n!}{k!(n-k)!} \cdot \frac{f^{(k)}(0)g^{(n-k)}(0)}{n!} \\ &= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} \cdot \frac{g^{(n-k)}(0)}{(n-k)!} \\ &= \sum_{k=0}^n a_k b_{n-k}\end{aligned}$$



## Example

- 1 The Taylor series for  $e^{4x} \ln(1+x)$  is

$$\begin{aligned} & \left(1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \dots\right) \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right) \\ &= x + \left(-\frac{1}{2} + 4\right)x^2 + \left(\frac{1}{3} + 4 \cdot \left(-\frac{1}{2}\right) + 8\right)x^3 + \dots \\ &= x + \frac{7x^2}{2} + \frac{19x^3}{3} + \dots \end{aligned}$$

- 2 The Taylor series for  $\frac{\tan^{-1} x}{\sqrt{1-x^2}}$  is

$$\begin{aligned} & \left(x - \frac{x^3}{3} + \frac{x^5}{5} + \dots\right) \left(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \dots\right) \\ &= x + \left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{3}{4} - \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{5}\right)x^5 + \dots \\ &= x + \frac{x^3}{6} + \frac{49x^5}{120} + \dots \end{aligned}$$



## Theorem

Suppose  $f(x)$  and  $g(x)$  are infinitely differentiable functions and the Taylor series of  $f(x)$  and  $g(x)$  at  $x = 0$  are

$$a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \dots$$

and

$$b_k x^k + b_{k+1} x^{k+1} + b_{k+2} x^{k+2} + \dots$$

where  $b_k \neq 0$ . Then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{a_k + a_{k+1}x + a_{k+2}x^2 + \dots}{b_k + b_{k+1}x + b_{k+2}x^2 + \dots} \\ &= \frac{a_k}{b_k} \end{aligned}$$

## Proof.

The assumptions on  $f(x)$  and  $g(x)$  imply that

$$\begin{aligned} f(0) = f'(0) = f''(0) = \dots = f^{(k-1)}(0) = 0; \quad f^{(k)}(0) = a_k \\ g(0) = g'(0) = g''(0) = \dots = g^{(k-1)}(0) = 0; \quad g^{(k)}(0) = b_k \end{aligned}$$

Therefore, by L'Hopital's rule, we have

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0} \frac{f''(x)}{g''(x)} = \dots = \lim_{x \rightarrow 0} \frac{f^{(k)}(x)}{g^{(k)}(x)} = \frac{a_k}{b_k}.$$



## Example

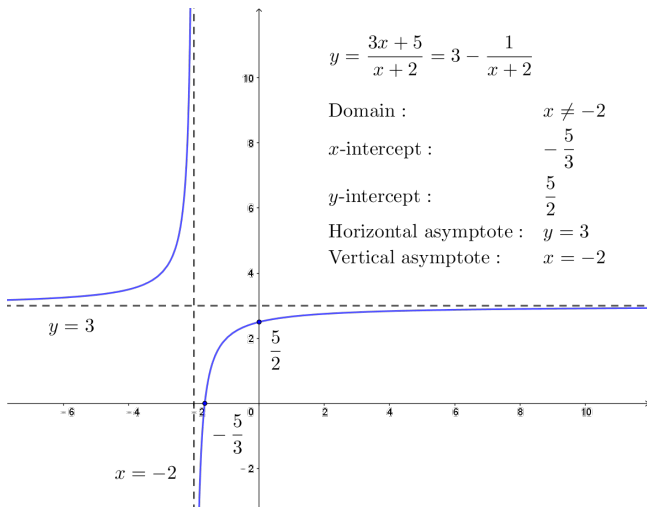
$$\begin{aligned} 1. \quad & \lim_{x \rightarrow 0} \frac{\ln(1+x) - x\sqrt{1-x}}{x - \sin x} \\ &= \lim_{x \rightarrow 0} \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots) - x(1 - \frac{x}{2} - \frac{x^2}{8} + \dots)}{x - (x - \frac{x^3}{6} + \dots)} \\ &= \lim_{x \rightarrow 0} \frac{\frac{11x^3}{24} + \dots}{\frac{x^3}{6} + \dots} \\ &= \frac{11}{4} \\ 2. \quad & \lim_{x \rightarrow 0} \left( \frac{e^x}{x} - \frac{1}{\tan x} \right) = \lim_{x \rightarrow 0} \frac{e^x \sin x - x \cos x}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{(1 + x + \frac{x^2}{2} + \dots)(x - \frac{x^3}{6} + \dots) - x(1 - \frac{x^2}{2} + \dots)}{x(x - \frac{x^3}{6} + \dots)} \\ &= \lim_{x \rightarrow 0} \frac{(x + x^2 + \frac{x^3}{3} + \dots) - (x - \frac{x^3}{2} + \dots)}{x^2 - \frac{x^4}{6} + \dots} \\ &= \lim_{x \rightarrow 0} \frac{x^2 + \frac{5x^3}{6} + \dots}{x^2 - \frac{x^4}{6} + \dots} \\ &= 1 \end{aligned}$$

# Curve sketching

To sketch the graph of  $y = f(x)$ , one first finds

- Domain: The values of  $x$  where  $f(x)$  is defined.
- $x$ -intercepts: The values of  $x$  such that  $f(x) = 0$ .
- $y$ -intercept:  $f(0)$
- Horizontal asymptotes:  
If  $\lim_{x \rightarrow -\infty / +\infty} f(x) = b$ , then  $y = b$  is a horizontal asymptote.
- Vertical asymptotes:  
If  $\lim_{x \rightarrow a^- / a^+} f(x) = -\infty / +\infty$ , then  $x = a$  is a vertical asymptote.

**Example 1:**  $f(x) = \frac{3x + 5}{x + 2}$



$$y = \frac{3x + 5}{x + 2} = 3 - \frac{1}{x + 2}$$

Domain :  $x \neq -2$

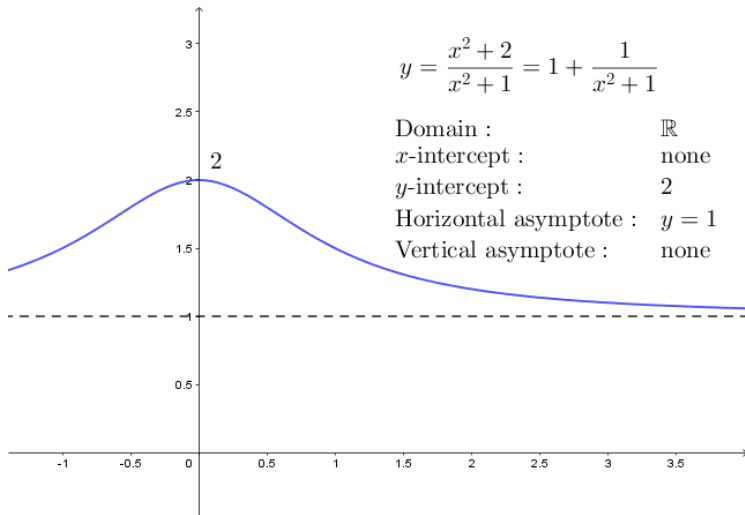
x-intercept :  $-\frac{5}{3}$

y-intercept :  $\frac{5}{2}$

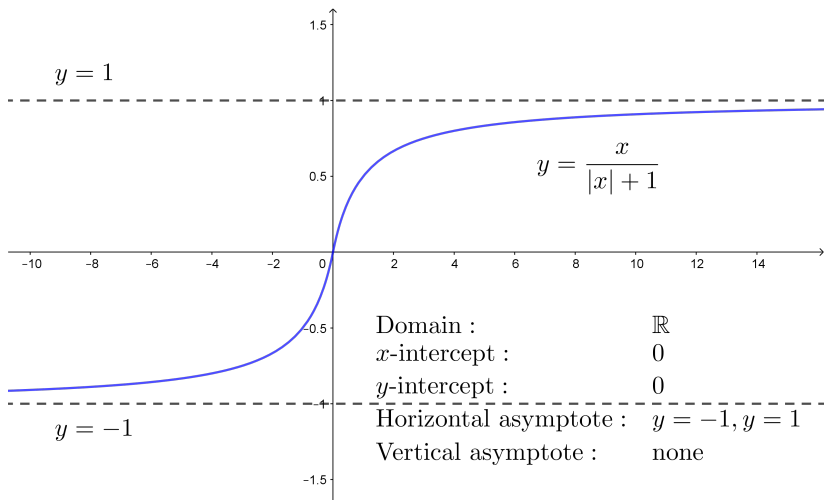
Horizontal asymptote :  $y = 3$

Vertical asymptote :  $x = -2$

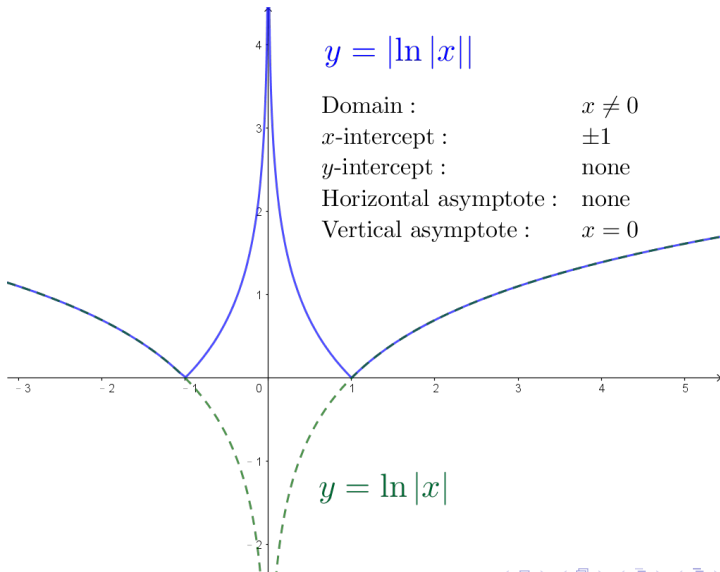
**Example 2:**  $f(x) = \frac{x^2 + 2}{x^2 + 1}$



**Example 3:**  $f(x) = \frac{x}{|x| + 1}$



**Example 4:**  $f(x) = |\ln |x||$



$$y = |\ln |x||$$

Domain :  $x \neq 0$

$x$ -intercept :  $\pm 1$

$y$ -intercept : none

Horizontal asymptote : none

Vertical asymptote :  $x = 0$

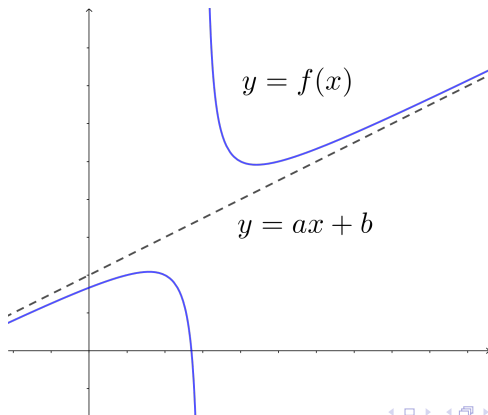
$$y = \ln |x|$$



## Definition (Oblique asymptote)

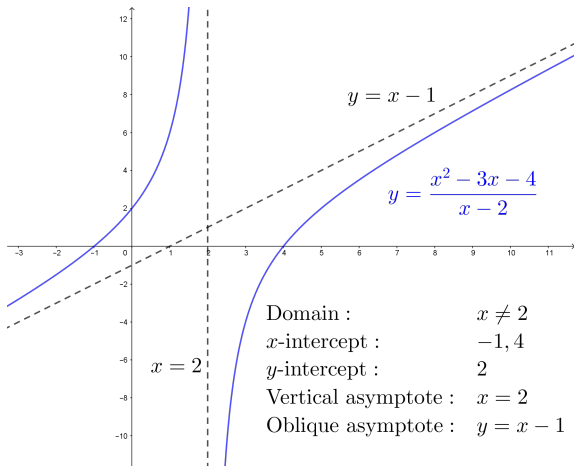
If

$$\lim_{x \rightarrow -\infty / +\infty} (f(x) - (ax + b)) = 0,$$

we say that  $y = ax + b$  is an oblique asymptote of  $y = f(x)$ .

**Example 5:**  $f(x) = \frac{x^2 - 3x - 4}{x - 2}$ .

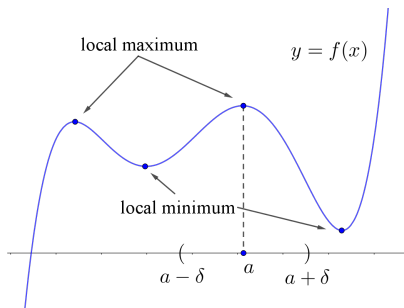
Note that  $\frac{x^2 - 3x - 4}{x - 2} = \frac{x^2 - 2x - (x - 2) - 6}{x - 2} = x - 1 - \frac{6}{x - 2}$ .



## Definition

Let  $f(x)$  be a continuous function. We say that  $f(x)$  has a

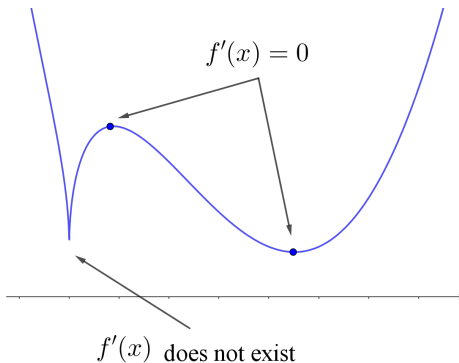
- 1 **local maximum** at  $x = a$  if there exists  $\delta > 0$  such that  $f(x) \leq f(a)$  for any  $x \in (a - \delta, a + \delta)$ .
- 2 **local minimum** at  $x = a$  if there exists  $\delta > 0$  such that  $f(x) \geq f(a)$  for any  $x \in (a - \delta, a + \delta)$ .



## Theorem

Let  $f(x)$  be a continuous function. Suppose  $f(x)$  has local maximum or local minimum at  $x = a$ . Then either

- 1  $f'(a) = 0$ , or
- 2  $f'(x)$  does not exist at  $x = a$ .



## Theorem (First derivative test)

Let  $f(x)$  be a continuous function and  $f'(a) = 0$  or  $f'(a)$  does not exist. Suppose there is  $\delta > 0$  such that

1

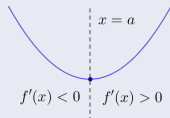
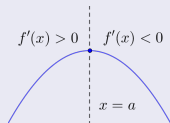
	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	+	-

Then  $f(x)$  has a local maximum at  $x = a$ .

2

	$a - \delta < x < a$	$a < x < a + \delta$
$f'(x)$	-	+

Then  $f(x)$  has a local minimum at  $x = a$ .



## Theorem (Second derivative test)

Let  $f(x)$  be a differentiable function and  $f'(a) = 0$ .

- ① If  $f''(a) < 0$ , then  $f(x)$  has a local maximum at  $x = a$ .

$$f''(a) < 0$$



- ② If  $f''(a) > 0$ , then  $f(x)$  has a local minimum at  $x = a$ .

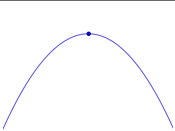
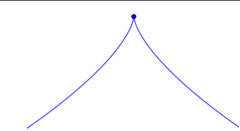
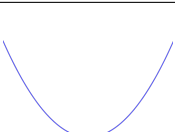
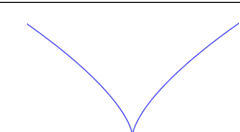
$$f''(a) > 0$$



## Definition (Turning point)

We say that  $f(x)$  has a **turning point** at  $x = a$  if  $f'(x)$  changes sign at  $x = a$ .

If  $f(x)$  has a turning point at  $x = a$ , then either  $f'(a) = 0$  or  $f'(x)$  does not exist.

Turning point	$f'(a) = 0$	$f'(a)$ does not exist
Relative maximum		
Relative minimum		

**Example 6:**  $f(x) = \frac{x-3}{x^2+4x-5}$

$$f(x) = \frac{x-3}{(x-1)(x+5)}, \quad x \neq -5, 1$$

$$f'(x) = \frac{(x^2+4x-5)(1) - (x-3)(2x+4)}{(x-1)^2(x+5)^2} = -\frac{(x+1)(x-7)}{(x-1)^2(x+5)^2}$$

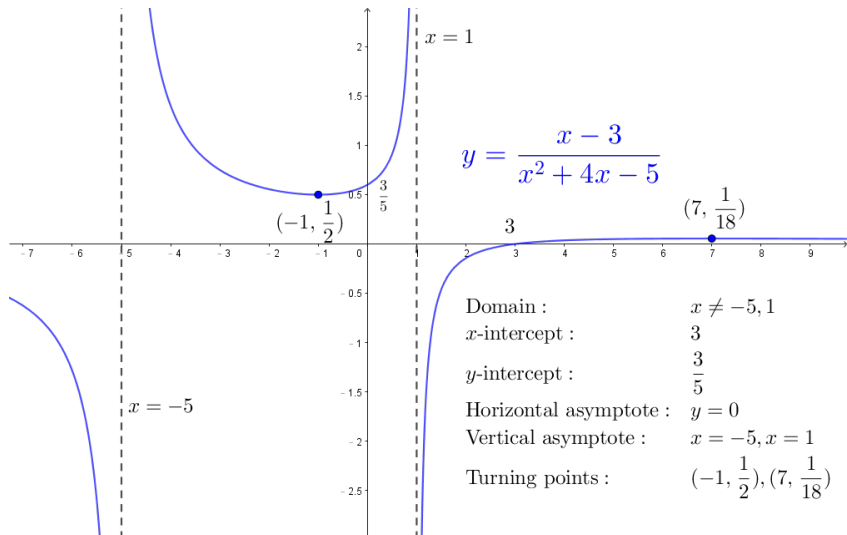
Thus  $f'(x) = 0$  when  $x = -1, 7$ .

	$x < -5$	$-5 < x < -1$	$-1 < x < 1$	$1 < x < 7$	$x > 7$
$f'(x)$	-	-	+	+	-

$(-1, \frac{1}{2})$  is a minimum point and  $(7, \frac{1}{18})$  is a maximum point.



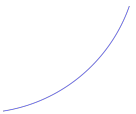
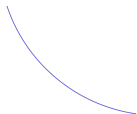
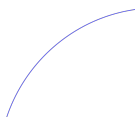

**Example:**  $f(x) = \frac{x-3}{x^2+4x-5}$ .



## Definition (Concavity)

We say that  $f(x)$  is

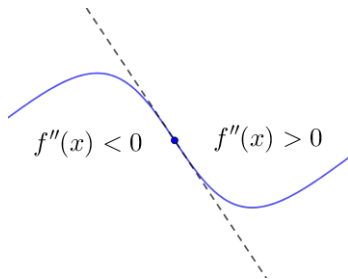
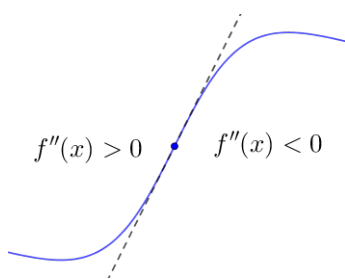
- 1 **Concave upward** on  $(a, b)$  if  $f''(x) > 0$  on  $(a, b)$ .
- 2 **Concave downward** on  $(a, b)$  if  $f''(x) < 0$  on  $(a, b)$ .

	$f'(x) > 0$	$f'(x) < 0$
Concave upward ( $f''(x) > 0$ )		
Concave downward ( $f''(x) < 0$ )		

## Definition (Inflection point)

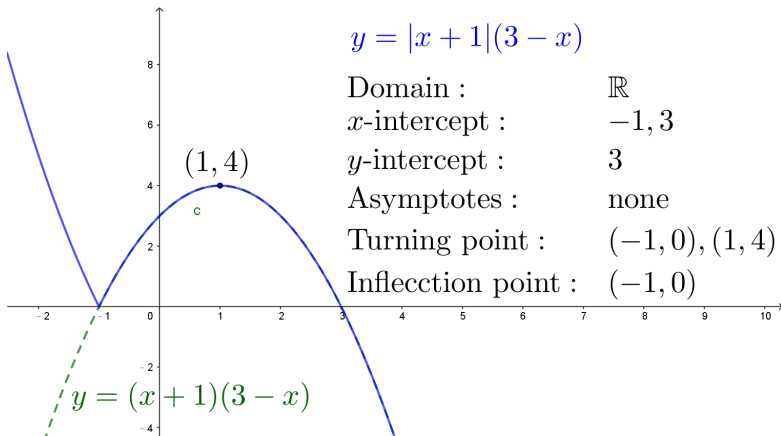
We say that  $f(x)$  has an **inflection point** at  $x = a$  if  $f''(x)$  changes sign at  $x = a$ .

If  $f(x)$  has an inflection point at  $x = a$ , then either  $f''(a) = 0$  or  $f''(a)$  does not exist.



**Example 7:**  $f(x) = |x + 1|(3 - x)$

$$f(x) = |x + 1|(3 - x) = \begin{cases} (x + 1)(x - 3) & \text{if } x < -1 \\ -(x + 1)(x - 3) & \text{if } x \geq -1 \end{cases}$$



**Example 8:**  $f(x) = x + \frac{1}{|x|}$

Since  $\lim_{x \rightarrow \pm\infty} (f(x) - x) = \lim_{x \rightarrow \pm\infty} \frac{1}{|x|} = 0$ ,

$y = f(x)$  has an oblique asymptote  $y = x$ .

When  $x < 0$ ,  $f(x) = x - \frac{1}{x}$ .

$$f'(x) = 1 + \frac{1}{x^2}$$

$$f''(x) = -\frac{2}{x^3}$$

When  $x > 0$ ,  $f(x) = x + \frac{1}{x}$ .

$$f'(x) = 1 - \frac{1}{x^2}$$

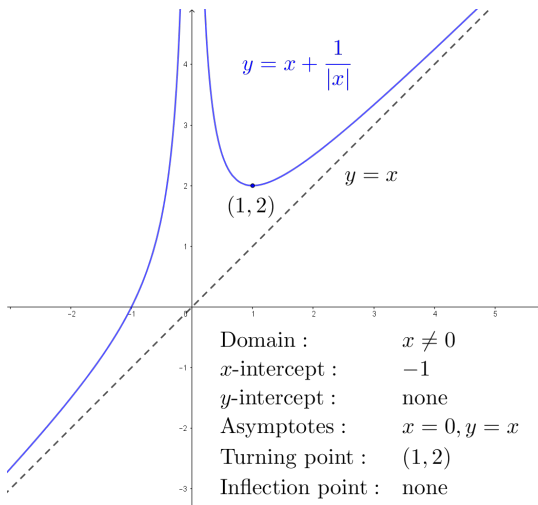
$$f''(x) = \frac{2}{x^3}$$

	$x < 0$	$0 < x < 1$	$x > 1$
$f'(x)$	+	-	+
$f''(x)$	+	+	+

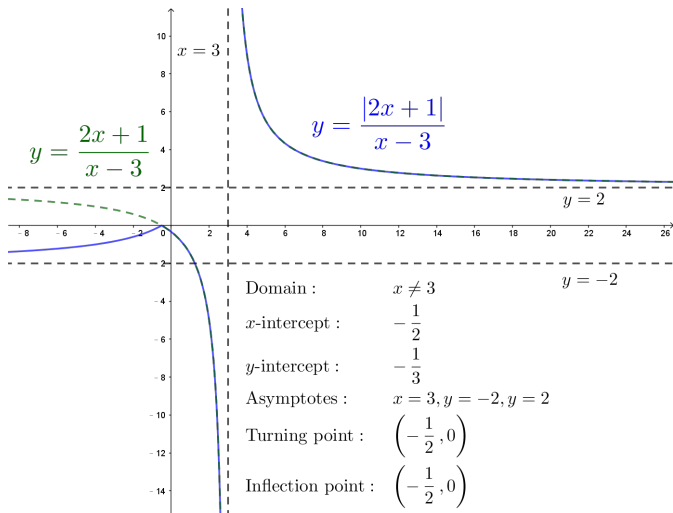
$f(x)$  has a minimum point at  $x = 1$ .

$f(x)$  has no inflection point.

**Example 8:**  $f(x) = x + \frac{1}{|x|}$



**Example 9:**  $f(x) = \frac{|2x + 1|}{x - 3}$



**Example 10:**  $f(x) = 2 - (x - 8)^{\frac{1}{3}}$

$$f'(x) = -\frac{1}{3(x-8)^{\frac{2}{3}}}$$

$$f''(x) = \frac{2}{9(x-8)^{\frac{5}{3}}}$$

$f'(x)$ ,  $f''(x)$  do not exist at  $x = 8$ .

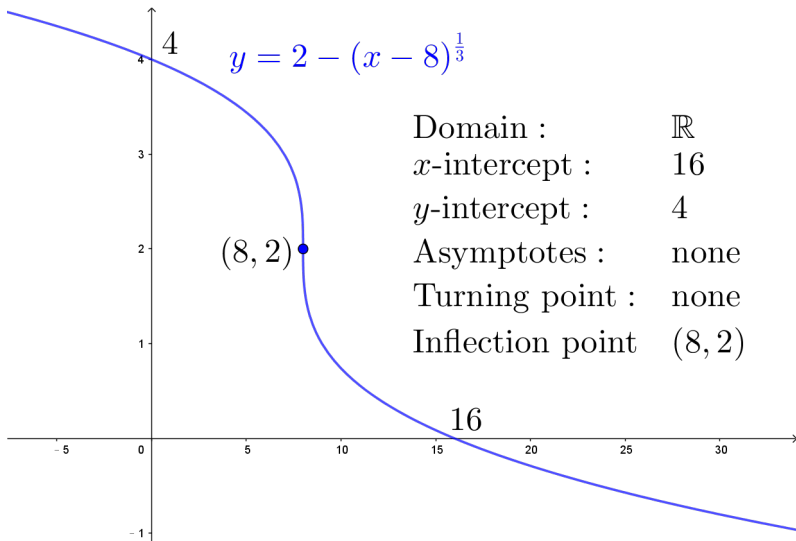
	$x < 8$	$x > 8$
$f'(x)$	-	-
$f''(x)$	-	+

$f(x)$  has no turning point.

$f(x)$  has an inflection point at  $x = 8$ .

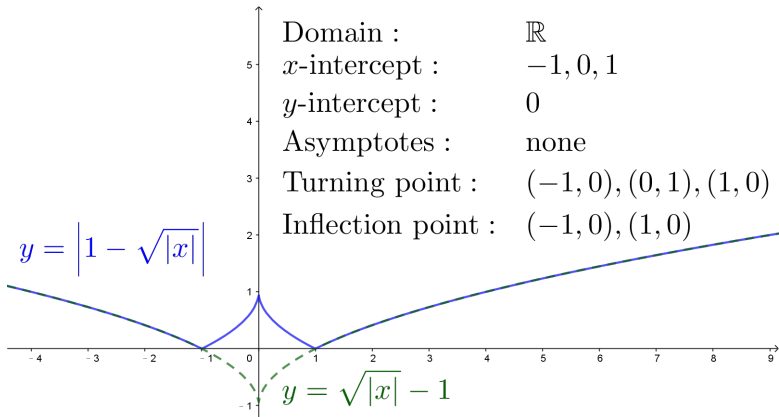


**Example 10:**  $f(x) = 2 - (x - 8)^{\frac{1}{3}}$



Domain :  $\mathbb{R}$   
 $x$ -intercept : 16  
 $y$ -intercept : 4  
Asymptotes : none  
Turning point : none  
Inflection point (8, 2)

**Example 11:**  $f(x) = |1 - \sqrt{|x|}|$



**Example 12:**  $f(x) = \frac{x^2 + x - 2}{x^2}$

Domain:  $x \neq 0$ 

$$f(x) = \frac{x^2 + x - 2}{x^2} = 1 + \frac{x - 2}{x^2}$$

 $f(x)$  has a horizontal asymptote  $y = 1$ .

$$f'(x) = \frac{x^2 - 2x(x - 2)}{x^4} = \frac{x - 2(x - 2)}{x^3} = -\frac{x - 4}{x^3}$$

 $f'(x) = 0$  when  $x = 4$ 

$$f''(x) = -\frac{x^3 - 3x^2(x - 4)}{x^6} = -\frac{x - 3(x - 4)}{x^6} = \frac{2(x - 6)}{x^4}$$

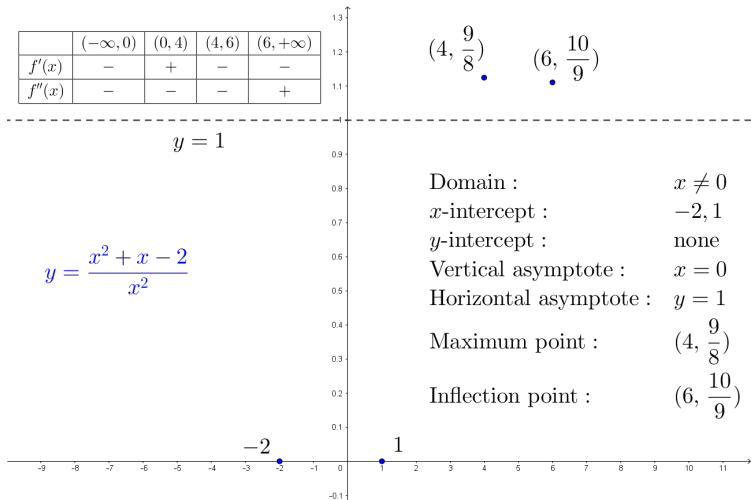
 $f''(x) = 0$  when  $x = 6$ .

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	-	+	-	-
$f''(x)$	-	-	-	+

 $(4, \frac{9}{8})$  is maximum point. $(6, \frac{10}{9})$  is an inflection point.

**Example 12:**  $f(x) = \frac{x^2 + x - 2}{x^2}$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	-	+	-	-
$f''(x)$	-	-	-	+



Domain :  $x \neq 0$

$x$ -intercept :  $-2, 1$

$y$ -intercept : none

Vertical asymptote :  $x = 0$

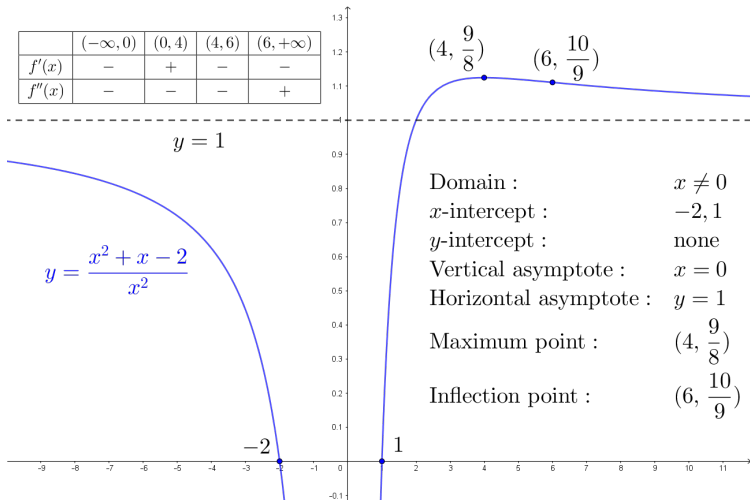
Horizontal asymptote :  $y = 1$

Maximum point :  $(4, \frac{9}{8})$

Inflection point :  $(6, \frac{10}{9})$

**Example 12:**  $f(x) = \frac{x^2 + x - 2}{x^2}$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	-	+	-	-
$f''(x)$	-	-	-	+



Domain :  $x \neq 0$

$x$ -intercept :  $-2, 1$

$y$ -intercept : none

Vertical asymptote :  $x = 0$

Horizontal asymptote :  $y = 1$

Maximum point :  $(4, \frac{9}{8})$

Inflection point :  $(6, \frac{10}{9})$

**Example 13:**  $f(x) = \frac{x^3}{(x-2)^2}$

$$f(x) = x + 4 + \frac{12x - 16}{(x-2)^2}, \quad x \neq 2$$

$f(x)$  has an oblique asymptote  $y = x + 4$

$$f'(x) = \frac{3x^2(x-2)^2 - 2(x-2)x^3}{(x-2)^4} = \frac{3x^2(x-2) - 2x^3}{(x-2)^3} = \frac{x^3 - 6x^2}{(x-2)^3}$$

$f'(x) = 0$  when  $x = 0, 6$

$$f''(x) = \frac{(3x^2 - 12x)(x-2)^3 - 3(x-2)^2(x^3 - 6x^2)}{(x-2)^6} = \frac{24x}{(x-2)^4}$$

$f''(x) = 0$  when  $x = 0$ .

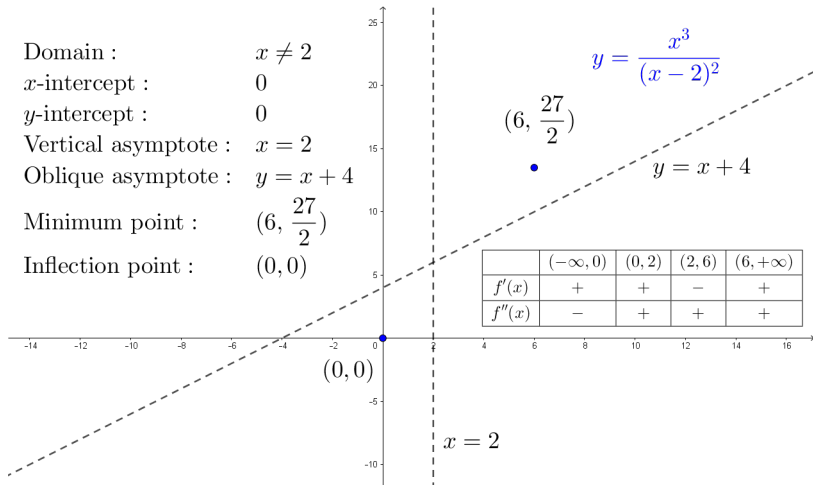
	$(-\infty, 0)$	$(0, 2)$	$(2, 6)$	$(6, +\infty)$
$f'(x)$	+	+	-	+
$f''(x)$	-	+	+	+

$(6, \frac{27}{2})$  is minimum point.

$(0, 0)$  is an inflection point.

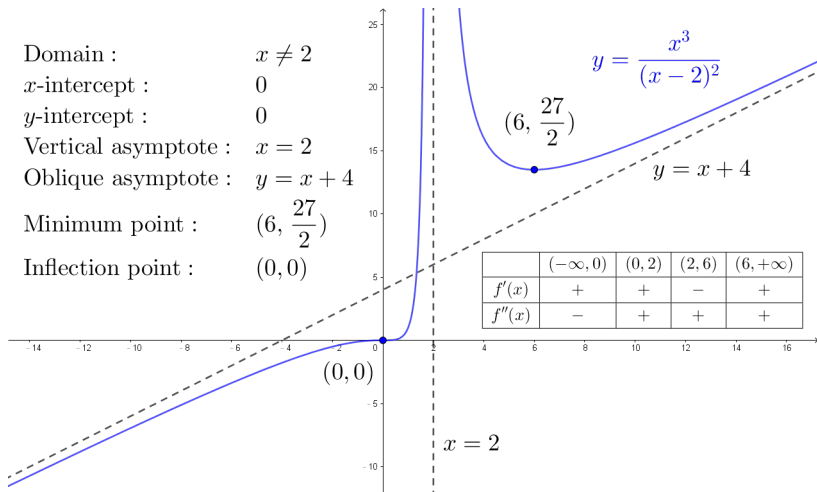
**Example 13:**  $f(x) = \frac{x^3}{(x-2)^2}$

Domain :	$x \neq 2$
$x$ -intercept :	0
$y$ -intercept :	0
Vertical asymptote :	$x = 2$
Oblique asymptote :	$y = x + 4$
Minimum point :	$(6, \frac{27}{2})$
Inflection point :	$(0, 0)$



**Example 13:**  $f(x) = \frac{x^3}{(x-2)^2}$

- Domain :  $x \neq 2$   
 $x$ -intercept : 0  
 $y$ -intercept : 0  
 Vertical asymptote :  $x = 2$   
 Oblique asymptote :  $y = x + 4$   
 Minimum point :  $(6, \frac{27}{2})$   
 Inflection point :  $(0, 0)$





**Example 14:**  $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

First

$$\lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}}{x} = \lim_{x \rightarrow \pm\infty} \left(1 - \frac{3}{x}\right)^{\frac{2}{3}} = 1$$

and

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} (f(x) - x) &= \lim_{x \rightarrow \pm\infty} x \left( \left(1 - \frac{3}{x}\right)^{\frac{2}{3}} - 1 \right) \\ &= \lim_{h \rightarrow 0} \frac{(1 - 3h)^{\frac{2}{3}} - 1}{h} \\ &= -2 \end{aligned}$$

Thus  $y = x - 2$  is an oblique asymptote.

**Example 14:**  $f(x) = x^{\frac{1}{3}}(x-3)^{\frac{2}{3}}$

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-\frac{2}{3}}(x-3)^{\frac{2}{3}} + \frac{2}{3}x^{\frac{1}{3}}(x-3)^{-\frac{1}{3}} \\ &= \frac{x-1}{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}}} \end{aligned}$$

$f'(x) = 0$  when  $x = 1$  and  $f'(x)$  does not exist when  $x = 0, 3$ .

$$\begin{aligned} f''(x) &= \frac{x^{\frac{2}{3}}(x-3)^{\frac{1}{3}} - \left(\frac{2}{3}x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}}\right)(x-1)}{x^{\frac{4}{3}}(x-3)^{\frac{2}{3}}} \\ &= \frac{3x(x-3) - (2(x-3) + x)(x-1)}{3x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}} \\ &= -\frac{2}{x^{\frac{5}{3}}(x-3)^{\frac{4}{3}}} \end{aligned}$$

$f''(x)$  does not exist when  $x = 0, 3$ .

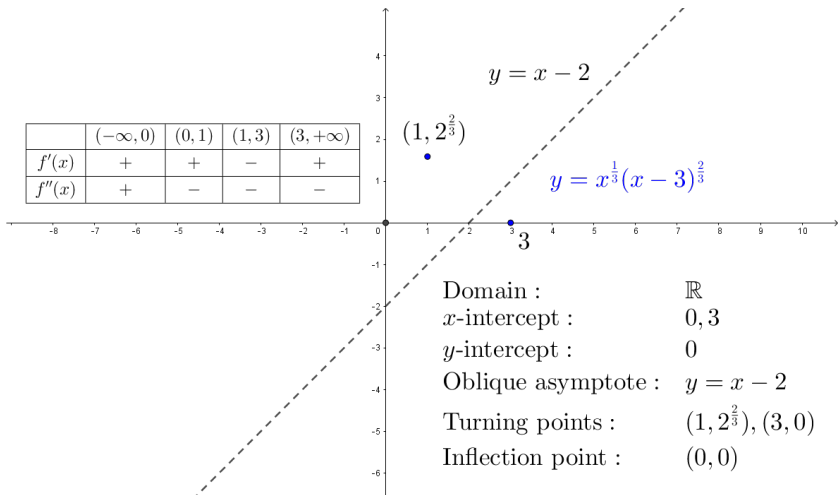
	$(-\infty, 0)$	$(0, 1)$	$(1, 3)$	$(3, +\infty)$
$f'(x)$	+	+	-	+
$f''(x)$	+	-	-	-

$(1, 2^{\frac{2}{3}})$  is a maximum point.

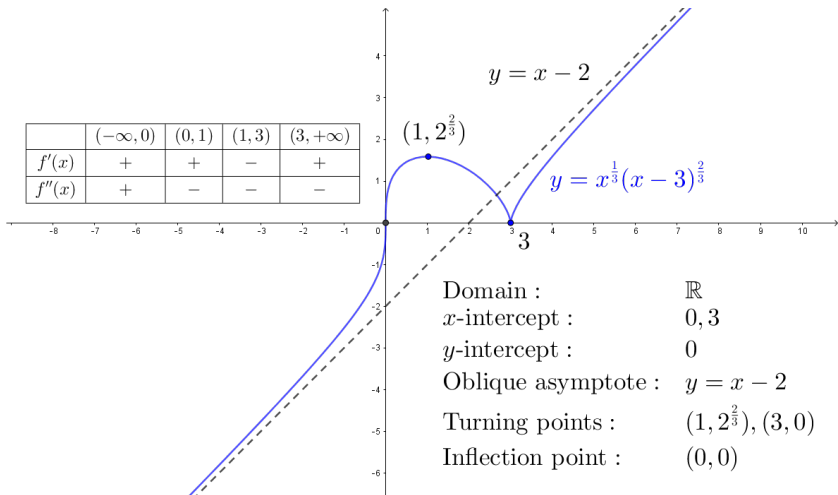
$(3, 0)$  is a minimum point.

$(0, 0)$  is an inflection point.

**Example 14:**  $f(x) = x^{\frac{1}{3}}(x - 3)^{\frac{2}{3}}$



**Example 14:**  $f(x) = x^{\frac{1}{3}}(x - 3)^{\frac{2}{3}}$



# Indefinite integral and substitution

## Definition

Let  $f(x)$  be a continuous function. A **primitive function**, or an **anti-derivative**, of  $f(x)$  is a function  $F(x)$  such that

$$F'(x) = f(x).$$

The collection of all anti-derivatives of  $f(x)$  is called the **indefinite integral** of  $f(x)$  and is denoted by

$$\int f(x)dx.$$

The function  $f(x)$  is called the **integrand** of the integral.

Note: Anti-derivative of a function is not unique. If  $F(x)$  is an anti-derivative of  $f$ , then  $F(x) + C$  is an anti-derivative of  $f(x)$  for any constant  $C$ . Moreover, any anti-derivative of  $f(x)$  is of the form  $F(x) + C$  and we write

$$\int f(x)dx = F(x) + C$$

where  $C$  is arbitrary constant called the **integration constant**. Note that  $\int f(x)dx$  is not a single function but a collection of functions.

## Theorem

Let  $f(x)$  and  $g(x)$  be continuous functions and  $k$  be a constant.

$$\textcircled{1} \int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

$$\textcircled{2} \int kf(x)dx = k \int f(x)dx$$

## Theorem (formulas for indefinite integrals)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$\int e^x dx = e^x + C;$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \cos x dx = \sin x + C;$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C;$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C;$$

$$\int \csc x \cot x dx = -\csc x + C$$

## Example

$$1. \int (x^3 - x + 5) dx = \frac{x^4}{4} - \frac{x^2}{2} + 5x + C$$

$$\begin{aligned} 2. \int \frac{(x+1)^2}{x} dx &= \int \frac{x^2 + 2x + 1}{x} dx \\ &= \int \left( x + 2 + \frac{1}{x} \right) dx \\ &= \frac{x^2}{2} + 2x + \ln|x| + C \end{aligned}$$

$$\begin{aligned} 3. \int \frac{3x^2 + \sqrt{x} - 1}{\sqrt{x}} dx &= \int \left( 3x^{3/2} + 1 - x^{-1/2} \right) dx \\ &= \frac{6}{5}x^{5/2} + x - 2x^{1/2} + C \end{aligned}$$

$$\begin{aligned} 4. \int \left( \frac{3 \sin x}{\cos^2 x} - 2e^x \right) dx &= \int (3 \sec x \tan x - 2e^x) dx \\ &= 3 \sec x - 2e^x + C \end{aligned}$$



## Example

Suppose we want to compute

$$\int x\sqrt{x^2+4} dx$$

First we let

$$u = x^2 + 4.$$

We may formally write

$$du = \frac{du}{dx} dx = \left[ \frac{d}{dx}(x^2 + 4) \right] dx = 2x dx$$

Here  $du$  is called the differential of  $u$  defined as  $\frac{du}{dx} dx$ . Thus the integral is

$$\begin{aligned} \int x\sqrt{x^2+4} dx &= \frac{1}{2} \int \sqrt{x^2+4}(2x dx) = \frac{1}{2} \int \sqrt{u} du \\ &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{(x^2+4)^{\frac{3}{2}}}{3} + C \end{aligned}$$

## Example

$$\begin{aligned}\int x\sqrt{x^2+4} dx &= \int \sqrt{x^2+4} d\left(\frac{x^2}{2}\right) \\ &= \frac{1}{2} \int \sqrt{x^2+4} dx^2 \\ &= \frac{1}{2} \int \sqrt{x^2+4} d(x^2+4) \\ &= \frac{(x^2+4)^{\frac{3}{2}}}{3} + C\end{aligned}$$

## Theorem

Let  $f(x)$  be a continuous function defined on  $[a, b]$ . Suppose there exists a differentiable function  $u = \varphi(x)$  and continuous function  $g(u)$  such that  $f(x) = g(\varphi(x))\varphi'(x)$  for any  $x \in (a, b)$ . Then

$$\begin{aligned}\int f(x)dx &= \int g(\varphi(x))\varphi'(x)dx \\ &= \int g(u)du\end{aligned}$$

## Example

$$\begin{aligned} & \int x^2 e^{x^3+1} dx \\ & \text{Let } u = x^3 + 1, \\ & \text{then } du = 3x^2 dx \\ & = \frac{1}{3} \int e^u du \\ & = \frac{e^u}{3} + C \\ & = \frac{e^{x^3+1}}{3} + C \end{aligned} \qquad \begin{aligned} & \int x^2 e^{x^3+1} dx \\ & = \int e^{x^3+1} d\left(\frac{x^3}{3}\right) \\ & = \frac{1}{3} \int e^{x^3+1} dx^3 \\ & = \frac{1}{3} \int e^{x^3+1} d(x^3 + 1) \\ & = \frac{e^{x^3+1}}{3} + C \end{aligned}$$

## Example

$$\begin{aligned} & \int \cos^4 x \sin x dx & \int \cos^4 x \sin x dx \\ \text{Let } u = \cos x, & & = \int \cos^4 x d(-\cos x) \\ \text{then } du = -\sin x dx & & = -\int \cos^4 x d \cos x \\ = -\int u^4 du & & = -\frac{\cos^5 x}{5} + C \\ = -\frac{u^5}{5} + C & & \\ = -\frac{\cos^5 x}{5} + C & & \end{aligned}$$

## Example

$$\begin{aligned} & \int \frac{dx}{x \ln x} & \int \frac{dx}{x \ln x} \\ \text{Let } u &= \ln x, & = \int \frac{d \ln x}{\ln x} \\ \text{then } du &= \frac{dx}{x} & = \ln |\ln x| + C \\ & & \\ & & \\ = & \int \frac{du}{u} & \\ & = \ln |u| + C & \\ & = \ln |\ln x| + C & \end{aligned}$$

## Example

$$\begin{aligned} & \int \frac{dx}{e^x + 1} & & \int \frac{dx}{e^x + 1} \\ \text{Let } u &= 1 + e^{-x}, & & = \int \left( 1 - \frac{e^x}{1 + e^x} \right) dx \\ \text{then } du &= -e^{-x} dx & & = x - \int \frac{de^x}{1 + e^x} \\ & & & = x - \ln(1 + e^x) + C \\ & = \int \frac{e^{-x} dx}{1 + e^{-x}} & & \\ & = - \int \frac{du}{u} & & \\ & = -\ln u + C & & \\ & = -\ln(1 + e^{-x}) + C & & \\ & = x - \ln(1 + e^x) + C & & \end{aligned}$$

## Example

$$\begin{aligned} & \int \frac{dx}{1 + \sqrt{x}} \\ \text{Let } u &= 1 + \sqrt{x}, \\ \text{then } du &= \frac{dx}{2\sqrt{x}} \\ &= 2 \int \frac{(u-1)du}{u} \\ &= 2 \int \left(1 - \frac{1}{u}\right) du \\ &= 2u - 2 \ln u + C' \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C \end{aligned}$$
$$\begin{aligned} & \int \frac{dx}{1 + \sqrt{x}} \\ &= \int \frac{\sqrt{x} dx}{\sqrt{x}(1 + \sqrt{x})} \\ &= 2 \int \frac{\sqrt{x} d\sqrt{x}}{1 + \sqrt{x}} \\ &= 2 \int \left(1 - \frac{1}{1 + \sqrt{x}}\right) d\sqrt{x} \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C \end{aligned}$$



# Definite integral

## Definition

Let  $f(x)$  be a function on  $[a, b]$ . A **Partition** of  $[a, b]$  is a set of finite points

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

and we define

$$\begin{aligned}\Delta x_k &= x_k - x_{k-1}, \text{ for } k = 1, 2, \dots, n \\ \|P\| &= \max_{1 \leq k \leq n} \{\Delta x_k\}\end{aligned}$$

## Definition

Let  $f(x)$  be a function on  $[a, b]$ . The **lower** and **upper Riemann sums** with respect to partition  $P$  are

$$\mathcal{L}(f, P) = \sum_{k=1}^n m_k \Delta x_k, \text{ and } \mathcal{U}(f, P) = \sum_{k=1}^n M_k \Delta x_k$$

where

$$m_k = \inf\{f(x) : x_{k-1} \leq x \leq x_k\}, \text{ and } M_k = \sup\{f(x) : x_{k-1} \leq x \leq x_k\}$$

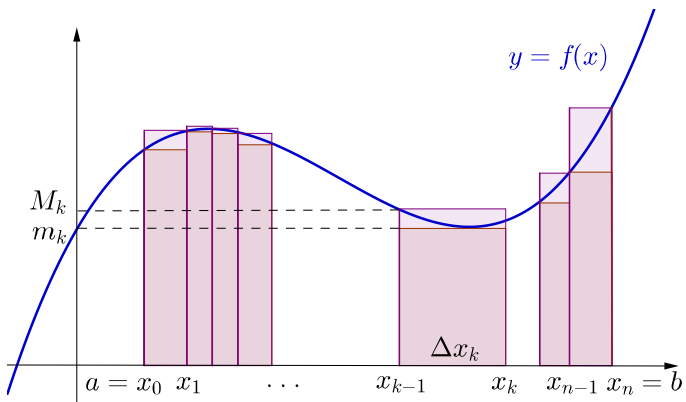


Figure: Upper and lower Riemann sum

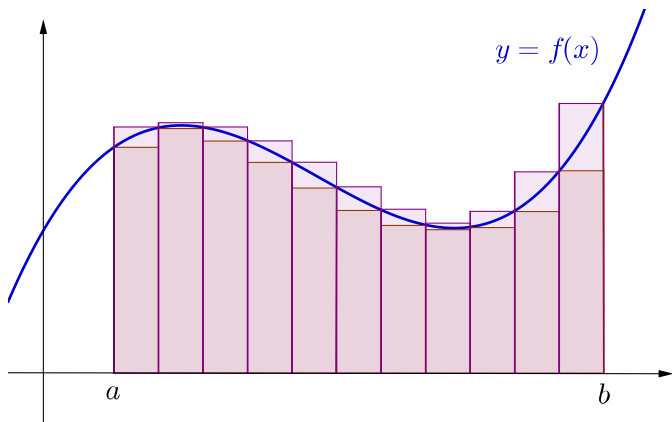


Figure: Upper and lower Riemann sum

## Definition (Riemann integral)

Let  $[a, b]$  be a closed and bounded interval and  $f : [a, b] \rightarrow \mathbb{R}$  be a real valued function defined on  $[a, b]$ . We say that  $f(x)$  is **Riemann integrable** on  $[a, b]$  if the limits of  $\mathcal{L}(f, P)$  and  $\mathcal{U}(f, P)$  exist as  $\|P\|$  tends to 0 and are *equal*. In this case, we define the **Riemann integral** of  $f(x)$  over  $[a, b]$  by

$$\int_a^b f(x)dx = \lim_{\|P\| \rightarrow 0} \mathcal{L}(f, P) = \lim_{\|P\| \rightarrow 0} \mathcal{U}(f, P).$$

Note: We say that  $\lim_{\|P\| \rightarrow 0} \mathcal{L}(f, P) = L$  if for any  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that if  $\|P\| < \delta$ , then  $|\mathcal{L}(f, P) - L| < \varepsilon$ .

## Theorem

Let  $f(x)$  and  $g(x)$  be integrable functions on  $[a, b]$ ,  $a < c < b$  and  $k$  be constants.

$$\textcircled{1} \int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\textcircled{2} \int_a^b kf(x)dx = k \int_a^b f(x)dx$$

$$\textcircled{3} \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\textcircled{4} \int_b^a f(x)dx = - \int_a^b f(x)dx$$

## Theorem

Suppose  $f(x)$  is a continuous function on  $[a, b]$ . Then  $f(x)$  is Riemann integrable on  $[a, b]$  and we have

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x_k \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(a + \frac{k}{n}(b-a)\right) \left(\frac{b-a}{n}\right).\end{aligned}$$

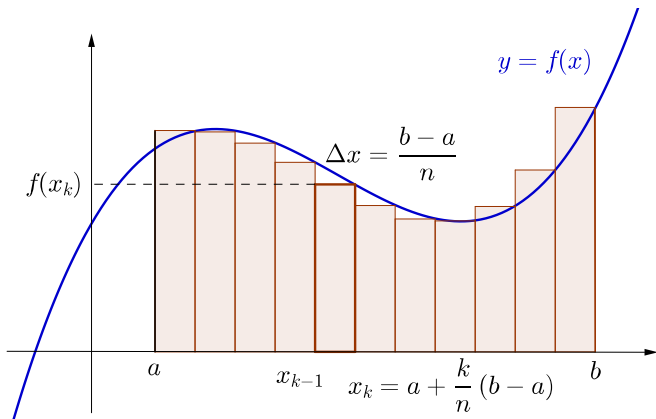


Figure: Formula for Riemann integral



## Example

Use the formula for definite integral of continuous function to evaluate

$$\int_0^1 x^2 dx$$

## Solution

$$\begin{aligned}\int_0^1 x^2 dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(0 + \frac{k}{n}(1-0)\right)^2 \left(\frac{1-0}{n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{n^3} \\ &= \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \frac{1}{3}\end{aligned}$$

# Fundamental theorem of calculus

## Theorem (Fundamental theorem of calculus)

**First part:** Let  $f(x)$  be a function which is continuous on  $[a, b]$ . Let  $F : [a, b] \rightarrow \mathbb{R}$  be the function defined by

$$F(x) = \int_a^x f(t)dt$$

Then  $F(x)$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  and

$$F'(x) = f(x).$$

for any  $x \in (a, b)$ . Put in another way, we have

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \quad \text{for } x \in (a, b).$$

## Theorem (Fundamental theorem of calculus)

**Second part:** Let  $f(x)$  be a function which is continuous on  $[a, b]$ . Let  $F(x)$  be a primitive function of  $f(x)$ , in other words,  $F(x)$  is a continuous function on  $[a, b]$  and  $F'(x) = f(x)$  for any  $x \in (a, b)$ .

Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

## Example

Let  $f(x) = \sqrt{1-x^2}$ . The graph of  $y = f(x)$  is a unit semicircle centered at the origin. Using the formula for area of circular sectors, we calculate

$$F(x) = \int_0^x f(t)dt = \int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2}.$$

By fundamental theorem of calculus, we know that  $F(x)$  is an anti-derivative of  $f(x)$ . One may check this by differentiating  $F(x)$  and get

$$\begin{aligned} F'(x) &= \frac{1}{2} \left( \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ &= \frac{1}{2} \left( \frac{1-x^2-x^2+1}{\sqrt{1-x^2}} \right) \\ &= \sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

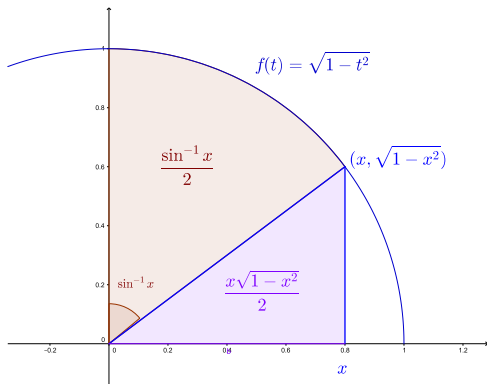


Figure: 
$$\int_0^x \sqrt{1-t^2} dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2}$$

## Example

$$\begin{aligned} 1. \int_1^3 (x^3 - 4x + 5) dx &= \left[ \frac{x^4}{4} - 2x^2 + 5x \right]_1^3 \\ &= \left[ \left( \frac{3^4}{4} - 2(3^2) + 5(3) \right) - \left( \frac{1^4}{4} - 2(1^2) + 5(1) \right) \right] \\ &= 14 \end{aligned}$$

$$\begin{aligned} 2. \int_0^{\pi^2} \frac{\sin \sqrt{x}}{\sqrt{x}} dx &= 2 \int_0^{\pi^2} \sin \sqrt{x} d\sqrt{x} = 2 [-\cos \sqrt{x}]_0^{\pi^2} \\ &= 2 [-\cos \sqrt{\pi^2} - (-\cos 0)] = 4 \end{aligned}$$

$$\begin{aligned} 3. \int_3^5 x \sqrt{x^2 - 9} dx &= \frac{1}{2} \int_3^5 \sqrt{x^2 - 9} d(x^2 - 9) \\ &= \frac{1}{3} \left[ (x^2 - 9)^{\frac{3}{2}} \right]_3^5 \\ &= \frac{64}{3} \end{aligned}$$

## Example

We have the following formulas for derivatives of functions defined by integrals.

$$\textcircled{1} \quad \frac{d}{dx} \int_a^x f(t) dt = f(x)$$

$$\textcircled{2} \quad \frac{d}{dx} \int_x^b f(t) dt = -f(x)$$

$$\textcircled{3} \quad \frac{d}{dx} \int_a^{v(x)} f(t) dt = f(v) \frac{dv}{dx}$$

$$\textcircled{4} \quad \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$$

## Proof.

1. This is the first part of fundamental theorem of calculus.

$$2. \frac{d}{dx} \int_x^b f(t) dt = \frac{d}{dx} \left( - \int_b^x f(t) dt \right)$$

$$= -f(x)$$

$$3. \frac{d}{dx} \int_a^{v(x)} f(t) dt = \left( \frac{d}{dv} \int_a^{v(x)} f(t) dt \right) \frac{dv}{dx}$$

$$= f(v) \frac{dv}{dx}$$

$$4. \frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = \frac{d}{dx} \left( \int_c^{v(x)} f(t) dt + \int_{u(x)}^c f(t) dt \right)$$

$$= \frac{d}{dx} \left( \int_c^{v(x)} f(t) dt - \int_c^{u(x)} f(t) dt \right)$$

$$= f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$$





## Example

Find  $F'(x)$  for the the functions.

$$\textcircled{1} F(x) = \int_1^x \sqrt{t} e^t dt$$

$$\textcircled{2} F(x) = \int_x^\pi \frac{\sin t}{t} dt$$

$$\textcircled{3} F(x) = \int_0^{\sin x} \sqrt{1+t^4} dt$$

$$\textcircled{4} F(x) = \int_{-x}^{x^2} e^{t^2} dt$$

## Solution

$$\begin{aligned}1. \frac{d}{dx} \int_1^x \sqrt{t} e^t dt &= \sqrt{x} e^x \\2. \frac{d}{dx} \int_x^\pi \frac{\sin t}{t} dt &= -\frac{\sin x}{x} \\3. \frac{d}{dx} \int_0^{\sin x} \sqrt{1+t^4} dt &= \sqrt{1+\sin^4 x} \frac{d}{dx} \sin x \\&= \cos x \sqrt{1+\sin^4 x} \\4. \frac{d}{dx} \int_{-x}^{x^2} e^{t^2} dt &= e^{(x^2)^2} \frac{d}{dx} x^2 - e^{(-x)^2} \frac{d}{dx} (-x) \\&= 2xe^{x^4} + e^{x^2}\end{aligned}$$

# Trigonometric integrals

## Techniques

*Useful identities for trigonometric integrals.*

- 1
  - $\cos^2 x + \sin^2 x = 1$
  - $\sec^2 x = 1 + \tan^2 x$
  - $\csc^2 x = 1 + \cot^2 x$
  
- 2
  - $\cos^2 x = \frac{1 + \cos 2x}{2}$
  - $\sin^2 x = \frac{1 - \cos 2x}{2}$
  - $\cos x \sin x = \frac{\sin 2x}{2}$
  
- 3
  - $\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$
  - $\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$
  - $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$

## Techniques

To evaluate

$$\int \cos^m x \sin^n x dx$$

where  $m, n$  are non-negative integers,

- Case 1. If  $m$  is odd, use  $\cos x dx = d \sin x$ . (Substitute  $u = \sin x$ .)
- Case 2. If  $n$  is odd, use  $\sin x dx = -d \cos x$ . (Substitute  $u = \cos x$ .)
- Case 3. If both  $m, n$  are even, then use double angle formulas to reduce the power.

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos x \sin x = \frac{\sin 2x}{2}$$

## Techniques

$$\textcircled{1} \int \tan x dx = \ln |\sec x| + C$$

$$\textcircled{2} \int \cot x dx = \ln |\sin x| + C$$

$$\textcircled{3} \int \sec x dx = \ln |\sec x + \tan x| + C$$

$$\textcircled{4} \int \csc x dx = \ln |\csc x - \cot x| + C$$

## Proof

We prove (1), (3) and the rest are left as exercise.

$$\begin{aligned} 1. \int \tan x dx &= \int \frac{\sin x dx}{\cos x} \\ &= - \int \frac{d \cos x}{\cos x} \\ &= - \ln |\cos x| + C \\ &= \ln |\sec x| + C \\ 3. \int \sec x dx &= \int \frac{\sec x (\sec x + \tan x) dx}{(\sec x + \tan x)} \\ &= \int \frac{(\sec^2 x + \sec x \tan x) dx}{(\sec x + \tan x)} \\ &= \int \frac{d(\tan x + \sec x)}{(\sec x + \tan x)} \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

## Techniques

To evaluate

$$\int \sec^m x \tan^n x dx$$

where  $m, n$  are non-negative integers,

- Case 1. If  $m$  is even, use  $\sec^2 x dx = d \tan x$ . (Substitute  $u = \tan x$ .)
- Case 2. If  $n$  is odd, use  $\sec x \tan x dx = d \sec x$ . (Substitute  $u = \sec x$ .)
- Case 3. If both  $m$  is odd and  $n$  is even, use  $\tan^2 x = \sec^2 x - 1$  to write everything in terms of  $\sec x$ .

## Example

Evaluate the following integrals.

$$\textcircled{1} \int \sin^2 x dx$$

$$\textcircled{2} \int \cos^4 3x dx$$

$$\textcircled{3} \int \cos 2x \cos x dx$$

$$\textcircled{4} \int \cos 3x \sin 5x dx$$



## Solution

$$1. \int \sin^2 x dx = \int \left( \frac{1 - \cos 2x}{2} \right) dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\begin{aligned} 2. \int \cos^4 x dx &= \int \left( \frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \int \left( \frac{1 + 2 \cos 2x + \cos^2 2x}{4} \right) dx \\ &= \frac{x}{4} + \frac{\sin 2x}{4} + \int \left( \frac{1 + \cos 4x}{8} \right) dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \end{aligned}$$

$$3. \int \cos 2x \cos x dx = \frac{1}{2} \int (\cos 3x + \cos x) dx = \frac{\sin 3x}{6} + \frac{\sin x}{2} + C$$

$$4. \int \cos 3x \sin 5x dx = \frac{1}{2} \int (\sin 8x + \sin 2x) dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$$

## Example

Evaluate the following integrals.

$$\textcircled{1} \int \cos x \sin^4 x dx$$

$$\textcircled{2} \int \cos^2 x \sin^3 x dx$$

$$\textcircled{3} \int \cos^4 x \sin^2 x dx$$

## Solution

$$1. \int \cos x \sin^4 x dx = \int \sin^4 x d \sin x = \frac{\sin^5 x}{5} + C$$

$$\begin{aligned} 2. \int \cos^2 x \sin^3 x dx &= - \int \cos^2 x (1 - \cos^2 x) d \cos x \\ &= - \int (\cos^2 x - \cos^4 x) d \cos x \\ &= - \frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C \end{aligned}$$

$$\begin{aligned} 3. \int \cos^4 x \sin^2 x dx &= \int \left( \frac{1 + \cos 2x}{2} \right) \left( \frac{\sin 2x}{2} \right)^2 dx \\ &= \frac{1}{8} \int (\sin^2 2x + \cos 2x \sin^2 2x) dx \\ &= \frac{1}{8} \int \left( \frac{1 - \cos 4x}{2} \right) dx + \frac{1}{16} \int \sin^2 2x d \sin 2x \\ &= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C \end{aligned}$$

## Example

Evaluate the following integrals.

$$\textcircled{1} \int \sec^2 x \tan^2 x dx$$

$$\textcircled{2} \int \sec x \tan^3 x dx$$

$$\textcircled{3} \int \tan^3 x dx$$

## Solution

$$1. \int \sec^2 x \tan^2 x dx = \int \tan^2 x d \tan x = \frac{\tan^3 x}{3} + C$$

$$\begin{aligned} 2. \int \sec x \tan^3 x dx &= \int \tan^2 x d \sec x = \int (\sec^2 x - 1) d \sec x \\ &= \frac{\sec^3 x}{3} - \sec x + C \end{aligned}$$

$$\begin{aligned} 3. \int \tan^3 x dx &= \int \tan x (\sec^2 x - 1) dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x d \tan x - \ln |\sec x| \\ &= \frac{\tan^2 x}{2} - \ln |\sec x| + C \end{aligned}$$

# Integration by parts

## Techniques

*Suppose the integrand is of the form  $u(x)v'(x)$ . Then we may evaluate the integration using the formula*

$$\int uv' dx = uv - \int u'v dx.$$

*The above formula is called integration by parts. It is usually written in the form*

$$\int u dv = uv - \int v du.$$

## Example

Evaluate the following integrals.

$$\textcircled{1} \int x e^{3x} dx$$

$$\textcircled{2} \int x^2 \cos x dx$$

$$\textcircled{3} \int x^3 \ln x dx$$

$$\textcircled{4} \int \ln x dx$$

## Solution

$$\begin{aligned} 1. \int x e^{3x} dx &= \frac{1}{3} \int x d e^{3x} = \frac{x e^{3x}}{3} - \frac{1}{3} \int e^{3x} dx \\ &= \frac{x e^{3x}}{3} - \frac{e^{3x}}{9} + C \end{aligned}$$

$$\begin{aligned} 2. \int x^2 \cos x dx &= \int x^2 d \sin x \\ &= x^2 \sin x - \int \sin x dx^2 \\ &= x^2 \sin x - 2 \int x \sin x dx \\ &= x^2 \sin x + 2 \int x d \cos x \\ &= x^2 \sin x + 2x \cos x - 2 \int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$



## Solution

$$\begin{aligned} 3. \int x^3 \ln x dx &= \frac{1}{4} \int \ln x dx^4 \\ &= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^4 d \ln x \\ &= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^4 \left( \frac{1}{x} \right) dx \\ &= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 dx \\ &= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C \end{aligned}$$

$$\begin{aligned} 4. \int \ln x dx &= x \ln x - \int x d \ln x \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

## Example

Evaluate the following integrals.

$$\textcircled{1} \int \sin^{-1} x dx$$

$$\textcircled{2} \int \ln(1 + x^2) dx$$

$$\textcircled{3} \int \sec^3 x dx$$

$$\textcircled{4} \int e^x \sin x dx$$

## Solution

$$\begin{aligned} 1. \int \sin^{-1} x dx &= x \sin^{-1} x - \int x d \sin^{-1} x \\ &= x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C \end{aligned}$$
$$\begin{aligned} 2. \int \ln(1+x^2) dx &= x \ln(1+x^2) - \int x d \ln(1+x^2) \\ &= x \ln(1+x^2) - 2 \int \frac{x^2 dx}{1+x^2} \\ &= x \ln(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= x \ln(1+x^2) - 2x + 2 \tan^{-1} x + C \end{aligned}$$

## Solution

$$\begin{aligned} 3. \quad \int \sec^3 x dx &= \int \sec x d \tan x \\ &= \sec x \tan x - \int \tan x d \sec x \\ &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ 2 \int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \\ \int \sec^3 x dx &= \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C \end{aligned}$$

## Solution

$$\begin{aligned} 4. \quad \int e^x \sin x dx &= \int \sin x de^x \\ &= e^x \sin x - \int e^x d \sin x \\ &= e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - \int \cos x de^x \\ &= e^x \sin x - e^x \cos x + \int e^x d \cos x \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\ 2 \int e^x \sin x dx &= e^x \sin x - e^x \cos x + C' \\ \int e^x \sin x dx &= \frac{1}{2} (e^x \sin x - e^x \cos x) + C \end{aligned}$$

# Reduction formula

## Techniques

For integral of the forms

$$I_n = \int \cos^n x dx, \int \sin^n x dx, \int x^n \cos x dx, \int x^n \sin x dx, \\ \int \sec^n x dx, \int \csc^n x dx, \int x^n e^x dx, \int (\ln x)^n dx, \\ \int e^x \cos^n x dx, \int e^x \sin^n x dx, \int \frac{dx}{(x^2 + a^2)^n}, \int \frac{dx}{(a^2 - x^2)^n},$$

we may use integration by parts to find a formula to express  $I_n$  in terms of  $I_k$  with  $k < n$ . Such a formula is called reduction formula.

## Example

Let

$$I_n = \int x^n \cos x dx$$

for positive integer  $n$ . Prove that

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}, \text{ for } n \geq 2$$

Proof.

$$\begin{aligned} I_n = \int x^n \cos x dx &= \int x^n d \sin x \\ &= x^n \sin x - \int \sin x dx^n \\ &= x^n \sin x - n \int x^{n-1} \sin x dx \\ &= x^n \sin x + n \int x^{n-1} d \cos x \\ &= x^n \sin x + nx^{n-1} \cos x - n \int \cos x dx^{n-1} \\ &= x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx \\ &= x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2} \end{aligned}$$





## Example

Let

$$I_n = \int \frac{dx}{(x^2 + a^2)^n}$$

where  $a > 0$  is a positive real number for positive integer  $n$ . Prove that

$$I_n = \frac{x}{2a^2(n-1)(x^2 + a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \text{ for } n \geq 2$$

## Proof

$$\begin{aligned} I_n &= \int \frac{dx}{(x^2 + a^2)^n} = \frac{x}{(x^2 + a^2)^n} - \int x d\left(\frac{1}{(x^2 + a^2)^n}\right) \\ &= \frac{x}{(x^2 + a^2)^n} + \int \frac{2nx^2 dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{(x^2 + a^2 - a^2) dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{dx}{(x^2 + a^2)^n} - 2na^2 \int \frac{dx}{(x^2 + a^2)^{n+1}} \\ &= \frac{x}{(x^2 + a^2)^n} + 2nI_n - 2na^2 I_{n+1} \\ I_{n+1} &= \frac{x}{2na^2(x^2 + a^2)^n} + \frac{2n-1}{2na^2} I_n \end{aligned}$$

Replacing  $n$  by  $n-1$ , we have

$$I_n = \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.$$

## Alternative proof.

$$\begin{aligned}I_n &= \frac{1}{a^2} \int \frac{x^2 + a^2 - x^2}{(x^2 + a^2)^n} dx \\&= \frac{1}{a^2} \int \left( \frac{1}{(x^2 + a^2)^{n-1}} - \frac{x^2}{(x^2 + a^2)^n} \right) dx \\&= \frac{1}{a^2} I_{n-1} - \frac{1}{2a^2} \int \frac{x}{(x^2 + a^2)^n} d(x^2 + a^2) \\&= \frac{1}{a^2} I_{n-1} + \frac{1}{2(n-1)a^2} \int x d \left( \frac{1}{(x^2 + a^2)^{n-1}} \right) \\&= \frac{1}{a^2} I_{n-1} + \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} - \frac{1}{2(n-1)a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}} \\&= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \left( \frac{1}{a^2} - \frac{1}{2(n-1)a^2} \right) I_{n-1} \\&= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}\end{aligned}$$



## Example

Prove the following reduction formula

$$\int \sin^n x dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

for  $n \geq 2$ . Hence show that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{(n-1) \cdot (n-3) \cdots 6 \cdot 4 \cdot 2}{n \cdot (n-2) \cdots 7 \cdot 5 \cdot 3} & \text{when } n \text{ is odd} \\ \frac{(n-1) \cdot (n-3) \cdots 7 \cdot 5 \cdot 3}{n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

## Proof

$$\begin{aligned}\int \sin^n x dx &= - \int \sin^{n-1} x d \cos x \\ &= - \cos x \sin^{n-1} x + \int \cos x d \sin^{n-1} x \\ &= - \cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x dx \\ &= - \cos x \sin^{n-1} x + (n-1) \int (1 - \sin^2 x) \sin^{n-2} x dx \\ n \int \sin^n x dx &= - \cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x dx \\ \int \sin^n x dx &= -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x dx\end{aligned}$$

## Proof

Hence when  $n$  is odd

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n} \cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\ &\vdots \\ &= \frac{(n-1) \cdot (n-3) \cdots 6 \cdot 4 \cdot 2}{n \cdot (n-2) \cdots 7 \cdot 5 \cdot 3} \int_0^{\frac{\pi}{2}} \sin x dx \\ &= \frac{(n-1) \cdot (n-3) \cdots 6 \cdot 4 \cdot 2}{n \cdot (n-2) \cdots 7 \cdot 5 \cdot 3}\end{aligned}$$

## Proof.

when  $n$  is even

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n} \cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\ &\vdots \\ &= \frac{(n-1) \cdot (n-3) \cdots 7 \cdot 5 \cdot 3}{n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2} \int_0^{\frac{\pi}{2}} dx \\ &= \frac{(n-1) \cdot (n-3) \cdots 7 \cdot 5 \cdot 3}{n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}\end{aligned}$$



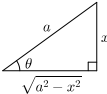
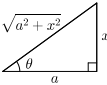
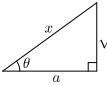
## Example

$$\begin{aligned} I_n &= \int x^n e^x dx; & I_n &= x^n e^x - nI_{n-1}, \quad n \geq 1 \\ I_n &= \int (\ln x)^n dx; & I_n &= x(\ln x)^n - nI_{n-1}, \quad n \geq 1 \\ I_n &= \int x^n \sin x dx; & I_n &= -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}, \quad n \geq 2 \\ I_n &= \int \cos^n x dx; & I_n &= \frac{\cos^{n-1} x \sin x}{n} + (n-1)I_{n-2}, \quad n \geq 2 \\ I_n &= \int \sec^n x dx; & I_n &= \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 2 \\ I_n &= \int e^x \cos^n x dx; & I_n &= \frac{e^x \cos^{n-1} x (\cos x + n \sin x)}{n^2 + 1} + \frac{n(n-1)}{n^2 + 1} I_{n-2}, \quad n \geq 2 \\ I_n &= \int e^x \sin^n x dx; & I_n &= \frac{e^x \sin^{n-1} x (\sin x - n \cos x)}{n^2 + 1} + \frac{n(n-1)}{n^2 + 1} I_{n-2}, \quad n \geq 2 \\ I_n &= \int x^n \sqrt{x+a} dx; & I_n &= \frac{2x^n (x+a)^{\frac{3}{2}}}{2n+3} - \frac{2na}{2n+3} I_{n-1}, \quad n \geq 1 \\ I_n &= \int \frac{x^n}{\sqrt{x+a}} dx; & I_n &= \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2na}{2n+1} I_{n-1}, \quad n \geq 1 \end{aligned}$$



# Trigonometric substitution

## Techniques (Trigonometric substitution)

Expression	Substitution	$dx$	Trigonometric ratios
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos \theta d\theta$	 $\cos \theta = \frac{\sqrt{a^2 - x^2}}{a}$ $\sin \theta = \frac{x}{a}$ $\tan \theta = \frac{x}{\sqrt{a^2 - x^2}}$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$dx = a \sec^2 \theta d\theta$	 $\cos \theta = \frac{a}{\sqrt{a^2 + x^2}}$ $\sin \theta = \frac{x}{\sqrt{a^2 + x^2}}$ $\tan \theta = \frac{x}{a}$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec \theta \tan \theta d\theta$	 $\cos \theta = \frac{a}{x}$ $\sin \theta = \frac{\sqrt{x^2 - a^2}}{x}$ $\tan \theta = \frac{\sqrt{x^2 - a^2}}{a}$

## Theorem

$$\textcircled{1} \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

$$\textcircled{2} \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

$$\textcircled{3} \int \frac{dx}{x\sqrt{x^2 - a^2}} = \cos^{-1} \frac{a}{x} + C$$

## Proof

1. Let  $x = a \sin \theta$ . Then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta \\ dx &= a \cos \theta d\theta\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{1}{a \cos \theta} (a \cos \theta d\theta) \\ &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1} \frac{x}{a} + C\end{aligned}$$

## Proof

2. Let  $x = a \tan \theta$ . Then

$$\begin{aligned}a^2 + x^2 &= a^2 + a^2 \tan^2 \theta = a^2 \sec^2 \theta \\ dx &= a \sec^2 \theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{a^2 + x^2} dx &= \int \frac{1}{a^2 \sec^2 \theta} (a \sec^2 \theta d\theta) \\ &= \frac{1}{a} \int d\theta \\ &= \frac{\theta}{a} + C \\ &= \frac{1}{a} \tan^{-1} \frac{x}{a} + C\end{aligned}$$

Proof.

3. Let  $x = a \sec \theta$ . Then

$$\begin{aligned}x\sqrt{x^2 - a^2} &= a \sec \theta \sqrt{a^2 \sec^2 \theta - a^2} = a^2 \sec \theta \tan \theta \\dx &= a \sec \theta \tan \theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2 - a^2}} dx &= \int \frac{1}{a^2 \sec \theta \tan \theta} (a \sec \theta \tan \theta d\theta) \\&= \frac{1}{a} \int d\theta \\&= \frac{\theta}{a} + C \\&= \frac{1}{a} \cos^{-1} \frac{a}{x} + C\end{aligned}$$

Note that  $\theta = \cos^{-1} \frac{a}{x}$  since  $\cos \theta = \frac{1}{\sec \theta} = \frac{a}{x}$ .



## Example

Use trigonometric substitution to evaluate the following integrals.

$$1 \quad \int \sqrt{1-x^2} \, dx$$

$$2 \quad \int \frac{1}{\sqrt{1+x^2}} \, dx$$

$$3 \quad \int \frac{x^3}{\sqrt{4-x^2}} \, dx$$

$$4 \quad \int \frac{1}{(9+x^2)^2} \, dx$$

## Solution

1. Let  $x = \sin \theta$ . Then

$$\begin{aligned}\sqrt{1-x^2} &= \sqrt{1-\sin^2 \theta} = \cos \theta \\ dx &= \cos \theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \sqrt{1-x^2} dx &= \int \cos^2 \theta d\theta \\ &= \int \frac{\cos 2\theta + 1}{2} d\theta \\ &= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C \\ &= \frac{\sin \theta \cos \theta}{2} + \frac{\sin^{-1} x}{2} + C \\ &= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2} + C\end{aligned}$$

## Solution

2. Let  $x = \tan \theta$ . Then

$$\begin{aligned}1 + x^2 &= 1 + \tan^2 \theta = \sec^2 \theta \\ dx &= \sec^2 \theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}} dx &= \int \frac{1}{\sec \theta} (\sec^2 \theta d\theta) \\ &= \int \sec \theta d\theta \\ &= \ln |\tan \theta + \sec \theta| + C \\ &= \ln(x + \sqrt{1+x^2}) + C\end{aligned}$$



## Solution

3. Let  $x = 2 \sin \theta$ . Then

$$\begin{aligned}\sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} = 2\cos\theta \\ dx &= 2\cos\theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^3}{\sqrt{4-x^2}} dx &= \int \frac{8\sin^3\theta}{2\cos\theta} (2\cos\theta d\theta) \\ &= 8 \int \sin^3\theta d\theta \\ &= -8 \int (1-\cos^2\theta) d\cos\theta \\ &= 8 \left( \frac{\cos^3\theta}{3} - \cos\theta \right) + C \\ &= \frac{(4-x^2)^{\frac{3}{2}}}{3} - 4(4-x^2)^{\frac{1}{2}} + C\end{aligned}$$

## Solution

4. Let  $x = 3 \tan \theta$ . Then

$$\begin{aligned}9 + x^2 &= 9 + 9 \tan^2 \theta = 9 \sec^2 \theta \\ dx &= 3 \sec^2 \theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{(9 + x^2)^2} dx &= \int \frac{1}{81 \sec^4 \theta} (3 \sec^2 \theta d\theta) = \frac{1}{27} \int \cos^2 \theta d\theta \\ &= \frac{1}{54} \int (\cos 2\theta + 1) d\theta = \frac{1}{54} \left( \frac{\sin 2\theta}{2} + \theta \right) + C \\ &= \frac{1}{54} (\cos \theta \sin \theta + \theta) + C \\ &= \frac{1}{54} \left( \frac{3}{\sqrt{9 + x^2}} \cdot \frac{x}{\sqrt{9 + x^2}} + \tan^{-1} \frac{x}{3} \right) + C \\ &= \frac{x}{18(9 + x^2)} + \frac{1}{54} \tan^{-1} \frac{x}{3} + C\end{aligned}$$

# Integration of rational functions

## Definition (Rational functions)

A rational function is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where  $f(x), g(x)$  are polynomials with real coefficients with  $g(x) \neq 0$ .

## Techniques

We can integrate a rational function  $R(x)$  with the following two steps.

- 1 Find the partial fraction decomposition of  $R(x)$ , that is, express

$$R(x) = q(x) + \sum \frac{A}{(x - \alpha)^k} + \sum \frac{B(x + a)}{((x + a)^2 + b^2)^k} + \sum \frac{C}{((x + a)^2 + b^2)^k}$$

where  $q(x)$  is a polynomial,  $A, B, C, \alpha, a, b$  represent real numbers and  $k$  represents positive integer.

- 2 Integrate the partial fraction.

## Theorem

Let  $R(x) = \frac{f(x)}{g(x)}$  be a rational function. We may assume that the leading coefficient of  $g(x)$  is 1.

- ① (Division algorithm for polynomials) There exists polynomials  $q(x), r(x)$  with  $\deg(r(x)) < \deg(g(x))$  or  $r(x) = 0$  such that

$$R(x) = q(x) + \frac{r(x)}{g(x)}.$$

$q(x)$  and  $r(x)$  are the quotient and remainder of the division  $f(x)$  by  $g(x)$ .

- ② (Fundamental theorem of algebra for real polynomials)  $g(x)$  can be written as a product of linear or quadratic polynomials. More precisely, there exists real numbers  $\alpha_1, \dots, \alpha_m, a_1, \dots, a_n, b_1, \dots, b_n$  and positive integers  $k_1, \dots, k_m, l_1, \dots, l_n$  such that

$$g(x) = (x - \alpha_1)^{k_1} \cdots (x - \alpha_k)^{k_m} ((x + a_1)^2 + b_1^2)^{l_1} \cdots ((x + a_n)^2 + b_n^2)^{l_n}.$$

## Techniques

Partial fractions can be integrated using the formulas below.

$$\bullet \int \frac{dx}{(x - \alpha)^k} = \begin{cases} \ln|x - \alpha| + C, & \text{if } k = 1 \\ -\frac{1}{(k-1)(x - \alpha)^{k-1}} + C, & \text{if } k > 1 \end{cases}$$

$$\bullet \int \frac{x dx}{(x^2 + a^2)^k} = \begin{cases} \frac{1}{2} \ln(x^2 + a^2) + C, & \text{if } k = 1 \\ -\frac{1}{2(k-1)(x^2 + a^2)^{k-1}} + C, & \text{if } k > 1 \end{cases}$$

$$\bullet \int \frac{dx}{(x^2 + a^2)^k} = \begin{cases} \frac{1}{a} \tan^{-1} \frac{x}{a} + C, & \text{if } k = 1 \\ \frac{x}{2a^2(k-1)(x^2 + a^2)^{k-1}} + \frac{2k-3}{2a^2(k-1)} \int \frac{dx}{(x^2 + a^2)^{k-1}}, & \text{if } k > 1 \end{cases}$$

## Theorem

Suppose  $\frac{f(x)}{g(x)}$  is a rational function such that the degree of  $f(x)$  is smaller than the degree of  $g(x)$  and  $g(x)$  has only simple real roots, i.e.,

$$g(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$$

for distinct real numbers  $\alpha_1, \alpha_2, \dots, \alpha_k$  and  $a \neq 0$ . Then

$$\frac{f(x)}{g(x)} = \frac{f(\alpha_1)}{g'(\alpha_1)(x - \alpha_1)} + \frac{f(\alpha_2)}{g'(\alpha_2)(x - \alpha_2)} + \cdots + \frac{f(\alpha_k)}{g'(\alpha_k)(x - \alpha_k)}$$

## Proof

First, observe that

$$g'(x) = \sum_{j=1}^k a(x - \alpha_1)(x - \alpha_2) \cdots (\widehat{x - \alpha_j}) \cdots (x - \alpha_k)$$

where  $(\widehat{x - \alpha_i})$  means the factor  $x - \alpha_i$  is omitted. Thus we have

$$\begin{aligned} g'(\alpha_i) &= \sum_{j=1}^k a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots (\widehat{\alpha_i - \alpha_j}) \cdots (\alpha_i - \alpha_k) \\ &= a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots (\widehat{\alpha_i - \alpha_i}) \cdots (\alpha_i - \alpha_k) \end{aligned}$$

Since  $g(x)$  has distinct real zeros, the partial fraction decomposition takes the form

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_k}{x - \alpha_k}.$$

## Proof.

Multiplying both sides by  $g(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_k)$ , we get

$$f(x) = \sum_{i=1}^k A_i a(x - \alpha_1)(x - \alpha_2) \cdots \widehat{(x - \alpha_i)} \cdots (x - \alpha_k)$$

For  $i = 1, 2, \dots, k$ , substituting  $x = \alpha_i$ , we obtain

$$\begin{aligned} f(\alpha_i) &= \sum_{j=1}^k A_j a(\alpha_j - \alpha_1)(\alpha_j - \alpha_2) \cdots \widehat{(\alpha_j - \alpha_i)} \cdots (\alpha_j - \alpha_k) \\ &= A_i a(\alpha_i - \alpha_1)(\alpha_i - \alpha_2) \cdots \widehat{(\alpha_i - \alpha_i)} \cdots (\alpha_i - \alpha_k) \\ &= A_i g'(\alpha_i) \end{aligned}$$

and the result follows. □



## Example

Evaluate the following integrals.

$$\textcircled{1} \int \frac{x^5 + 2x - 1}{x^3 - x} dx$$

$$\textcircled{2} \int \frac{9x - 2}{2x^3 + 3x^2 - 2x} dx$$

$$\textcircled{3} \int \frac{x^2 - 2}{x(x - 1)^2} dx$$

$$\textcircled{4} \int \frac{x^2}{x^4 - 1} dx$$

$$\textcircled{5} \int \frac{8x^2}{x^4 + 4} dx$$

$$\textcircled{6} \int \frac{2x + 1}{x^4 + 2x^2 + 1} dx$$

## Solution

1. By division and factorization  $x^3 - x = x(x - 1)(x + 1)$ , we obtain the partial fraction decomposition

$$\frac{x^5 + 4x - 3}{x^3 - x} = x^2 + 1 + \frac{5x - 3}{x^3 - x} = x^2 + 1 + \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}.$$

Multiply both sides by  $x(x - 1)(x + 1)$  and obtain

$$\begin{aligned} 5x - 3 &= A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1) \\ \Rightarrow A &= 3, B = 1, C = -4. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^5 + 4x - 3}{x^3 - x} dx &= \int \left( x^2 + 1 + \frac{3}{x} + \frac{1}{x - 1} - \frac{4}{x + 1} \right) dx \\ &= \frac{x^3}{3} + x + 3 \ln |x| + \ln |x - 1| - 4 \ln |x + 1| + C. \end{aligned}$$

## Solution

2. By factorization  $2x^3 + 3x^2 - 2x = x(x + 2)(2x - 1)$ , we obtain the partial fraction decomposition

$$\frac{9x - 2}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{x + 2} + \frac{C}{2x - 1}.$$

Multiply both sides by  $x(x + 2)(2x - 1)$  and obtain

$$\begin{aligned} 9x - 2 &= A(x + 2)(2x - 1) + Bx(2x - 1) + Cx(x + 2) \\ \Rightarrow A &= 1, B = -2, C = 2. \end{aligned}$$

Therefore

$$\begin{aligned} &\int \frac{9x - 2}{2x^3 + 3x^2 - 2x} dx \\ &= \int \left( \frac{1}{x} - \frac{2}{x + 2} + \frac{2}{2x - 1} \right) dx \\ &= \ln|x| - 2 \ln|x + 2| + \ln|2x - 1| + C. \end{aligned}$$

## Solution

3. The partial fraction decomposition is

$$\frac{x^2 - 2}{x(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x}.$$

Multiply both sides by  $x(x-1)^2$  and obtain

$$\begin{aligned}x^2 - 2 &= Ax + Bx(x-1) + C(x-1)^2 \\ \Rightarrow A &= -1, B = 3, C = -2.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^2 - 2}{x(x-1)^2} dx &= \int \left( -\frac{1}{(x-1)^2} + \frac{3}{x-1} - \frac{2}{x} \right) dx \\ &= \frac{1}{x-1} + 3 \ln|x-1| - 2 \ln|x| + C.\end{aligned}$$

## Solution

4. The partial fraction decomposition is

$$\begin{aligned}\frac{x^2}{x^4 - 1} &= \frac{x^2}{(x^2 - 1)(x^2 + 1)} \\ &= \frac{1}{2} \left( \frac{1}{x^2 - 1} + \frac{1}{x^2 + 1} \right) \\ &= \frac{1}{2(x - 1)(x + 1)} + \frac{1}{2(x^2 + 1)} \\ &= \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x^2 + 1)}\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^2 dx}{x^4 - 1} &= \int \left( \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x^2 + 1)} \right) dx \\ &= \frac{1}{4} \ln |x - 1| - \frac{1}{4} \ln |x + 1| + \frac{1}{2} \tan^{-1} x + C\end{aligned}$$

## Solution

5. By factorization  $x^4 + 4 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$ ,

$$\begin{aligned} & \int \frac{8x^2}{x^4 + 4} dx \\ &= \int \frac{8x^2 dx}{(x^2 - 2x + 2)(x^2 + 2x + 2)} dx \\ &= \int 2x \left( \frac{4x}{(x^2 - 2x + 2)(x^2 + 2x + 2)} \right) dx \\ &= \int 2x \left( \frac{1}{x^2 - 2x + 2} - \frac{1}{x^2 + 2x + 2} \right) dx \\ &= \int \left( \frac{2x}{(x-1)^2 + 1} - \frac{2x}{(x+1)^2 + 1} \right) dx \\ &= \int \left( \frac{2(x-1)}{(x-1)^2 + 1} + \frac{2}{(x-1)^2 + 1} - \frac{2(x+1)}{(x+1)^2 + 1} + \frac{2}{(x+1)^2 + 1} \right) dx \\ &= \ln(x^2 - 2x + 2) + 2 \tan^{-1}(x-1) - \ln(x^2 + 2x + 2) + 2 \tan^{-1}(x+1) + C \end{aligned}$$

## Solution

$$\begin{aligned} 6. & \int \frac{2x+1}{x^4+2x^2+1} dx \\ &= \int \frac{2xdx}{(x^2+1)^2} + \int \frac{dx}{(x^2+1)^2} \\ &= \int \frac{d(x^2+1)}{(x^2+1)^2} + \int \frac{x^2+1}{(x^2+1)^2} dx - \int \frac{x^2 dx}{(x^2+1)^2} \\ &= -\frac{1}{x^2+1} + \int \frac{dx}{x^2+1} - \frac{1}{2} \int \frac{xd(x^2+1)}{(x^2+1)^2} \\ &= -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2} \int xd\left(\frac{1}{x^2+1}\right) \\ &= -\frac{1}{x^2+1} + \tan^{-1} x + \frac{1}{2} \left(\frac{x}{x^2+1}\right) - \frac{1}{2} \int \frac{dx}{x^2+1} \\ &= \frac{x-2}{2(x^2+1)} + \frac{1}{2} \tan^{-1} x + C \end{aligned}$$

## Example

Find the partial fraction decomposition of the following functions.

1  $\frac{5x - 3}{x^3 - x}$

2  $\frac{9x - 2}{2x^3 + 3x^2 - 2x}$



## Solution

① For  $g(x) = x^3 - x = x(x-1)(x+1)$ ,  $g'(x) = 3x^2 - 1$ . Therefore

$$\begin{aligned}\frac{5x-3}{x^3-x} &= \frac{-3}{g'(0)x} + \frac{5(1)-3}{g'(1)(x-1)} + \frac{5(-1)-3}{g'(-1)(x+1)} \\ &= \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1}\end{aligned}$$

② For  $g(x) = 2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$ ,  $g'(x) = 6x^2 + 6x - 2$ .  
Therefore

$$\begin{aligned}&\frac{9x-2}{2x^3+3x^2-2x} \\ &= \frac{-2}{g'(0)x} + \frac{9(-2)-2}{g'(-2)(x+2)} + \frac{9(\frac{1}{2})-2}{g'(\frac{1}{2})(2x-1)} \\ &= \frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1}\end{aligned}$$

□

# *t*-substitution

## Techniques

To evaluate

$$\int R(\cos x, \sin x, \tan x) dx$$

where  $R$  is a rational function, we may use  $t$ -substitution

$$t = \tan \frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1-t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad \sin x = \frac{2t}{1+t^2};$$

$$dx = d(2 \tan^{-1} t) = \frac{2dt}{1+t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x) dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, \frac{2t}{1-t^2}\right) \frac{2dt}{1+t^2}$$

which is an integral of rational function.

## Example

Use  $t$ -substitution to evaluate the following integrals.

$$① \int \frac{dx}{1 + \cos x}$$

$$② \int \frac{\sin x dx}{\cos x + \sin x}$$

$$③ \int \frac{dx}{1 + \cos x + \sin x}$$

## Solution

1. Let  $t = \tan \frac{x}{2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$ ,  $dx = \frac{2dt}{1+t^2}$ . We have

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \left( \frac{1}{1+\frac{1-t^2}{1+t^2}} \right) \frac{2dt}{1+t^2} = \int dt = t + C = \tan \frac{x}{2} + C \\ &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} + C = \frac{2 \cos \frac{x}{2} \sin \frac{x}{2}}{2 \cos^2 \frac{x}{2}} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

*Alternatively*

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \frac{dx}{2 \cos^2 \frac{x}{2}} = \frac{1}{2} \int \sec^2 \frac{x}{2} dx \\ &= \tan \frac{x}{2} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

## Solution

2. Let  $t = \tan \frac{x}{2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$ ,  $\sin x = \frac{2t}{1+t^2}$ ,  $dx = \frac{2dt}{1+t^2}$ . We have

$$\begin{aligned}\int \frac{\sin x dx}{\cos x + \sin x} &= \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \frac{2dt}{1+t^2} \\ &= \int \left( \frac{1}{1+t^2} + \frac{t}{1+t^2} + \frac{t-1}{1+2t-t^2} \right) dt \\ &= \tan^{-1} t + \frac{1}{2} \ln |1+t^2| - \frac{1}{2} \ln |1+2t-t^2| + C \\ &= \tan^{-1} t - \frac{1}{2} \ln \left| \frac{1+2t-t^2}{1+t^2} \right| + C \\ &= \tan^{-1} t - \frac{1}{2} \ln \left| \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} \right| + C \\ &= \frac{x}{2} - \frac{1}{2} \ln |\cos x + \sin x| + C\end{aligned}$$

## Solution

*Alternatively*

$$\begin{aligned}\int \frac{\sin x dx}{\cos x + \sin x} &= \frac{1}{2} \int \left( 1 - \frac{\cos x - \sin x}{\cos x + \sin x} \right) dx \\ &= \frac{x}{2} - \frac{1}{2} \int \frac{d(\sin x + \cos x)}{\cos x + \sin x} \\ &= \frac{x}{2} - \frac{1}{2} \ln |\cos x + \sin x| + C\end{aligned}$$

## Solution

3. Let  $t = \tan \frac{x}{2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$ ,  $\sin x = \frac{2t}{1+t^2}$ ,  $dx = \frac{2dt}{1+t^2}$ . We have

$$\begin{aligned} \int \frac{dx}{1 + \cos x + \sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1 + \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \\ &= \int \frac{dt}{1+t} \\ &= \ln |1+t| + C \\ &= \ln \left| 1 + \tan \frac{x}{2} \right| + C \\ &= \ln \left| 1 + \frac{\sin x}{1 + \cos x} \right| + C \\ &= \ln \left| \frac{1 + \cos x + \sin x}{1 + \cos x} \right| + C \end{aligned}$$