MAT4250 Game Theory

A "game" is being played whenever people interact with each other. Games like chess, poker, majong are called parlor games. Real life games include auctions, choosing a platform in election, war games, economics (part of microeconomic). This type of games typically offer opportunities for conflicts, cooperations and bargaining, and to take into account of rational and irrational behavior.

Game theory was first considered by economists for strategics and decision making, and by mathematicians for poker, chess type of games. The classic book by Von Neumann and Morgenstern (1944), *The theory of games and economic behavior*, established the foundation of the theory. They gave detail study of the zero-sum games, and started the non-zero-sum game and formulated the concept of cooperative and non-cooperative strategies. The theory was completed by the mathematician John Nash who proved the existence of the now call Nash-equilibrium for the non-zero sum game (1951). He received the Nobel prize in economics in 1994 for this important contribution. Nash's discovery and his unusual life were detailed in the best seller *A beautiful mind*.

Text:

Introduction to Game Theory, P. Morris, Springer, 1994.

References:

Game Theory, G, Owen, Saunders Co., 1969.

Games and Strategies, P. Straffin, Math. Assoc. AMS, 1993.

A Beautiful Mind, S. Nasar, Touchstone, 1998.

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Chapter 1

The setup of the games

§1.1 Game trees

Example 1.1 (Matching coin) A and B each conceal a \$1 or \$2 coin in their hand, and open at the same time. If the coins are the same, then A wins; otherwise B wins.

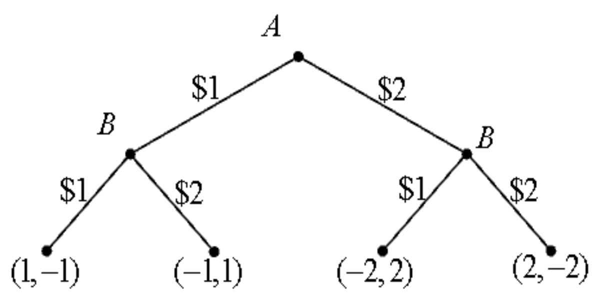


Figure 1.1: A game tree of "matching coin".

Example 1.2 (Two finger morra) A and B each hold up one or two fingers and call for "1" or "2" simultaneously of the opponents finger. If one player is right and the other is wrong, the one is right wins an amount equal the sum of the fingers; otherwise, it is a draw and no one wins.

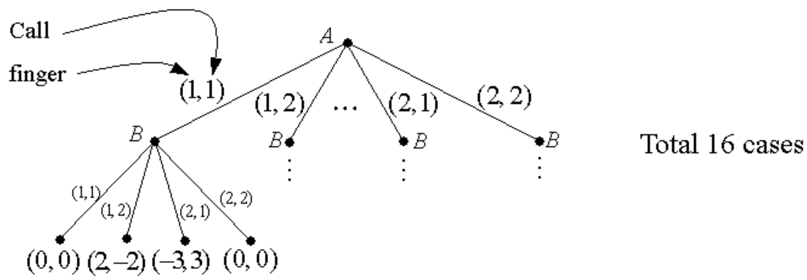


Figure 1.2: A game tree of "two finger morra".

Example 1.3 (Tic-Tac-Tot)

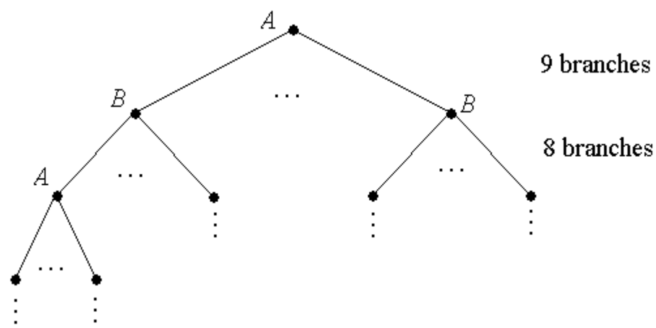


Figure 1.3: A game tree of "Tic-Tac-Tot".

For a game tree for N players $P_1, \cdots, P_N$ (as in Figure 1.4), we call the starting node a *root* and a node u a *vertex*. If u is labelled P_i, we call u *belongs* to P_i. A line segment (u, v) joining two vertices is called an *edge*, v is called a *child* of u. A *path* from u_1 to u_n, denoted by $(u_1, \cdots, u_n)$, is a string of edges joining u_1 and u_n; u_n is called a *descendant* of u_1. At the terminal vertex w, we use $\mathbf{p}(w) = (x_1, \cdots, x_N)$ to denote the *payoff*; $x_i = p_i(w)$ is the payoff for P_i.

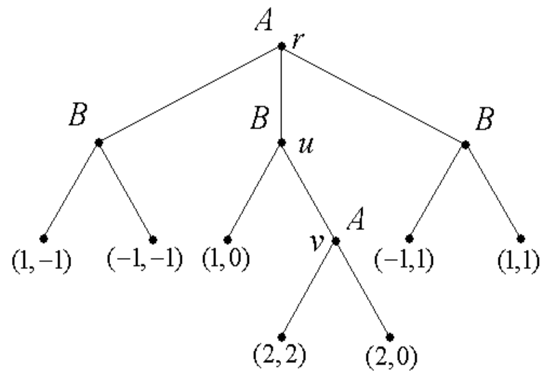


Figure 1.4: An example of game tree.

The above tree expression does not completely specified the game yet. If we compare Examples 1.1, 1.2 and Example 1.3, we see that in the first two examples, the players do not know what the opponent is at (*imperfect information*), whereas in the third example he knows exactly what has happened after each play (*perfect information*). To resolve this we need the following definition.

Definition 1.1 Let V_i be the set of vertices belong to P_i. Let $\{V_i^j\}_j$ be a partition of V_i. If each V_i^j satisfies

(i) any two vertices has the same number of children and

(ii) for $u, v \in V_i^j$ neither one is the descendant of the other.

We call each V_i^j an *information set*. The game is said to have *perfect information* if each V_i^j is a singleton.

In the definition, (i) means that player P_i knows which information set he is in, but he does not know which vertex he is at (unless the set contains only one point); (ii) implies that a path will pass each information set only once. In practice, the vertices in an information set is in the same level of the player. The following figure indicates two possible information sets.

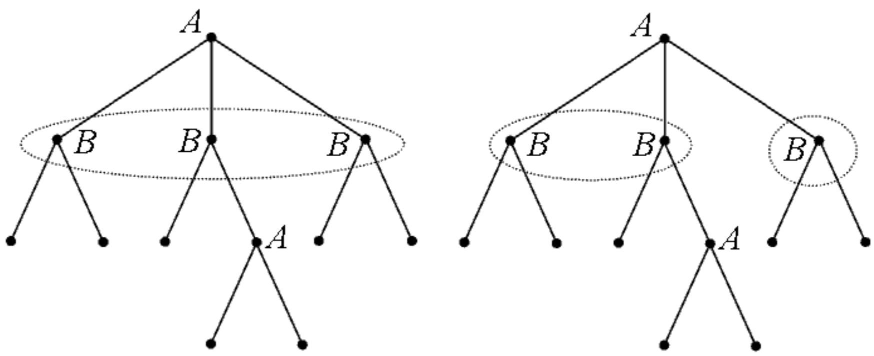


Figure 1.5: Two cases of information sets

In Example 1.1, the information set for B consists of the two vertices belong to B. In Example 1.3, the information sets for B are each individual vertex.

Let us consider Example 1.1 more carefully:

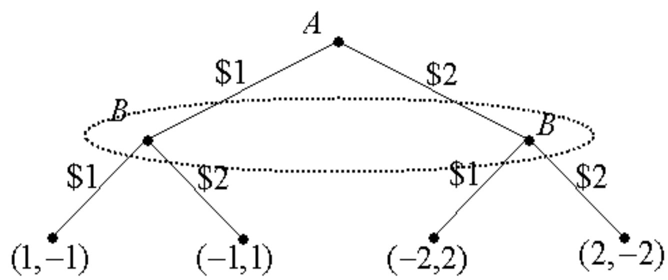


Figure 1.6: Information set of Example 1.1

$V_A = \{\text{r}\}$: there are two choices: 1 or 2.
$V_B = \{u_1, u_2\}$: u_1, u_2 belong to the same information set (as B does not know what A has chosen). We regard u_1, u_2 the same, hence there are only two choices, namely 1 or 2.

We can write down the payoff as

A \ B	1	2
1	(1,−1)	(−1,1)
2	(−2,2)	(2,−2)

or

$$\begin{bmatrix} (1,-1) & (-1,1) \\ (-2,2) & (2,-2) \end{bmatrix}.$$

Note that in the bi-matrix, the second coordinates are the negative of the first coordinates (zero sum game), we can simply express it as

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}.$$

Example 1.4 Consider the following game tree,

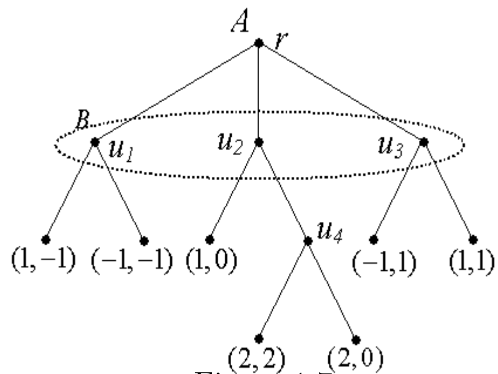


Figure 1.7

$V_A = \{r, u_4\}$: The information sets are $\{r\}$ and $\{u_4\}$. r has three choices L, M, R; u_4 has two choices L, R. Hence there are 6 choices for A: LL, LR, ML, MR, RL, RR.

$V_B = \{u_1, u_2, u_3\}$ The three vertices belong to the same information set, there are two choices for the three u_i: l or r.

Hence we can write the payoffs in a matrix form

$$\begin{array}{c} \\ LL \\ LR \\ ML \\ MR \\ RL \\ RR \end{array} \begin{array}{c} \begin{array}{cc} \quad l \quad & \quad r \quad \end{array} \\ \begin{bmatrix} (1,-1) & (-1,-1) \\ (1,-1) & (-1,-1) \\ (1,0) & (2,2) \\ (1,0) & (2,0) \\ (-1,1) & (1,1) \\ (-1,1) & (1,1) \end{bmatrix} \end{array}$$

Note that there are choices that will give the same paths and hence the same payoffs (e.g. check the choices (LL, l) and (LR, l), etc.). We can simplify the matrix as

$$\begin{array}{r} \\ LL \simeq LR \\ ML \\ MR \\ RL \simeq RR \end{array} \begin{array}{c} \begin{array}{cc} \quad l \quad & \quad r \quad \end{array} \\ \begin{bmatrix} (1,-1) & (-1,-1) \\ (1,0) & (2,2) \\ (1,0) & (2,0) \\ (-1,1) & (1,1) \end{bmatrix} \end{array}.$$

Definition 1.2 Let $\{V_i^j\}_j$ be the information set of P_i. By a *strategy* σ_i for player P_i. We mean a function which assigns to each information set V_i^j, one of the child of a representative u in V_i^j. Let Σ_i denote the set of strategies of P_i. A *game in extensive form* is a game tree T together with the strategic sets $\Sigma_1, \cdots, \Sigma_N$.

The strategies $\sigma_1, \cdots, \sigma_N$ for $P_1, \cdots, P_N$ determine one terminal vertex w, we use $\pi(\sigma_1, \cdots, \sigma_N)$ to denote the payoff $\mathbf{p}(w)$. The i-th component is $\pi_i(\sigma_1, \cdots, \sigma_N)$. This sets up an N-dimensional array of the game. It is called the *normal form.*

In the case $N = 2$ (2-person game), the normal form can be represented as a bi-matrix (or a matrix) as in the previous examples. As in Example 1.4, $\Sigma_A = \{LL, LR, \cdots, RR\}$, $\Sigma_B = \{l, r\}$; for $\sigma_A = LL$, $\sigma_B = l$, then $\pi_A(LL, l) = 1$, $\pi_B(LL, l) = -1$.

In many cases, a game has "chances" moves. For example, in Example 1.1, we change that B flip a coin to determine to use \$1 or \$2, then the game tree can be expressed as

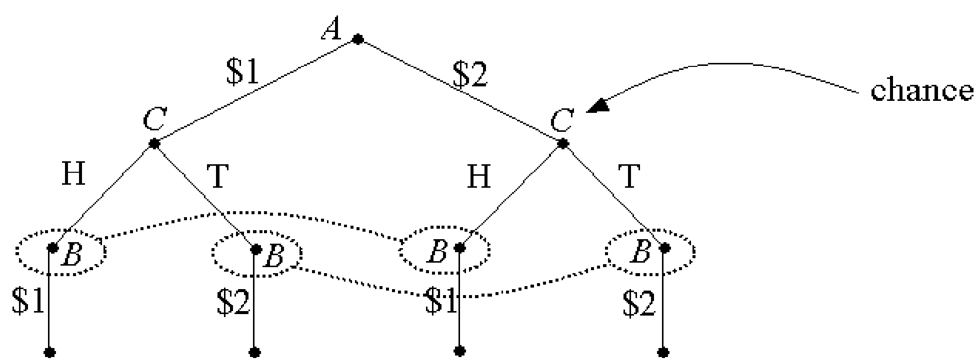


Figure 1.8

In this case there are two information sets for B: the 1-st and 3-rd vertices, and 2-nd and 4-th vertices.

Example 1.5 (A simplified Poker game) Both P_1, P_2 put \$1 into the pot, then each is dealt with a card A or K. P_1 can choose to bet (add \$2) or drop (give up and P_2 wins); P_2 can call (to match) or fold (give up, then P_1 wins). Then they will compare the cards for win, lose or draw.

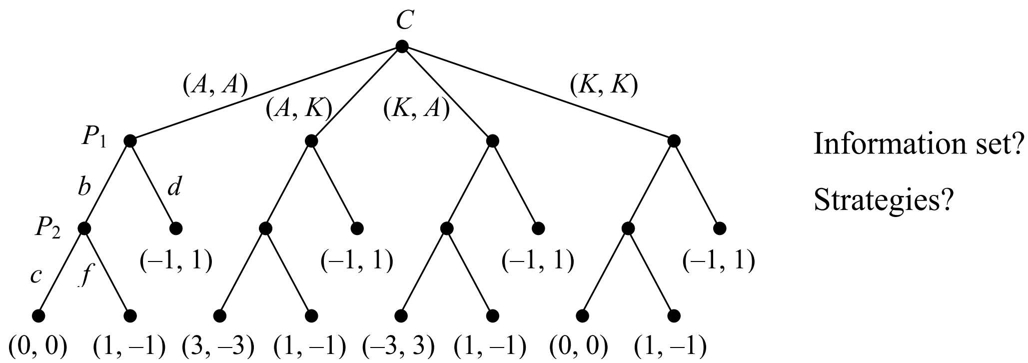


Figure 1.9

If u is a vertex belongs to chance, let $E(u)$ denote all the edges from u. For $v \in E(u)$, the probability for $v \in E(u)$ (u fixed) satisfies

$$\Pr(u,v) \geq 0 \qquad \text{and} \qquad \sum_{v \in E(u)} \Pr(u,v) = 1.$$

Let $\sigma_1, \cdots, \sigma_N$ be the strategies of $P_1, \cdots, P_N$. It determines certain paths R from r to the terminal vertices w:

$$\Pr(\sigma_1, \cdots, \sigma_N; w) = \prod \{\Pr(u,v) : u \text{ belongs to chance}, (u,v) \in R\}.$$

The expected payoff of $\sigma_1, \cdots, \sigma_N$ for P_i is

$$\pi_i(\sigma_1, \cdots, \sigma_N) = \sum_w p_i(w) \Pr(\sigma_1, \cdots \sigma_N; w)$$

where $\mathbf{p}(w) = \big(p_1(w), \cdots, p_N(w)\big)$ is the payoff for each w.

Example 1.6 .

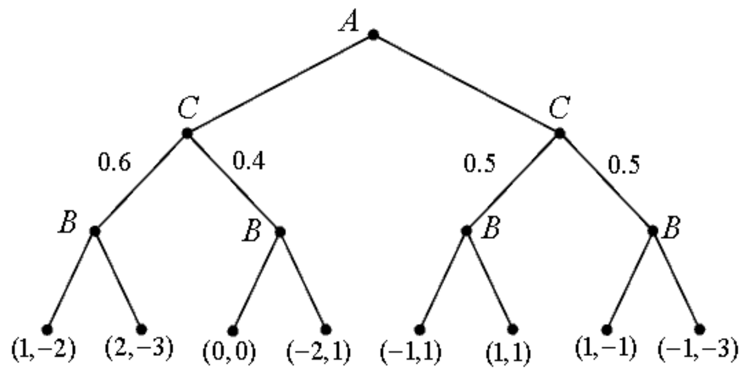


Figure 1.10

For the game tree in Figure 1.10, if we consider the information set for B to be singletons (perfect information), then there are 2 choices for A (L or R) and there are two choices for each vertex for B. We can trace the paths by the strategies to get, for example,

$$\begin{aligned} \pi_{\mathrm{A}}(L, LRLR) &= 0.6 \times 1 + 0.4 \times (-2) = -0.2 \\ \pi_A(R, RLRL) &= 0.5 \times 1 + 0.5 \times 1 = 1. \end{aligned}$$

We can also consider B to have two information sets; the first two vertices and the remaining two vertices. The reader can work out the strategies, the expected payoff and the corresponding matrix.

To conclude this introductory section, we give a proposition which is conceptually simple. Let T be a game tree of N-players. For a vertex u we let T_u consist of u and the descendants of u, then T_u is again a game tree called a *subtree* with root u. For a strategy σ_i, let σ_i/T_u be the restriction of σ_i on T_u. (The reader should try to formulate the meaning of this.)

Proposition 1.1 *Let T be a game tree with N players $P_1, \cdots, P_N$. Let $\sigma_1, \cdots, \sigma_N$ be strategies for $P_1, \cdots, P_N$.*

(i) *If the root r belongs to P_i and (r, u) is in the path determined by*

$\sigma_1, \cdots, \sigma_N$. Then for any $1 \le j \le N$,

$$\pi_j(\sigma_1, \cdots, \sigma_N) = \pi_j(\sigma_1/T_u, \cdots, \sigma_N/T_u).$$

(ii) *If r belongs to chance, then for any $1 \leq j \leq N$,*

$$\pi_j(\sigma_1, \cdots, \sigma_N) = \sum_{u \in E(r)} \pi_j(\sigma_1/T_u, \cdots, \sigma_N/T_u) \Pr(r, u).$$

§1.2 Equilibrium of strategies

Definition 2.1 A choice of strategies $(\sigma_1^*, \cdots, \sigma_N^*)$ for $P_1, \cdots, P_N$ is called an *equilibrium* N-tuple if for any $\sigma_i \in \Sigma_i$,

$$\pi_i(\sigma_1^*, \cdots, \sigma_i, \cdots, \sigma_N^*) \leq \pi_i(\sigma_1^*, \cdots, \sigma_i^*, \cdots, \sigma_N^*).$$

It means that any single player who is departing from the strategies $\sigma_1^*, \cdots, \sigma_N^*$ will hurt himself. Hence they have good reason to stay with the strategies in the equilibrium N-tuple.

Example 2.1 (Prisoners' dilemma) Two criminals A, B commit a crime and are arrested. The penalty is to be prisoned as indicated in the following table:

A \ B	*confess*	*deny*
confess	$(-5, -5)$	$(-1, -10)$
deny	$(-10, -1)$	$(-2, -2)$

They do not know what the other will do, hence both A and B have strategies c, d. By direct observation the equilibrium strategy is (c, c) with payoff $(-5, -5)$, i.e., both confess and get five years in prison. Of course if they can cooperate, it is better to deny. We will discuss later on the cooperative/non-cooperative, non-zero sum games.

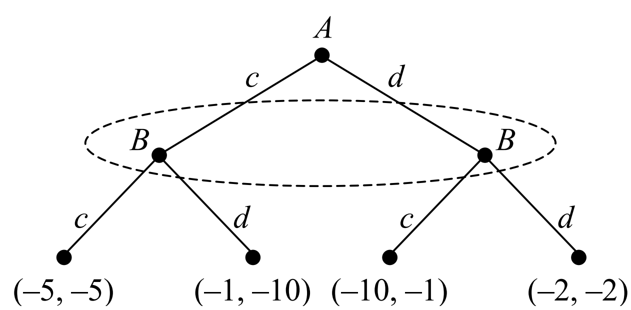


Figure 2.1: game tree of the "Prisoners' dilemma"

A game may have more than one equilibrium strategies. For example in Figure 2.2. A has strategies L and R; B has strategies l and r. Then the two equilibrium strategies are (L, l) and (R, r) which correspond to payoff $(-9, 5)$ and $(10, 4)$ respectively.

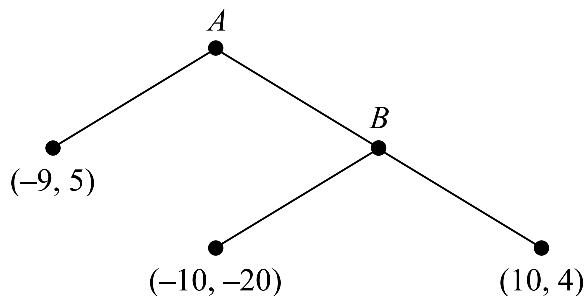


Figure 2.2

We have another simple example from the bi-matrix representation where (L, l) and (R, r) are equilibrium strategies

$A \backslash B$	l	r
L	$(2, 1)$	$(0, 0)$
R	$(0, 0)$	$(1, 2)$

Figure 2.3

In general the existence of an equilibrium N-tuple is not guaranteed. For example, consider the game with bimatrix representation in Figure 2.4, it is direct to check that it has no equilibrium pair.

$A \backslash B$	l	r
L	$(1, 0)$	$(0, 1)$
R	$(0, 1)$	$(1, 0)$

Figure 2.4

One of the main contribution of Nash is that he proved even though there is no "pure" equilibrium strategy, any non-cooperative game has a "mixed" equilibrium strategy (Chapter 4). Nevertheless, for the special case of *perfect information*, we can prove the existence of such an equilibrium strategies.

Theorem 2.2 *For a N-person game with perfect information, there exists an equilibrium N-tuple of strategies.*

We will first consider a decomposition of a game tree T of perfect information (information sets are singletons). Let T_u be the subtree with vertex u and let T/u be the *quotient tree* consisted of the remaining vertices, plus u as terminal vertices.

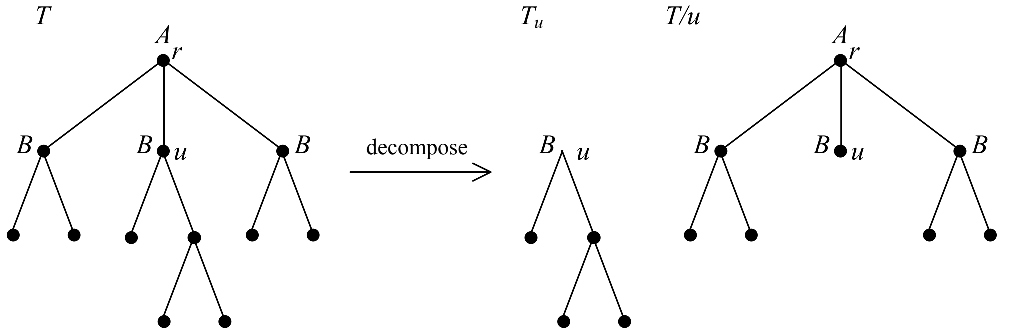


Figure 2.5

Let $\sigma = (\sigma_1, \cdots, \sigma_N)$ be an N-tuple of strategies for $P_1, \cdots, P_N$. Let

$$\sigma' = \sigma/T_u \quad \text{and} \quad \sigma'' = \sigma/(T/u)$$

be the restrictions of σ on T_u and T/u respectively. Since T is of perfect information, the decomposition

$$\sigma \to (\sigma', \sigma'')$$

is a bijection. Let $\mathbf{p}(w)$ be the payoff of T at a terminal vertex w. In terms of σ we write it as $\pi(\sigma)$. It induces a payoff on T_u denoted by $\pi'(\sigma')$. For the payoff $\pi''(\sigma'')$ for T/u, we define the payoff at u by $\pi'(\sigma')$ (if the path determined by σ passed through u). Then it is clear that at u

$$\mathbf{p}(u) = \pi''(\sigma'') = \pi'(\sigma') = \pi(\sigma). \tag{2.1}$$

Lemma 2.3 *Let σ' be an equilibrium N-tuple for T_u and let σ'' be an equilibrium N-tuple for T/u. Then the corresponding σ is an equilibrium N-tuple of T.*

Proof: Consider σ'' in T/u and the terminal vertex is w. If $w \neq u$, we take σ in T to be σ''. Then it is directed to check that for another strategy $\hat{\sigma_i} \in \Sigma_i$, by (2.1) and that σ'' is an equilibrium tuple:

$$\begin{aligned}\pi_i(\sigma_1, \cdots, \hat{\sigma_i}, \cdots, \sigma_N) &= \pi_i''(\sigma_1'', \cdots, \hat{\sigma_i}'', \cdots, \sigma_N'') \\ &\leq \pi_i''(\sigma_1'', \cdots, \sigma_i'', \cdots, \sigma_N'') \\ &= \pi_i(\sigma_1, \cdots, \sigma_i, \cdots, \sigma_N).\end{aligned}$$

Hence σ is an equilibrium N-tuple. If $w = u$, then put σ' and σ'' together to form σ, a similar argument implies σ is an equilibrium N-tuple. □

Proof of the theorem: We use induction on the length of the game, i.e. the maximum length of the paths.

If T has length 1, there is only one move for one player. The theorem is obviously true. As induction hypothesis, suppose the statement is true for T with length less than or equal to m. For T has length $m+1$, we decompose it into several subtrees of length less than or equal to m. We apply the induction hypothesis to obtain the equilibrium N-tuples for the subtrees and the quotient tree, and use Lemma 2.3 to put them together to form the equilibruim N-tuple on T. □

This is a simple algorithm called *Zermelo algorithm* to determine an equilibrium strategy for the game tree of perfect information: starting from the bottom, the players try to make move in the largest payoff to himself.

Consider the following example in Figure 2.6. Let us start from the subtrees for B and pick up the best strategy for B in each case. The next step is to consider the quotient tree. A picks up the left move. the equilibrium pair strategies is (L, rm) with payoff $(6, 10)$.

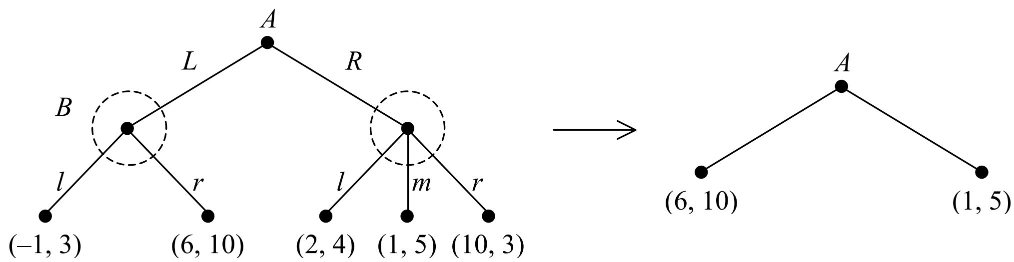


Figure 2.6

You should check this strategy is equilibrium directly from the definition. Also you should compare this concrete example with the proof of the theorem, it has the same idea.

Exercises

1. Concerning the game pictured in the following, answer the following question.

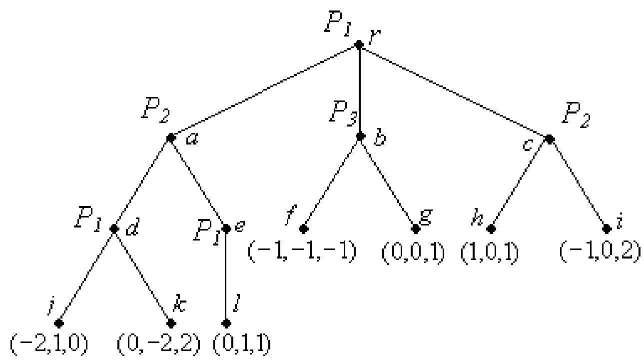


Figure 1

   (a) What is the amount player P_3 is guaranteed to win, assuming that all players play rationally.

   (b) What choice would you advice P_1 to make on his first move?

   (c) If the rules of the game allow P_1 to offer a bribe to another player, how much should he offer to whom for doing what?

2. There are two players, and, at the start, three piles on the table in front of them, each containing two matches. In turns, the players take any (positive) number of matches from *one* of the piles. The player taking the last match loses. Sketch a game tree. Show that the first player has a sure win.

3. Consider the game of perfect information shown in Figure 2. What strategies would you advise A and B adopt?

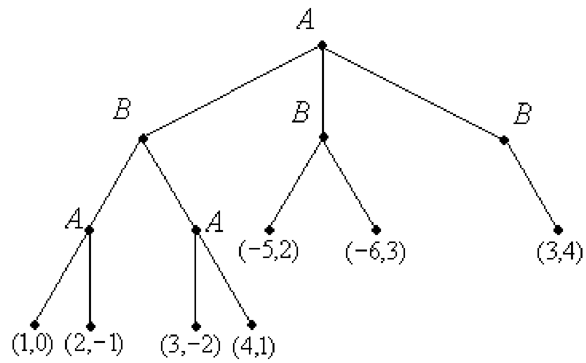


Figure 2

4. The game of Sevens is played between two players (called A and B) as follows. Each player rolls a fair die in such a way that the other cannot see which number came up. Player A must then bet \$1 that either: (i) The *total* on the two dice is less than seven, or (ii) the total is greater then seven . Then, B can either (i) accept the bet, or (ii) reject it. If the bet is rejected, the payoff to each is zero. Otherwise, both dice are revealed. It the total is exactly seven, then both payoffs are zero. Otherwise, one of the players wins the other's dollar. Describe a tree for Sevens, including information sets.

5. Write down the normal form for the game shown in Figure 3. Find the equilibrium pairs (if any).

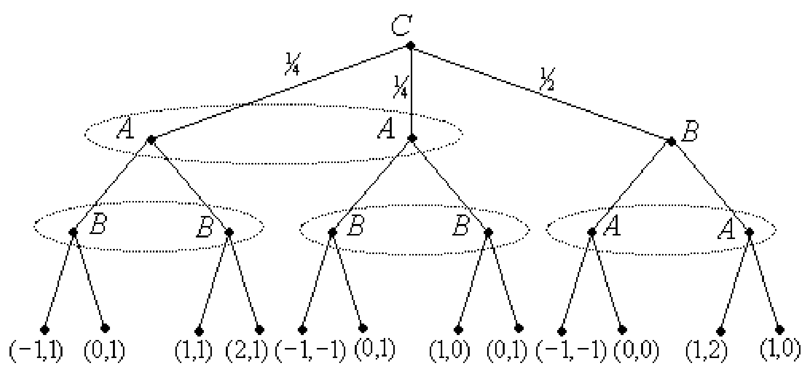


Figure 3

Appendix.

The game Hex is a board game play by two person (see Figure 3.1). To start the game each player has his own territory (black or white) on the opposite side of the board. The players take term to put in black and white pieces on the board. The first person connects his territory by the pieces of his color will win the game.

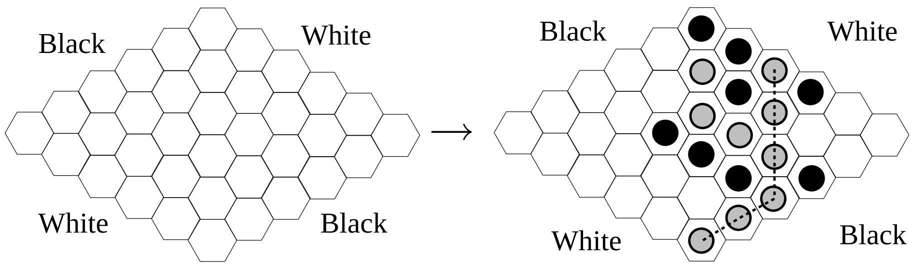


Figure 3.1

The game was invented by a Danish mathematician P. Hein in 1942 and it became popular under the name of Hex. Nash (1948) proved that the game cannot end in a draw and also the first player can always win theoretically, though he may not know such strategy (the proof of existence of the winning strategy is not constructive). We explain the idea of his proof in the following

Theorem 3.1 *The game cannot end in a draw.*

Proof: Assume the black and white are arranged as indicated and renamed for convenience.

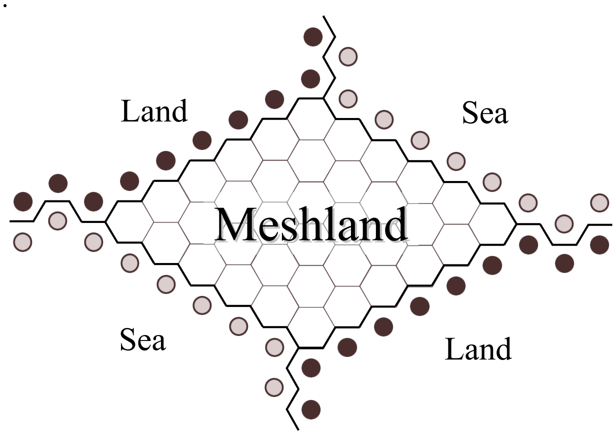


Figure 3.2

The game is to stop if a dam is built to connect the lands or a channel is built to connect the seas. Hence we only need to consider the case all hexagons are filled (the only possible case for draw).

Assume someone starts to walk from the point of enter, walk with land on his left side and sea on his right side.

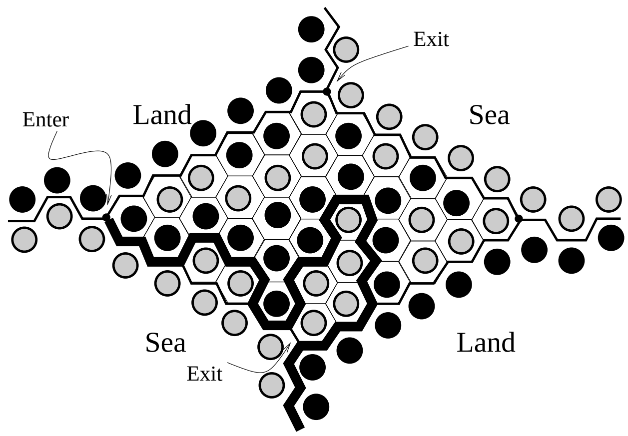


Figure 3.3

We claim that

(i) There is a uniquely determined path: Assume that walker walks in between the two hexagons at the bottom, he has a unique choice to go left or right according to the color of the top hexagon, and this situation will continue in his next step.

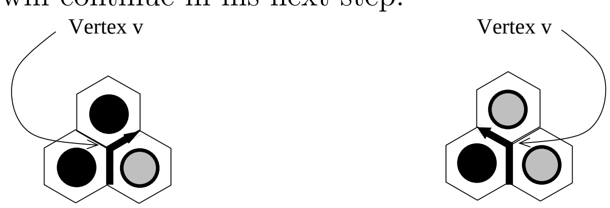


Figure 3.4

(ii) The path cannot be repeated (i.e. he will not walk in circle): Let $(v_1, v_2, v_3, \cdots)$ be the vertices of the path and let k be the first vertex that the path return at time n, i.e., $v_k = v_n$. We see that it is impossible by inspecting the following diagram.

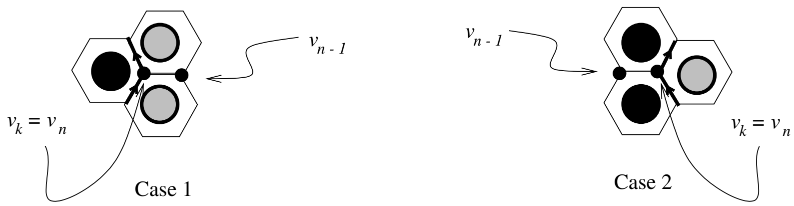


Figure 3.5

Since there is only finitely many vertices and the path cannot be repeated, the walk cannot continue indefinitely, he must exit somewhere. By the rule "land on the left and sea on the right", the walker has only two places to exit, the top and the bottom. If he exits in the bottom, the land is connected (see Figure 3.3). Similarly if he exists at the top, then the sea is connected. In other word, the game cannot end with a draw. □

Theorem 3.2 *The first person can always win.*

Proof: Since this is a game of perfect information, there is equilibrium strategy (may be more than one) which results with white wins or black wins.

Assume that white move first. If black has a winning strategy and the moves are $v_1v_2v_3\cdots v_n$ as follows:

○ v_1 ● v_2 ○ v_3 ● v_4 $\cdots$ ● v_n.

Since white moves first, he can use black's strategy to be his own

○ v_2 ● v_3 ○ v_4 $\cdots$ ○ v_n.

This implies that white always has a winning strategy.

□

The *strategy stealing method* is not constructive, as in the proof neither side has a "concrete" strategy. You can find this game on `www.mazeworks.com/hex7/`.

Question 1. What is the winning strategy for the Hex game if the size of the board are 1×1, 2×2, 3×3 respectively?

Question 2. Which game can you apply the argument in Theorem 3.2?

Chapter 2

Two person zero-sum games

A game is called *zero-sum* if the payoffs of the strategies satisfy

$$\sum_{i=1}^{N} \pi_i(\sigma_1, \cdots, \sigma_n) = 0 \qquad \forall\ \sigma_i \in \Sigma_i,\ 1 \leq i \leq N.$$

This means that for each terminal vertex w, the sum of the components of $\mathbf{p}(w)$ is $\mathbf{0}$, i.e., one player's gain is another player's lost. Many recreational or parlor games are zero-sum. Economics and international competitions are often not of this type; players can do better by playing appropriately and jointly, or all do worse if someone play stupidly.

For a two person game, the normal form can be represented as a payoff bi-matrix. Since it is zero-sum, only one matrix, $M = [m_{ij}]$, needs to be used. The two players are referred to *row player* and *column player*. The entries m_{ij} are the payoff for the row player when he chooses strategy i and the opponent uses strategy j; the column player's payoff is $-m_{ij}$. Hence for the row player, larger number is favored, and for the column player, smaller number is favored.

2.1 Saddle point

Definition 1.1 Let $M = [m_{ij}]$ be an $m \times n$ real matrix, an entry m_{pq} is called a *saddle point* of M if

$$m_{iq} \leq m_{pq} \qquad \forall\ 1 \leq i \leq m,$$

and

$$m_{pj} \geq m_{pq} \qquad \forall\ 1 \leq j \leq n.$$

In other word, m_{pq} is the maximum in the q-th column, and is the minimum in the p-th row.

Example 1.1 Consider the following matrices

$$\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}.$$

In the first matrix, $m_{21} = -1$ is the saddle point; in the second matrix, the four entries of 1 are saddle points. In the third matrix, there is no saddle point.

In connection with the equilibrium pair defined in the last section, we have

Proposition 1.2 *In a zero-sum game, m_{pq} is a saddle point if and only if (p, q) is an equilibrium pair.*

Proof: Let (σ_1^*, σ_2^*) be the equilibrium pair corresponding to the coordinate (p, q) in M, then

$$\pi(\sigma_1^*, \sigma_2^*) = (m_{pq}, -m_{pq}).$$

For any strategy σ_1 for the row player, $\pi_1(\sigma_1, \sigma_2^*)$ corresponds to a row entry m_{iq}. Hence

$$\pi_1(\sigma_1, \sigma_2^*) \leq \pi_1(\sigma_1^*, \sigma_2^*)$$

is equivalent to m_{pq} is the maximum in the q-th column.

Similarly for a strategy σ_2 for the column player, $\pi_2(\sigma_1^*, \sigma_2)$ corresponding to a column entry m_{pj},

$$\pi_2(\sigma_1^*, \sigma_2) \leq \pi_2(\sigma_1^*, \sigma_2^*)$$

is equivalent to (using zero-sum)

$$\pi_1(\sigma_1^*, \sigma_2) \geq \pi_1(\sigma_1^*, \sigma_2^*),$$

which means m_{pq} is the minimum in the p-th row. The proposition follows from this observation. □

Proposition 1.3 *If m_{kl} and m_{pq} are saddle points of the matrix M. Then m_{kq}, m_{pl} are also saddle points and these four values are equal.*

Proof: By using the maximum and minimum properties of m_{kl} and m_{pq}, it is easy to check the following inequalities.

$$\begin{array}{ccc} m_{kl} & \leq & m_{kq} \\ \vee\!\mid & & \wedge\!\mid \\ m_{pk} & \geq & m_{pq} \end{array}$$

This implies the proposition. □

Definition 1.4 Let M be as above. The *value* to the row player and the *value* to the column player are defined by

$$u_r(M) = \max_i \Big(\min_j m_{ij} \Big), \quad u_c(M) = \min_j \Big(\max_i m_{ij} \Big).$$

If they are equal, we call it the *value* of the game.

Intuitively if both players play "conservatively", the row player cannot do worst than $u_r(M)$ (i.e., he is guaranteed for a return from $u_r(M)$), and will not do better than $u_c(M)$.

For the matrices

$$M_1 = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -2 & -3 \\ 0 & 3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & -3 \\ 0 & 2 \\ -5 & 10 \end{bmatrix}.$$

It is direct to check that $u_r(M_1) = u_c(M_1) = -1$; $u_r(M_2) = 0$, $u_c(M_2) = 2$ and $u_r(M_3) = 0$, $u_c(M_3) = 2$.

The basic relations of the two values are:

Theorem 1.5 *Let $u_r(M)$ and $u_c(M)$ be defined as above, then*

(i) $u_r(M) \leq u_c(M)$;

(ii) $u_r(M) = u_c(M)$ *if and only if* M *has a saddle point.*

Proof: (i) Note that $m_{ij} \leq \max_i m_{ij}$, hence taking minimum on j, we have

$$\min_j m_{ij} \leq \min_j(\max_i m_{ij}) = u_c(M) \quad \forall\ 1 \leq i \leq m.$$

Now taking maximum on the left side on i, the inequality $u_r(M) \leq u_c(M)$ follows.

(ii) Sufficiency: Let m_{pq} be a saddle point. Since m_{pq} is the minimum on the p-th row, $m_{pq} = \min_j m_{pj}$. This implies that

$$m_{pq} \leq \max_i(\min_j m_{ij}) \leq u_r(M).$$

Similarly, since m_{pq} is the maximum of the q-th column, $m_{pq} = \max_i m_{iq}$. This implies

$$m_{pq} \geq \min_j(\max_i m_{ij}) \geq u_c(M).$$

Hence $u_r(M) \geq u_c(M)$ and by (i), they are equal.

Necessity: Since $u_r(M)$ is maximum over $1 \leq i \leq m$, we can choose p such that $u_r(M)$ attains the maximum, i.e.,

$$u_r(M) = \min_j m_{pj}.$$

Then choose l such that

$$m_{pl} = \min_j m_{pj} = u_r(M) = u_c(M). \tag{1.1}$$

By the definition of $u_c(M)$, there exists q such that

$$\max_i m_{iq} = u_c(M) = m_{pl}. \tag{1.2}$$

Thus

$$m_{pl} \geq m_{pq}. \tag{1.3}$$

By (1.1), m_{pl} is the minimum of the row, (1.3) is actually an equality. i.e., m_{pq} is the minimum of the p-th row. Hence (1.2) implies that m_{pq} is also the maximum of the q-th column so that m_{pq} is a saddle point. □

By using this theorem, it is easy to determine a saddle point of a matrix. Consider the matrices in Example 1.1:

$$\begin{array}{cc} & \begin{array}{c}\text{row min.}\end{array} \\ \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & 1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ -1 \\ 1 \end{array} \\ \text{col. max.} \quad \begin{array}{ccc} 3 & 1 & 1 \end{array} & \end{array} \qquad \begin{array}{cc} & \begin{array}{c}\text{row min.}\end{array} \\ \begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} & \begin{array}{c} -1 \\ 1 \\ -1 \end{array} \\ \text{col. max.} \quad \begin{array}{ccc} 2 & 2 & 3 \end{array} & \end{array}$$

For the first matrix, $u_r(M) = u_c(M) = 1$, and there are four saddle points. In the second matrix, $1 = u_r(M) < u_c(M) = 2$, hence there is no saddle point.

2.2 Mixed strategies

Consider $M = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, there is no saddle point, and it is unwise to use a "pure" strategy. It is conceivable that each player should choose at each play of the game a strategy at random.

Definition 2.1 Let M be an $m \times n$ matrix game. A *mixed strategy* of the row player is an m-tuple $\mathbf{p} = [p_1, \cdots, p_m]$ of probability $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$. A mixed strategy $\mathbf{q}$ for the column player can be defined similarly.

For the *expected payoffs*, we let

$$E(\mathbf{p}, \mathbf{q}) = \mathbf{p} M \mathbf{q}^t = \sum_{i,j} p_i q_j m_{ij} \ .$$

For a pure strategy j, we can regard it as $\mathbf{q} = [0, \cdots, 0, 1, 0, \cdots, 0]$ where the 1 is at the j-th coordinate, hence the expected payoff is

$$E(\mathbf{p}, j) = \sum_{i=1}^m p_i m_{ij} \ ;$$

The meaning for

$$E(i, \mathbf{q}) = \sum_{j=1}^n q_i m_{ij}$$

can be defined by the same way.

Note that $E(\mathbf{p}, j)$ is the j-th coordinate of $\mathbf{p}M$; $E(i, \mathbf{q})$ is the i-th coordinate of $M\mathbf{q}^t$.

Example 2.1 Let $M = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Consider $\mathbf{p} = [\frac{1}{2}, \frac{1}{2}]$, $\mathbf{q} = [\frac{2}{3}, \frac{1}{3}]$. Then

$$E(\mathbf{p}, \mathbf{q}) = \begin{bmatrix} \frac{1}{2}, & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{1}{3}.$$

Now suppose the column player use $[\frac{2}{3}, \frac{1}{3}]$, how should the row player play to maximize the payoff?

We let $\mathbf{p} = [p, 1-p]$, then

$$E(\mathbf{p},\mathbf{q}) = \begin{bmatrix} p, & 1-p \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{4p}{3} - \frac{1}{3}.$$

The maximum is 1 at $p = 1$, i.e., he should play the first strategy.

We can also solve the problem more directly. Note that

$$M\mathbf{q}^t = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}.$$

i.e., $E(1,\mathbf{q}) = 1$, $E(2,\mathbf{q}) = -\frac{1}{3}$. The row player should play the first strategy to get the maximum payoff.

Definition 2.2 *We define the row value and the column value as*

$$\begin{aligned} v_r(M) &= \max_{\mathbf{p}} \min_{\mathbf{q}} E(\mathbf{p},\mathbf{q}), \\ v_c(M) &= \min_{\mathbf{q}} \max_{\mathbf{p}} E(\mathbf{p},\mathbf{q}). \end{aligned}$$

By using the same explanation as for the pure strategy, the row player will not do worst than $v_r(M)$ and will not do better then $v_c(M)$ if both players play rationally. By elementary analysis of continuous functions applied to $\varphi(\mathbf{p}) = \min_{\mathbf{q}} E(\mathbf{p},\mathbf{q})$ and $\psi(\mathbf{q}) = \max_{\mathbf{p}} E(\mathbf{p},\mathbf{q})$, there exists $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$ such that

$$u_r(M) = \varphi(\hat{\mathbf{p}})$$

$$u_c(M) = \psi(\hat{\mathbf{q}}).$$

Definition 2.3 *We call* $\hat{\mathbf{p}}$*,* $\hat{\mathbf{q}}$ *optimal mixed strategies for the two players respectively if*

$$\begin{aligned} v_r(M) &= \min_{\mathbf{q}} E(\hat{\mathbf{p}},\mathbf{q}), \\ v_c(M) &= \max_{\mathbf{p}} E(\mathbf{p},\hat{\mathbf{q}}). \end{aligned}$$

We compare the payoffs for the pure and mixed strategies:

Proposition 2.4 *With the above notations,*

(i) $u_r(M) \leq v_r(M) \leq E(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \leq v_c(M) \leq u_c(M)$;

(ii) *if M has a saddle point m_{pq}, then the optimal mixed strategies are the pure strategy:* $\hat{\mathbf{p}} = [0, \cdots, 0, 1, 0, \cdots, 0]$, $\hat{\mathbf{q}} = [0, \cdots, 0, 1, 0, \cdots, 0]$ *and* $u_r(M) = u_c(M) = v_r(M) = v_c(M)$.

The fundamental theorem for the two person zero-sum game is the following Minimax Theorem. The proof is due to Von Neumann and Morgenstern. (see the reference by G. Owen).

Theorem 2.5 (Minimax Theorem) *There exists optimal pair* $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ *of mixed strategies such that*

$$v_r(M) = v_c(M) = E(\hat{\mathbf{p}}, \hat{\mathbf{q}}).$$

The optimal mixed strategy may not be unique, indeed it is analogous to the situation for the saddle points (Proposition 1.3):

Proposition 2.6 *If* $(\mathbf{s}, \mathbf{t})$ *is another pair of optimal mixed strategy. Then*

$$E(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = E(\mathbf{s}, \hat{\mathbf{q}}) = E(\hat{\mathbf{p}}, \mathbf{t}) = E(\mathbf{s}, \mathbf{t}).$$

Proof: By repeatedly using the minimum and maximum properties of $(\mathbf{s}, \mathbf{t})$ and $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$, we have

$$E(\hat{\mathbf{p}}, \hat{\mathbf{q}}) \geq E(\mathbf{s}, \hat{\mathbf{q}}) \geq E(\mathbf{s}, \mathbf{t}) \geq E(\hat{\mathbf{p}}, \mathbf{t}) \geq E(\hat{\mathbf{p}}, \hat{\mathbf{q}}).$$

□

To prove Theorem 2.5, we need some preparations on the convex sets. A subset $C \subseteq \mathbb{R}^m$ is called a *convex set* if,

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C, \quad \forall\ \mathbf{x}, \mathbf{y} \in C \ \text{ and } \ 0 \leq \lambda \leq 1.$$

The above expression is called a *convex combination* of $\mathbf{x}$ and $\mathbf{y}$. More generally, for $x_1, \cdots, x_n \in \mathbb{R}^m$, the convex combination of $x_1, \cdots, x_n$ is

$$\sum_{i=1}^{n} \lambda_i x_i \qquad \text{where} \quad \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1.$$

The set C of all convex combinations of $x_1, \cdots, x_n$ is a convex set and is called the *convex hull* of $x_1, \cdots, x_n$. (The reader may notice that $(\lambda_1, \cdots, \lambda_n)$ is in fact a probability vector.)

A well known theorem in convex analysis is:

If C is a closed convex set and $\mathbf{x} \notin C$. Then there exists a hyperplane separating C and $\mathbf{x}$ on the two sides of the hyperplane.

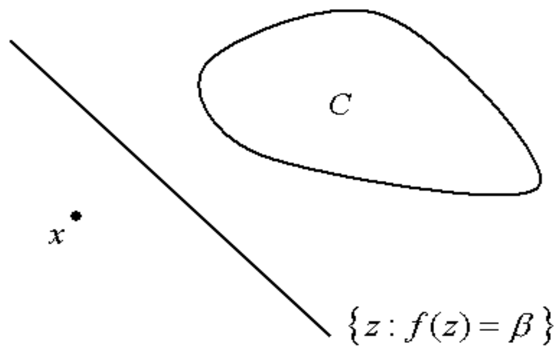


Figure 2.1

More precisely, the assertion says that we can find a linear function $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ and β such that

$$f(\mathbf{x}) < \beta < f(\mathbf{y}) \qquad \forall \; \mathbf{y} \in C.$$

(Note that the hyperplane in the above statement is $\{\mathbf{z} : f(\mathbf{z}) = \beta\}$). Recall that a linear functional f can be represented as $\boldsymbol{\alpha} = [\alpha_1, \cdots, \alpha_m]$ and

$$f(\mathbf{x}) = \boldsymbol{\alpha} \cdot \mathbf{x} = \sum_{i=1}^{m} \alpha_i x_i \; .$$

This implies

Lemma 2.7 *Let C be a closed convex set in $\mathbb{R}^m$ and $\mathbf{x} \notin C$. Then there exists $[\alpha_1, \cdots, \alpha_m]$ and γ such that* (i) $\sum_{i=1}^m \alpha_i x_i = \gamma$; (ii) $\sum_{i=1}^m \alpha_i y_i > \gamma$ *for all $\mathbf{y} \in C$.*

Another elementary but longer proof can be found in Owen's book.

Let

$$M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix},$$

consider the column vectors

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \cdots, \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}, \mathbf{a}_{n+1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}, \mathbf{a}_{n+m} = \begin{bmatrix} \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Lemma 2.8 *Let C be the convex hull of $\{\mathbf{a}_1, \cdots, \mathbf{a}_{n+m}\} \in \mathbb{R}^m$. Then either*

(i) $\mathbf{0} \in C$*; or*

(ii) *there exists* $\mathbf{p} = [p_1, \cdots, p_m]$*,* $p_i \geq 0$*,* $\sum_{i=1}^m p_i = 1$ *such that* $\mathbf{p}M > 0$*, i.e.,*

$$\sum_{i=1}^m p_i a_{ij} > 0, \quad \forall\ j = 1, \cdots, m.$$

Proof: If (i) is not true, then $\mathbf{0} \notin C$. By the above lemma, there exists $[\alpha_1, \cdots, \alpha_m]$ and γ such that

$$\sum_{i=1}^m \alpha_i \cdot \mathbf{0} = \gamma \ \ (= 0)$$

and

$$\boldsymbol{\alpha} \cdot \mathbf{y} = \sum_{i=1}^m \alpha_i y_i > 0 \qquad \forall\ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \in C.$$

In particular if $\mathbf{y} = \mathbf{a}_j$, $1 \leq j \leq n$, then

$$\sum_{i=1}^m \alpha_i a_{ij} > 0,$$

and if $\mathbf{y} = \mathbf{a}_{n+i}$ $(= \mathbf{e}_i)$, then $\alpha_i > 0$. Now let $p_i = \alpha_i / \sum_{i=1}^{m} \alpha_i$. Then $p_i > 0$, $\sum_{i=1}^{m} p_i = 1$ and

$$\sum_{i=1}^{m} p_i a_{ij} > 0, \quad j = 1, \cdots, m.$$

□

Proof of the Theorem: From Lemma 2.8, either (i) or (ii) holds. We will show that if (i) holds, then $v_c(M) \le 0$; if (ii) holds, then $v_r(M) > 0$.

If this is proved, then we can conclude either $v_c(M) \le 0$ or $v_r(M) > 0$. In other word, we have the statement:

$$\textit{It is impossible that } v_r(M) \le 0 < v_c(M). \tag{2.1}$$

Now if we replace M by $M' = [a'_{ij}]$ with $a'_{ij} = a_{ij} + k$. Then

$$v_c(M') = v_c(M) + k, \quad \text{and} \quad v_r(M') = v_r(M) + k.$$

By (2.1), it is impossible that

$$v_r(M') \le 0 < v_c(M'),$$

i.e. it is impossible that

$$v_r(M) \le -k < v_c(M).$$

Since k is arbitrary, we cannot have $v_r(M) < v_c(M)$. Hence $v_c(M) \le v_r(M)$ and by Proposition 2.4(i), equality follows.

It remain to prove the cases for (i) and (ii): If (i) holds, then there exists $s_1, \cdots, s_{m+n} \ge 0$, $\sum_{j=1}^{m+n} s_j = 1$ such that

$$\sum_{j=1}^{n} s_j \mathbf{a}_j + \sum_{j=1}^{m} s_{n+j} \mathbf{e}_j = 0 \ ,$$

i.e.,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} = - \begin{bmatrix} s_{n+1} \\ \vdots \\ s_{n+m} \end{bmatrix}. \tag{2.2}$$

It is easy to see that $s_j > 0$ for some $1 \le j \le n$. Let $s = \sum_{j=1}^{n} s_j > 0$, $q_i = s_i/s$ and $\mathbf{q} = [q_1, \cdots, q_n]$. Then $\sum_{i=1}^{n} q_i = 1$, $q_i \ge 0$, and (2.1) reduces to

$$M\mathbf{q}^t = -\begin{bmatrix} s_{n+1}/s \\ \vdots \\ s_{n+m}/s \end{bmatrix} \le \mathbf{0}.$$

This implies that $E(\mathbf{p}, \mathbf{q}) = \mathbf{p}M\mathbf{q}^t \le 0$. Hence

$$v_c(M) = \min_{\mathbf{q}} \big(\max_{\mathbf{p}} E(\mathbf{p}, \mathbf{q}) \big) \le 0.$$

Next if (ii) holds, then by Lemma 2.7, $\mathbf{p}M > 0$. Hence $E(\mathbf{p}, \mathbf{q}) = \mathbf{p}M\mathbf{q}^t > 0$ for all $q_i > 0$, $\sum_{i=1}^{n} q_i = 1$, so that

$$v_r(M) = \max_{\mathbf{p}} \big(\min_{\mathbf{q}} E(\mathbf{p}, \mathbf{q}) \big) > 0.$$

□

2.3 Some simple calculations

We can simplify the expression of $v_c(M)$ and $v_r(M)$ by the following:

Proposition 3.1 *Let M be an $m \times n$ matrix, then*

$$\begin{aligned} v_r(M) &= \max_{\mathbf{p}} \min_{j} \ E(\mathbf{p}, j); \\ v_c(M) &= \min_{\mathbf{q}} \max_{i} \ E(i, \mathbf{q}). \end{aligned}$$

Proof: It is clear that

$$v_r(M) = \max_{\mathbf{p}} \big(\min_{\mathbf{q}} E(\mathbf{p}, \mathbf{q}) \big) \leq \max_{\mathbf{p}} \big(\min_{j} E(\mathbf{p}, j) \big).$$

On the other hand, let l be such that

$$E(\mathbf{p}, l) = \min_{j} E(\mathbf{p}, j).$$

Then if $\mathbf{q}$ is a mixed strategy for the column player,

$$E(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{n} q_j E(\mathbf{p}, j) \geq E(\mathbf{p}, l).$$

It follows that

$$\min_{\mathbf{q}} E(\mathbf{p}, \mathbf{q}) \geq E(\mathbf{p}, l) = \min_{j} E(\mathbf{p}, j).$$

Now taking $\max_{\mathbf{p}}$ on both side, the first identity for $v_r(M)$ follows. The case for $v_c(M)$ can be proved by the same way. □

Example 3.1 Let $M = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$, find the optimal mixed strategies for the row player and the column player.

Let $\mathbf{p} = [p, 1-p]$, then

$$\mathbf{p}M = \begin{bmatrix} p, & 1-p \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3p-1, & -4p+1 \end{bmatrix}.$$

Hence

$$v_r(M) = \max_{\mathbf{p}} \ \min_j E(\mathbf{p}, j) = \max_{\mathbf{p}} \ \min\{3p - 1, -4p + 1\}$$

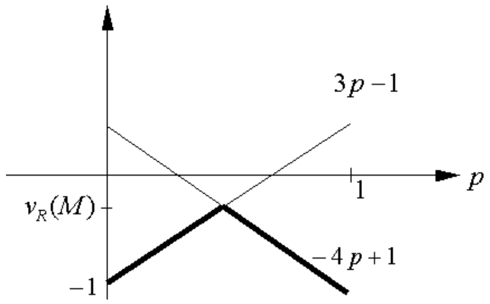


Figure 3.1

From the graph, we see that the solution is at the intersection $(\frac{2}{7}, -\frac{1}{7})$. Hence the optimal strategy for the row player is $\mathbf{p} = [\frac{2}{7}, \frac{5}{7}]$, and $v_r(M) = -\frac{1}{7}$.

For the column player, we let $\mathbf{q} = [q, 1 - q]$ and consider

$$\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q \\ 1 - q \end{bmatrix} = \begin{bmatrix} 5q - 3 \\ -2q + 1 \end{bmatrix}.$$

Then

$$v_c(M) = \min_{\mathbf{q}} \max_i E(i, \mathbf{q}) = \min_{\mathbf{q}} \max\{5q - 3, -2q + 1\},$$

By the same argument, the solution $\mathbf{q} = [\frac{4}{7}, \frac{3}{7}]$ and $v_c(M) = -\frac{1}{7}$.

In a matrix M, we say that row i *dominates* row k if

$$m_{ij} \geq m_{kj} \qquad \forall\ 1 \leq j \leq n,$$

and column j *dominates* column l if

$$m_{ij} \leq m_{il} \qquad \forall\ 1 \leq i \leq m.$$

It is clear that row k will not be used by the row player, and column l will not be used by the column player.

Example 3.2 Let

$$M = \begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Observe that the first row is dominated by the second row, we can omit the first row and the matrix is reduced to

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Now the third column can be ignored because it is dominated by the second column. Consequently, we need only consider the matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

The matrix has no saddle point. We consider the mixed strategies $\mathbf{p}, \mathbf{q}$. For the row player

$$[p,\ 1-p] \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = [3p-1,\ -2p+1].$$

By equating the two coordinates $p = \frac{2}{5}$. For the column player, we consider

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} 3q-1 \\ -2q+1 \end{bmatrix},$$

so that $q = \frac{2}{5}$. Putting these back to the matrix M, we see that the mixed strategies are $\mathbf{p} = [0, \frac{2}{5}, \frac{3}{5}]$, $\mathbf{q} = [\frac{2}{5}, \frac{3}{5}, 0]$ and the value of the game is $\frac{1}{5}$.

We can use a similar concept of dominance for the mixed strategies. Let $\mathbf{p}$ be a mixed strategy for the row player, row i is called *active* if $p_i > 0$, otherwise it is called *inactive*. Similarly we can define this for the column player.

It is easy to see that a row (column) being dominated is inactive in an optimal strategy for the row (column) player. The following proposition is useful in determining the optimal $\mathbf{p}$ and $\mathbf{q}$.

Proposition 3.2 *Let* $(\mathbf{p}, \mathbf{q})$ *be a pair of mixed strategies. Then* $\mathbf{p}, \mathbf{q}$ *are optimal if only if the following holds*

(i) *row k is inactive in* $\mathbf{p}$ *whenever*

$$E(k, \mathbf{q}) < \max_i E(i, \mathbf{q});$$

(ii) *column l is inactive in* $\mathbf{q}$ *whenever*

$$E(\mathbf{p}, l) > \min_j E(\mathbf{p}, j).$$

Proof: Necessity: Let $\mathbf{p}, \mathbf{q}$ be optimal. Suppose $p_k > 0$ and

$$E(k, \mathbf{q}) < \max_i E(i, \mathbf{q}) = v_c(M),$$

then

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} p_i E(i, \mathbf{q}) < \sum_{i=1}^{m} p_i v_c(M) = v_c(M).$$

This contradicts that $\mathbf{p}$ is optimal mixed strategy. (ii) can be proved similarly.

Sufficiency: By assumption (i), $p_i \neq 0$ only on those i such that $E(i, \mathbf{q})$ attains the maximum. This together with Proposition 3.1 imply that

$$E(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{m} p_i E(i, \mathbf{q}) = \max_i E(i, \mathbf{q}) \geq v_c(M).$$

Similarly, we have

$$E(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{n} E(\mathbf{p}, j) q_j = \min_j E(\mathbf{p}, j) \leq v_r(M).$$

Note that $v_c(M) = v_r(M)$, it must equal to $E(\mathbf{p}, \mathbf{q})$ also. $\square$

It follows from (i) that $M\mathbf{q}^t = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, if the u_k is less than the maximum, then $p_k = 0$, and a similar explanation holds for (ii). This can be used to justify whether a probability vector is optimal.

Example 3.3 Let $M = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix}$, let $\mathbf{p} = [0,0,1]$, $\mathbf{q} = [\frac{2}{3}, 0, \frac{2}{3}]$.

Then

$$\begin{bmatrix} 2 & -1 & -1 \\ -2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -2 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1, & 2, & 1 \end{bmatrix}.$$

Note that $p_2 = 0$, $q_2 = 0$. Hence the strategies $(\mathbf{p}, \mathbf{q})$ are optimal.

The theorem can also be used to reduce the matrix to simplify calculations.

Example 3.4 Let

$$M = \begin{bmatrix} 4 & -4 & 1 \\ -4 & 4 & -2 \end{bmatrix}.$$

Consider row player first:

$$[p,\ 1-p] \begin{bmatrix} 4 & -4 & 1 \\ -4 & 4 & 2 \end{bmatrix} = [8p-4,\ \ -8p+4,\ \ 3p-2]\,.$$

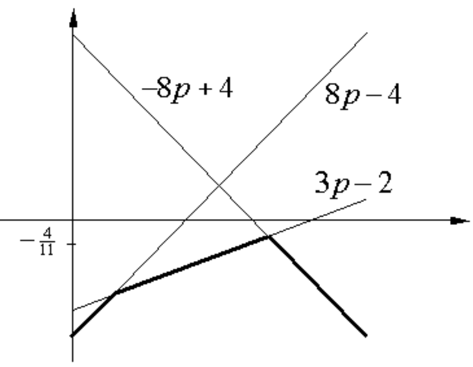


Figure 3.2

Take the minimum of the three lines, then take the maximum value, it is the point at $(\frac{6}{11}, -\frac{4}{11})$. It follows that the optimal row strategy is $[\frac{6}{11}, \frac{5}{11}]$ with $v(M) = \frac{-4}{11}$.

Now observe that

$$[\tfrac{6}{11},\ \tfrac{5}{11}]\begin{bmatrix} 4 & -4 & 1 \\ -4 & 4 & -2 \end{bmatrix} = [\tfrac{4}{11},\ -\tfrac{4}{11},\ -\tfrac{4}{11}]$$

By Proposition 3.2 (ii), the first column is inactive. Hence we need only consider

$$\begin{bmatrix} -4 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} q \\ 1-q \end{bmatrix} = \begin{bmatrix} -5q+1 \\ 6q-2 \end{bmatrix}.$$

The solution is $q = \frac{3}{11}$. The column strategy is $[0, \frac{3}{11}, \frac{8}{11}]$.

Example 3.5 Let

$$M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

We start with column strategy first to obtain $\mathbf{q} = [\frac{1}{2}, \frac{1}{2}]$ and $v_c(M) = \frac{1}{4}$, then apply the same method as the above to reduce to a 2×2 matrix. The solution is $\mathbf{p} = [0, 0, \frac{1}{2}, \frac{1}{2}]$. The detail is left for the reader.

A game is *symmetric* if the two players are indistinguishable. In this case, the representing matrix is a square matrix $M = [m_{ij}]$ and is skew symmetric, $M = -M^t$, i.e.,

$$m_{ij} = -m_{ji}, \quad 1 \le i, j \le n.$$

In particular the diagonal must be zero. For example

$$M = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

is a skew symmetric matrix.

Theorem 3.3 *The value of a symmetric game is zero. Moreover, if $\hat{\mathbf{p}}$ is an optimal strategy for the row player, then it is also optimal for the column player.*

Proof: For any strategy $\mathbf{p}$, consider $E(\mathbf{p},\mathbf{p}) = \mathbf{p}M\mathbf{p}^t$. then

$$\mathbf{p}M\mathbf{p}^t = -\mathbf{p}M^t\mathbf{p}^t = -(\mathbf{p}M\mathbf{p}^t)^t = -\mathbf{p}M\mathbf{p}^t.$$

Hence $E(\mathbf{p},\mathbf{p}) = \mathbf{p}M\mathbf{p}^t = 0$. It follow that for any $\mathbf{p}$,

$$\min_{\mathbf{q}} E(\mathbf{p},\mathbf{q}) \leq 0$$

so that $v_r(M) = \max_{\mathbf{p}} \min_{\mathbf{q}} E(\mathbf{p},\mathbf{q}) \leq 0$. Similarly we can show that $v_c(M) \geq 0$. Recall the Minimax Theorem, $v_r(M) = v_c(M)$, hence the value of the game is 0.

For the second assertion, if $(\hat{\mathbf{p}},\hat{\mathbf{q}})$ is optimal for M, then by symmetry $(\hat{\mathbf{q}},\hat{\mathbf{p}})$ is also optimal for M. By Proposition 2.6, $(\hat{\mathbf{p}},\hat{\mathbf{p}})$ is also optimal for M.

□

Example 3.6 Consider a zero-sum game with matrix

$$M = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}.$$

Let $\mathbf{q} = [x, y, z]$ be an optimal mixed strategy, then

$$M\mathbf{q}^t = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y - 2z \\ -x + 3y \\ 2x - 3z \end{bmatrix} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

By Theorem 3.3, the game value is 0, hence $v_i \leq 0$ for all i and some of the $v_i = 0$. Those with $v_i < 0$ are inactive (Proposition 3.2). We pick two of the equations, together with $x + y + z = 1$ to set up

$$\begin{cases} y - 2z &= 0 \\ -x + 3y &= 0 \\ x + y + z &= 1 \end{cases}.$$

The solution is $\mathbf{q} = [\frac{1}{2}, \frac{1}{3}, \frac{1}{6}]$.

Example 3.7 Consider the zero-sum game with skew symmetric matrix

$$M = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 1 & -2 & 1 & 0 \end{bmatrix}.$$

Let $\mathbf{q} = [w, x, y, z]$ be an optimal solution, then

$$M\mathbf{q}^t = \begin{bmatrix} -x - z \\ w - y + 2z \\ x - 2z \\ w - 2x + y \end{bmatrix} := \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

By the same reason as the previous example, we see that $v_1 < 0$ and the first row is inactive. We pick the other three values $v_2, v_3, v_4 = 0$ (the active ones) to set up

$$\begin{cases} w - y + 2z & = & 0 \\ x - 2z & = & 0 \\ w - 2x + y & = & 0 \\ w + x + y + z & = & 1 \end{cases}.$$

The solution is $\mathbf{q} = [0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}]$.

Exercises

1. For the following matrix, compute $u_r(M)$ and $u_c(M)$:

$$M = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & -1 & 3 & 1 \\ 2 & 0 & 0 & 2 \\ 3 & 2 & 1 & -1 \end{bmatrix}.$$

2. Let

$$M = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}.$$

   (a) Compute $E\big([\frac{1}{5}, \frac{2}{5}, \frac{2}{5}], [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]\big)$.

   (b) On the assumption that the row player continues to play $[\frac{1}{5}, \frac{2}{5}, \frac{2}{5}]$, what is the best way for the column player to play?

3. Let

$$M = \begin{bmatrix} 3 & 2 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

   Eliminate dominated rows and columns so as to reduce M to the smallest size possible.

4. Prove that if $\mathbf{p}$ and $\mathbf{q}$ are mixed strategies for the row player and column player respectively, such that

$$\min_j E(\mathbf{p}, j) = \max_i E(i, \mathbf{q}),$$

   then $\mathbf{p}$ and $\mathbf{q}$ are optimal.

5. Given the game matrix

$$\begin{bmatrix} 2 & -3 & 4 & -5 \\ -1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 3 \\ 1 & 2 & -3 & 4 \\ -3 & 4 & -5 & 6 \end{bmatrix},$$

   verify that $\mathbf{p} = [\frac{1}{2}, 0, \frac{1}{6}, 0, \frac{1}{3}]$ and $\mathbf{q} = [0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$ are optimal strategies.

6. Suppose that both $\mathbf{r}$ and $\mathbf{u}$ are optimal strategies for the row player for the matrix game M. Prove that if $0 \leq t \leq 1$ then $t\mathbf{r} + (1-t)\mathbf{u}$ is also an optimal strategy for the row player.

7. Solve
$$\begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & -1 & 2 & -1 \end{bmatrix}.$$

8. Solve
$$\begin{bmatrix} -1 & 3 \\ 4 & -1 \\ -3 & 5 \\ 3 & 1 \end{bmatrix}.$$

9. Solve the game scissors-paper-stone.

10. Find at least two optimal mixed strategies for two-finger morra.

Chapter 3

Linear programming and solving matrix game

Let M be a game matrix, consider the game value

$$v = \max_{\mathbf{p}} \min_{\mathbf{q}} E(\mathbf{p}, \mathbf{q}) = \max_{\mathbf{p}} \min_{j} E(\mathbf{p}, j).$$

We need to find a probability vector $\mathbf{p}$ so that

$$v = \min_{j} E(\mathbf{p}, j) = \min_{j} \sum_{i=1}^{m} p_i m_{ij}$$

is maximum, i.e. v is largest so that

(i) $p_i \geq 0$ for $1 \leq i \leq m$,

(ii) $\sum_{i=1}^{m} p_i = 1$,

(iii) $v \leq \sum_{i=1}^{m} p_i m_{ij}$, $j = 1, \cdots, n$.

By adding c to each entry (i.e. $m_{ij} + c$) and to v, there is nothing changed on $\mathbf{p}$. Hence we can assume that *all the entries are positive*, so that $v > 0$. Let $y_i = p_i / v \geq 0$ for $1 \leq i \leq m$, then $\sum_{i=1}^{n} y_i = 1/v$. The problem becomes

$$\text{(ii)}' \quad \text{minimize} \quad y_1 + \cdots + y_m,$$

subject to

$$\text{(iii)}' \quad 1 \leq \sum_{i=1}^{m} y_i m_{ij}, \quad j = 1, \cdots, n.$$

If we consider the column player, the corresponding problem for finding **q** becomes

$$\text{maximize} \quad x_1 + \cdots + x_n$$

subject to

$$\sum_{j=1}^{n} m_{ij} x_j \leq 1, \quad 1 \leq i \leq m$$

(all the entries are assumed non-negative).

3.1 Linear Programming

A linear programming (LP) is a problem of maximizing a linear function subject to linear constraints:

$$\text{maximize} \quad f(\mathbf{x}) = \textstyle\sum_{j=1}^{n} c_i x_i + d$$

$$\text{subject to} \quad \textstyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \cdots, m,$$

(we assume $x_i \geq 0$ without saying it). The function $f(\mathbf{x})$ is called the *objective function* and the inequalities are called the *constraints*. In matrix form it is

$$\text{maximize } f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} + d \qquad \text{subject to } A\mathbf{x} \leq \mathbf{b}.$$

(Here we consider $\mathbf{x}$ and $\mathbf{b}$ are column vectors.) We call the above LP problem the *primal problem*. The dual problem is

$$\text{minimize} \quad g(\mathbf{y}) = \textstyle\sum_{i=1}^{m} b_i y_i + d$$

$$\text{subject to} \quad \textstyle\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = 1, \cdots, n.$$

In matrix form, it is

$$\text{minimize} \quad g(\mathbf{y}) = \mathbf{b} \cdot \mathbf{y} + d \quad \text{subject to} \quad \mathbf{y}A \geq \mathbf{c}.$$

(Here $\mathbf{y}$ and $\mathbf{c}$ are row vectors.)

A (nonnegative) vector $\mathbf{x}$ is called *feasible* if it satisfies the constraints of the LP problem, and the LP problem is called feasible. A feasible vector $\mathbf{x}$ is called *optimal* if $f(\mathbf{x})$ is maximum. We say that the primal (dual) problem is *bounded* if the objective function is bounded above (below, respectively).

Proposition 1.1 *Let* $\mathbf{x}$, $\mathbf{y}$ *be feasible in the respective problems, then*

$$f(\mathbf{x}) \leq g(\mathbf{y}).$$

It follows that

$$\max_{\mathbf{x}} f(\mathbf{x}) \leq \min_{\mathbf{y}} g(\mathbf{y}).$$

Proof: Observe that

$$\begin{aligned} f(\mathbf{x}) &= \textstyle\sum_{j=1}^{n} c_j x_j + d \\ &\le \textstyle\sum_{j=1}^{n} \big(\sum_{i=1}^{m} a_{ij} y_i\big) x_j + d \\ &= \textstyle\sum_{i=1}^{m} \big(\sum_{j=1}^{n} a_{ij} x_j\big) y_i + d \\ &\le \textstyle\sum_{i=1}^{m} b_i y_i + d \\ &= g(\mathbf{y}). \end{aligned}$$

□

It follows from Proposition 1.1 that

Corollary 1.2 *The primal (dual) problem is bounded if and only if the dual (primal) problem is feasible.*

By using the theorem for symmetric game, we can show that the maximum and minimum are actually equal. It is the basic theorem of linear programming

Theorem 1.3 *If the two dual programs are feasible, then both of them will have optimal solutions* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *and* $\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*$*. It follows that*

$$f(\mathbf{x}^*) = \max_{\mathbf{x}} f(\mathbf{x}) = \min_{\mathbf{y}} g(\mathbf{y}) = g(\mathbf{y}^*),$$

i.e., both programs have the same value.

Proof: Consider the new matrix

$$M = \left[\begin{array}{cc|c} 0 & A & -\mathbf{b}^t \\ -A^t & 0 & \mathbf{c}^t \\ \hline \mathbf{b} & -\mathbf{c} & 0 \end{array}\right].$$

Note that A is an $m \times n$ matrix, M is an $(m+n+1) \times (m+n+1)$ matrix and is skew symmetric. By Theorem 3.3 in the last chapter, the game value is 0. Let

$$\mathbf{s} = (s_1, \cdots, s_m, s_{m+1}, \cdots, s_{m+n}, s_{m+n+1})$$

be an optimal strategy (for both players) of M, then $\max_i E(i, \mathbf{s}) = 0$. It follows that $M\mathbf{s}^t \leq \mathbf{0}$. If we let

$$\mathbf{y} = (s_1, \cdots, s_m), \quad \mathbf{x}^t = (s_{m+1}, \cdots, s_{m+n}),$$

then $\mathbf{s} = (\mathbf{y}, \mathbf{x}^t, s_{n+m+1})$, and $M\mathbf{s}^t \leq \mathbf{0}$ reduces to

$$\begin{cases} A\mathbf{x} - w\mathbf{b} & \leq & \mathbf{0} \\ -\mathbf{y}A + w\mathbf{c} & \leq & \mathbf{0} \\ \mathbf{b} \cdot \mathbf{y} - \mathbf{c} \cdot \mathbf{x} & \leq & 0 \ . \end{cases}$$

Either one of the following case will occur.

Case (I): there exists optimal $\mathbf{s}$ such that $w = s_{m+n+1} > 0$. Let $\mathbf{x}^* = \mathbf{x}/w$, $\mathbf{y}^* = \mathbf{y}/w$. Then

$$\begin{cases} A\mathbf{x}^* & \leq & \mathbf{b} \\ \mathbf{y}^*A & \geq & \mathbf{c} \\ \mathbf{b} \cdot \mathbf{y}^* & \leq & \mathbf{c} \cdot \mathbf{x}^* \ . \end{cases}$$

Together with Proposition 1.1 that $\mathbf{c} \cdot \mathbf{x}^* \leq \mathbf{b} \cdot \mathbf{y}^*$, we see that $\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*$ and $\mathbf{x}^*$ and $\mathbf{y}^*$ are the required solutions.

Case (II): for all optimal $\mathbf{s}$, $w = s_{n+m+1} = 0$. We show that this case cannot happen. By Proposition 3.2(ii) in Chapter II, the $(m + n + 1)$-th column is inactive and hence

$$\mathbf{s}M\mathbf{e}_{n+m+1} = E(\mathbf{s}, m + n + 1) > \min_j E(\mathbf{s}, j) = 0.$$

This implies

$$\begin{cases} -A\mathbf{x} & \geq & \mathbf{0} \\ \mathbf{y}A & \geq & \mathbf{0} \\ -\mathbf{b} \cdot \mathbf{y} + \mathbf{c} \cdot \mathbf{x} & > & 0 \ , \end{cases} \qquad \text{i.e.,} \qquad \begin{cases} A\mathbf{x} & \leq & \mathbf{0} \\ \mathbf{y}A & \geq & \mathbf{0} \\ \mathbf{c} \cdot \mathbf{x} & > & \mathbf{b} \cdot \mathbf{y} \ . \end{cases}$$

Consider the last inequality, it implies

$$\text{either} \quad \mathbf{c} \cdot \mathbf{x} > 0 \quad \text{or} \quad 0 > \mathbf{b} \cdot \mathbf{y} \ .$$

For the first case, let $\mathbf{x}'$ be in the constraint set (i.e., $A\mathbf{x}' \le \mathbf{b}$), then $\mathbf{x}' + \alpha\mathbf{x}$ is also in the constraint set for $\alpha > 0$ (i.e., $A(\mathbf{x}' + \alpha\mathbf{x}) \le \mathbf{b}$). This implies that $\mathbf{c} \cdot (\mathbf{x}' + \alpha\mathbf{x})$ can be as large as desired. Hence the primal problem is unbounded. By Corollary 1.2, the dual problem is infeasible. This contradicts the assumption.

By considering $0 > \mathbf{b} \cdot \mathbf{y}$, we can show that the primal problem is infeasible. This again contradicts the assumption and completes the proof that case (II) is impossible. □

Exercise. Apply the above theorem to prove the Minimax theorem in the last chapter.

Let us rewrite the primal problem as

$$\begin{array}{rl} \text{maximize} & f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j + d \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j - b_i = -x_{n+i}, \quad 1 \le i \le m \end{array} \tag{1.1}$$

with all $x_i, x_{i+n} \ge 0$.

By *pivoting* an independent variable x_j and a dependent variable x_{n+i}, we mean interchange the role of these two variables.

Example 1.1. Consider the problem

$$\text{maximize} \quad x_1 - x_4$$

$$\text{subject to} \quad \left\{ \begin{array}{rrrrcr} x_1 & +x_2 & +x_3 & -x_4 & \le & 2 \\ -x_1 & -3x_2 & -x_3 & +2x_4 & \le & -1. \end{array} \right.$$

(We assume $x_i \ge 0$ as convention). We introduce two (non-negative) variables x_5 and x_6 and rewrite it as

$$\text{maximize} \quad x_1 - x_4$$

$$\text{subject to} \quad \left\{ \begin{array}{rrrrrcr} x_1 & +x_2 & +x_3 & -x_4 & -2 & = & -x_5 \\ -x_1 & -3x_2 & -x_3 & +2x_4 & +1 & = & -x_6. \end{array} \right.$$

Pivot on x_1 and x_6, the problem becomes

$$\text{maximize} \quad -3x_2 - x_3 + x_4 + x_6 + 1$$

$$\text{subject to} \quad \left\{ \begin{array}{rrrrrcl} -2x_2 & & +x_4 & +x_6 & -1 & = & -x_5 \\ 3x_2 & +x_3 & -2x_4 & -x_6 & -1 & = & -x_1. \end{array} \right.$$

Again pivot on x_4 and x_5, the problem is further reduced to

$$\begin{array}{ll} \text{maximize} & -x_2 - x_3 - x_5 + 2 \\ \text{subject to} & \left\{ \begin{array}{rrrrrcl} -2x_2 & & +x_5 & +x_6 & -1 & = & -x_4 \\ -x_2 & +x_3 & +2x_5 & +x_6 & -3 & = & -x_1. \end{array} \right. \end{array} \tag{1.2}$$

The advantage of this form is that $x_2,\ x_3,\ x_5 \geq 0$, hence the objective function has maximum 2 at

$$x_2 = x_3 = x_5 = x_6 = 0$$

(recall that $x_i \geq 0$) and in this case

$$x_1 = 3 \quad \text{and} \quad x_4 = 1.$$

It follows that the maximum is attained at $x_1 = 3$, $x_2 = x_3 = 0$, $x_4 = 1$.

We call the expression (1.1) the *basic form.* The *basic solution* is obtained by letting the independent variables equal to 0, and solve for the other dependent variables. A basic solution $\mathbf{x}$ is *feasible* if all coordinates x_i are nonnegative. A feasible basic solution is *optimal* if the objective function attains maximum there.

In Example 1.1, the original problem has no feasible basic solution; but (1.2) has optimal feasible basic solution.

To examine the pivoting operation more systematically, we use the problem in the form of a tableau. We use Example 1.1 to illustrate the idea. First note that the constraints of (1.1) in matrix form is

$$A\mathbf{x} - \mathbf{b} = -I\mathbf{x}_s,$$

Putting Example 1.1 in a tableau and perform the pivot operation, we have

x_1	x_2	x_3	x_4	x_5	x_6	-1	
1	1	1	-1	1	0	2	$= -x_5$
-1^*	-3	-1	2	0	1	-1	$= -x_6$
1	0	0	-1	0	0	0	$= f$

$\xrightarrow[\text{on } x_1, x_6]{\text{pivot}}$

x_1	x_2	x_3	x_4	x_5	x_6	-1	
0	-2	0	1^*	1	1	1	$= -x_5$
1	3	1	-2	0	-1	1	$= -x_1$
0	-3	-1	1	0	1	-1	$= f$

$\xrightarrow[\text{on } x_4, x_5]{\text{pivot}}$

x_1	x_2	x_3	x_4	x_5	x_6	-1	
0	-2	0	1	1	1	1	$= -x_4$
1	-1	1	0	2	1	3	$= -x_1$
0	-1	-1	0	-1	0	-2	$= f$

(* means the two corresponding variables are to be pivoted). This is the same as (1.2), the maximum value is 2.

In general in the pivot operation of x_l and x_k, we have

x_j	x_l	x_k	-1	
a	b^*	1	b_k	$= -x_k$
c	d	0	b_i	$= -x_i$

$\longrightarrow$

x_j	x_l	x_k	-1	
a/b	1	$1/b$	b_k/b	$= -x_l$
$c - ad/b$	0	$-d/b$	$b_i - b_k d/b$	$= -x_i$

(1.3)

The pivoting algorithm (simplex algorithm): Given a feasible basic form, the corresponding tableau is:

x_1	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{n+m}	-1	
a_{11}	$\cdots$	a_{1n}	1	$\cdots$	0	b_1	$= -x_{n+1}$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
a_{m1}	$\cdots$	a_{mn}	0	$\cdots$	1	b_m	$= -x_{n+m}$
c_1	$\cdots$	c_n	0	$\cdots$	0	$-d$	$= f$

(Note that the existence of feasible basic solution implies $b_1, \cdots, b_m \geq 0$).

Step 1. (i) if $c_1, \cdots, c_n \le 0$, then d is the maximum attains at $x_1 = \cdots = x_n = 0$, $x_{n+i} = b_i$, $1 \le i \le m$.

(ii) Otherwise, go to Step 2.

Step 2. Choose one of the $c_l > 0$ and consider the rows i such that $a_{il} > 0$ (if all $a_{il} \le 0$, then the problem is unbounded, there is no optimal solution).

Choose row k such that

$$\frac{b_k}{a_{kl}} = \min\left\{\frac{b_i}{a_{il}} : 1 \le i \le m, \quad a_{il} > 0\right\}. \tag{1.4}$$

Step 3. Pivot on x_l and x_k, i.e., on the entry a_{kl} (Note that by the general expression in the pivot operation and by (1.3), (1.4), the column on the new b_i's are nonnegative. Moreover the new $d' \ge d$.)

Step 4. Back to Step 1.

Example 1.2. Maximize $f(\mathbf{x}) = 3x_1 + x_2 + 3x_3$

$$\text{subject to} \begin{cases} 2x_1 & +x_2 & +x_3 & \le & 2 \\ -3x_1 & & +x_3 & \le & 5 \\ 2x_1 & +2x_2 & +x_3 & \le & 6. \end{cases}$$

x_1	x_2	x_3	x_4	x_5	x_6	-1	
2	1	1^*	1	0	0	2	$= -x_4$
-3	0	1	0	1	0	5	$= -x_5$
2	2	1	0	0	1	6	$= -x_6$
3	1	3	0	0	0	0	$= f$

$\longrightarrow$

x_1	x_2	x_3	x_4	x_5	x_6	-1	
2	1	1	1	0	0	2	$= -x_3$
-5	-1	0	-1	1	0	3	$= -x_5$
0	1	0	-1	0	1	4	$= -x_6$
-3	-2	0	-3	0	0	-6	$= f$

The solution of the problem is

$$x_1 = x_2 = 0, \quad x_3 = 2, \quad \text{maximum value is } 6.$$

(Try start by pivoting x_1 and x_4 instead).

Example 1.3. Solve the dual problem of Example 1.2

Recall the dual problem of

$$\text{maximize} \quad f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} \qquad \text{subject to} \quad A\mathbf{x}^t \leq \mathbf{b}$$

is

$$\text{minimize} \quad g(\mathbf{y}) = \mathbf{b} \cdot \mathbf{y} \qquad \text{subject to} \quad \mathbf{y}A \geq \mathbf{c}.$$

Hence, the dual problem of Example 1.2 is

$$\text{minimize} \quad g(\mathbf{y}) = 2y_1 + 5y_2 + 6y_3$$

$$\text{subject to} \quad \begin{cases} 2y_1 & -3y_2 & +2y_3 & \geq & 3 \\ y_1 & & +2y_3 & \geq & 1 \\ y_1 & +y_2 & +y_3 & \geq & 3. \end{cases}$$

The tableau can be considered dually:

	x_1	x_2	x_3	-1	
y_1	2	1	1	2	$= -x_4$
y_2	-3	0	1	5	$= -x_5$
y_3	2	2	1	6	$= -x_6$
-1	3	1	3	0	$= f$
	$= y_4$	$= y_5$	$= y_6$	$= g$	

From Example 1.2, it reduces to

	x_1	x_2	x_4	-1	
y_6	2	1	1	2	$= -x_3$
y_2	-5	-1	-1	3	$= -x_5$
y_3	0	1	-1	4	$= -x_6$
-1	-3	-2	-3	-6	$= f$
	$= y_4$	$= y_5$	$= y_1$	$= g$	

The solution is $(3, 0, 0)$ with minimum 6.

Example 1.4. Maximize $f(x_1, x_2, x_3) = 5x_1 + 2x_2 + x_3$

$$\text{subject to} \quad \begin{cases} x_1 & +3x_2 & -x_3 & \leq & 6 \\ & x_2 & +x_3 & \leq & 4 \\ 3x_1 & +x_2 & & \leq & 7. \end{cases}$$

We set up the tableau as

x_1	x_2	x_3	x_4	x_5	x_6	-1	
1	3	-1	1	0	0	6	$= -x_4$
0	1	1	0	1	0	4	$= -x_5$
3^*	1	0	0	0	1	7	$= -x_6$
5	2	1	0	0	0	0	$= f$

After pivoting for two times, we have

x_1	x_2	x_3	x_4	x_5	x_6	-1	
0	$1/3$	0	1	1	$-1/3$	$23/3$	$= -x_4$
0	1	1	0	1	0	4	$= -x_3$
1	0	0	0	0	$1/3$	$7/3$	$= -x_1$
0	$-2/3$	0	0	-1	$-5/3$	$-47/3$	$= f$

The solution is $(x_1, x_2, x_3) = (\frac{7}{3}, 0, 4)$ and the maximum value is $\frac{47}{3}$.

If we consider the dual problem, the corresponding tableau is

	x_2	x_5	x_6	-1	
y_1	$1/3$	1	$-1/3$	$23/3$	$= -x_4$
y_6	1	1	0	4	$= -x_3$
y_4	0	0	$1/3$	$7/3$	$= -x_1$
-1	$-2/3$	-1	$-5/3$	$-47/3$	$= f$
	$= y_5$	$= y_2$	$= y_3$	$= g$	

the solution is $(y_1, y_2, y_3) = (0, 1, \frac{5}{3})$ and the minimum value is $\frac{47}{3}$.

In general it may happen that the problem is unbounded, hence there is no optimal solution.

Example 1.5. Maximize $f(x_1, x_2) = x_1 + x_2$

$$\text{subject to} \quad \begin{cases} -2x_1 & +x_2 & \leq & -3 \\ x_1 & -2x_2 & \leq & 4 \end{cases}$$

where $x_1, x_2 \geq 0$.

Consider

x_1	x_2	x_3	x_4	-1	
-2	1	1	0	-3	$= -x_3$
1	-2^*	0	1	4	$= -x_4$
1	1	0	0	0	$= f$

$\longrightarrow$

x_1	x_2	x_3	x_4	-1	
$-1/4$	0	1	$1/2$	5	$= -x_3$
$-1/2^*$	1	0	$-1/2$	-2	$= -x_2$
$3/2$	0	0	$1/2$	2	$= f$

.

$\longrightarrow$

x_1	x_2	x_3	x_4	-1	
1	0	$-4/7$	$-2/7$	$-20/7$	$= -x_1$
0	1	$-2/7$	$-9/14$	$-24/7$	$= -x_2$
0	0	$6/7$	$13/14$	$44/7$	$= f$

The corresponding matrix has negative entries, x_3, x_4 can be chosen as large as possible. This implies x_1, x_2 can be as large as possible. Hence $f(x_1, x_2)$ is unbounded, there is no optimal solution.

3.2 Solving matrix game

Example 2.1. Find the optimal strategy for the game matrix

$$M = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & -1 \\ -1 & 3 & 0 \end{bmatrix}.$$

First we modify the matrix by adding 3 to each entry.

$$M' = \begin{bmatrix} 1 & 5 & 4 \\ 3 & 2 & 6 \\ 5 & 4 & 2 \\ 2 & 6 & 3 \end{bmatrix}.$$

By using the reduction in the beginning of the chapter, we will consider the column player first

$$\text{maximize} \quad f = x_1 + x_2 + x_3$$

$$\text{subject to} \quad \begin{cases} x_1 & +5x_2 & +4x_3 & \leq & 1 \\ 3x_1 & +2x_2 & +6x_3 & \leq & 1 \\ 5x_1 & +4x_2 & +2x_3 & \leq & 1 \\ 2x_1 & +6x_2 & +3x_3 & \leq & 1 \end{cases}.$$

By using the pivoting algorithm two times, we can reduce

	x_1	x_2	x_3	-1	
y_1	1	5	4	1	$= -x_4$
y_2	3	2	6	1	$= -x_5$
y_3	5^*	4	2	1	$= -x_6$
y_4	2	6	3	1	$= -x_7$
-1	1	1	1	0	$= f$
	$= y_5$	$= y_6$	$= y_7$	$= g$	

to

	x_6	x_7	x_5	-1	
y_1	$18/55$	$-54/55$	$-3/10$	$1/11$	$= -x_4$
y_7	$-7/55$	$1/55$	$1/5$	$1/11$	$= -x_3$
y_5	$3/11$	$-2/11$	0	$1/11$	$= -x_1$
y_6	$-3/110$	$12/55$	$-1/10$	$1/11$	$= -x_2$
-1	$-13/110$	$-3/55$	$-1/10$	$-3/11$	$= f$
	$= y_3$	$= y_4$	$= y_2$	$= g$	

The solution of the LP problem is

$$(x_1, x_2, x_3) = (1/11, 1/11, 1/11) \quad \max f = 3/11.$$

$$(y_1, y_2, y_3, y_4) = (0, 1/10, 13/10, 3/55) \quad \min g = 3/11.$$

To put it back into the matrix M and $\mathbf{p}$, $\mathbf{q}$, we recall the relation of $\mathbf{p}$, $\mathbf{q}$ with the above solutions is

$$p_j = 11/3 y_j, \qquad q_i = 11/3 x_i.$$

Thus $\mathbf{p} = (0,\ 11/30,\ 13/30,\ 1/5)$, $\mathbf{q} = (1/3,\ 1/3,\ 1/3)$. The value of the game is 11/3 – 3 = 2/3.

Example 2.2. For the skew symmetric matrix

$$M = \begin{bmatrix} 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

It can be solved by the linear programming or by the method for skew symmetric from last chapter. The optimal strategy is then

$$\mathbf{p} = \left[1/9,\ 1/9,\ 1/3,\ 1/9,\ 1/3\right].$$

The most surprising part of the solution is the third strategy which is the least favorable one, but it will be used often in the mixed optimal strategy.

Example 2.3. (A war game) Suppose generals A and B are fighting for two positions. A has 4 regiments and B has 3 regiments. They have to decide how many regiments to deploy in each position. The payoff is computed as follows: if r regiments defeats $s(< r)$ regiments, the winner gains $s+1$ (equal number of regiment will be a draw). The game matrix is

	$(3,0)$	$(0,3)$	$(2,1)$	$(1,2)$
$(4,0)$	4	0	2	1
$(0,4)$	0	4	1	2
$(3,1)$	1	-1	3	0
$(1,3)$	-1	1	0	3
$(2,2)$	-2	-2	2	2

By using the linear programming, we can show that the optimal strategies are

$$\mathbf{p} = \Big[4/9,\ 4/9,\ 0,\ 0,\ 1/9\Big], \qquad \mathbf{q} = \Big[1/30,\ 7/90,\ 8/15,\ 16/45\Big].$$

From this solution, we see that A should concentrate his force, B should spread out his force. Moreover since the two positions for B are indistinguishable, then $\mathbf{q}' = [7/90, 1/30, 16/45, 8/15]$ is also an optimal strategy of B, hence the average $\mathbf{q}'' = [1/18, 1/18, 4/9, 4/9]$ is also an optimal strategy of B.

Exercises

1. Consider the problem

$$\text{maximize} \quad -x_1 + 2x_2 - 3x_3 + 4x_4$$

$$\text{subject to} \quad \begin{cases} & & x_3 & -x_4 & \leq & 0 \\ x_1 & & -2x_3 & & \leq & 1 \\ & 2x_2 & & +x_4 & \leq & 3 \\ -x_1 & +3x_2 & & & \leq & 5 \end{cases}$$

Is it feasible? Is it bounded?

2. Consider the problem

$$\text{maximize} \quad x_1 - 3x_2$$

$$\text{subject to} \quad \begin{cases} x_1 & +x_2 & \geq & 1 \\ x_1 & -x_2 & \leq & 1 \\ x_1 & -x_2 & \geq & -1 \end{cases}$$

Is it feasible? Is it bounded?

3. Solve

$$\text{maximize} \quad x_1 + x_2 + x_3$$

$$\text{subject to} \quad \begin{cases} x_1 & & +x_3 & & \leq & 3 \\ & x_2 & & -x_4 & \leq & 0. \end{cases}$$

4. Solve

$$\text{maximize} \quad x_1 + x_2 - x_3 - 2x_4$$

$$\text{subject to} \quad \begin{cases} x_1 & +2x_2 & +2x_3 & +x_4 & \leq & 10 \\ x_1 & & -x_3 & & \leq & 0 \\ & x_2 & & & \leq & 2. \end{cases}$$

5. Solve

$$\text{minimize} \quad y_1 + y_2 + y_3$$

$$\text{subject to} \quad \begin{cases} -y_1 & +2y_2 & & \geq & 1 \\ & -y_2 & +3y_3 & \geq & 1. \end{cases}$$

6. Solve the following primal problem and its dual

$$\text{maximize} \quad 5x_1 + x_2 + 5x_3$$

$$\text{subject to} \quad \begin{cases} x_1 & +2x_2 & +x_3 & \leq & 4 \\ x_1 & & & \leq & 2 \\ & & x_3 & \leq & 2. \end{cases}$$

7. Solve the game

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

8. Solve the game

$$\begin{bmatrix} -1 & 2 & -1 & 1 \\ 1 & 0 & 2 & -1 \\ -1 & 1 & -2 & 2 \end{bmatrix}.$$

9. In the war game in Example 2.3, suppose that general A does not entirely trust game theory — instead of playing the strategy we computed, he decides to flip a fair coin and send all his regiments to either one of the places, depending on whether the coin comes up heads or tails. In other words his mixed strategy is

$$\mathbf{p} = \Big(1/2,\ 1/2,\ 0,\ 0\Big).$$

Assuming that general B knows this, how would you advise him to respond, and what would be the result? Does general B do better than the value of the game predicts?

Chapter 4

Non-Zero-Sum Games

4.1 Non-cooperative Games

The normal form (payoffs) of a two person game is represented as a bimatrix $(A, B) = [(a_{ij}, b_{ij})]$.

Example 1.1 (Prisoner's dilemma) Two criminals commit a crime and are arrested. The penalty is to be prisoned as indicated in the following table. Should they confess or deny?

	$confess$	$deny$
$confess$	$(-5, -5)$	$(-1, -10)$
$deny$	$(-10, -1)$	$(-2, -2)$

Example 1.2 (Battles of the buddies). Two friends want to go together for an event. They have a choice between a "ball game" and an "opera". However they make independent decisions, and the "happiness rating" is as follows. How do they make the decision?

	$ball$	$opera$
$ball$	$(5, 1)$	$(0, 0)$
$opera$	$(0, 0)$	$(1, 5)$

We use the same concept of pure strategy and mixed strategy.

Definition 1.1 *Let*

$$\mathcal{P}_1 = \{\mathbf{x} : \; x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}, \quad \mathcal{P}_2 = \{\mathbf{y} : \; y_i \geq 0, \sum_{j=1}^{n} y_j = 1\}$$

be the sets of mixed strategies of the two players. For $\mathbf{p} \in \mathcal{P}_1, \; \mathbf{q} \in \mathcal{P}_2$*, the payoff* $\boldsymbol{\pi}(\mathbf{p}, \mathbf{q}) = (\pi_1(\mathbf{p}, \mathbf{q}), \pi_2(\mathbf{p}, \mathbf{q}))$ *is defined by*

$$\begin{aligned} \pi_1(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} = \mathbf{p} A \mathbf{q}^t \\ \pi_2(\mathbf{p}, \mathbf{q}) &= \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j b_{ij} = \mathbf{p} B \mathbf{q}^t \end{aligned}$$

The values

$$v_1 = \max_{\mathbf{p}} \min_{\mathbf{q}} \pi_1(\mathbf{p}, \mathbf{q}), \quad v_2 = \max_{\mathbf{q}} \min_{\mathbf{p}} \pi_2(\mathbf{p}, \mathbf{q})$$

are called the maximin values for the two players respectively.

Example 1.3 Consider the bimatrix

$$(A, B) = \begin{bmatrix} (1,1) & (0,1) & (2,0) \\ (1,2) & (-1,-1) & (1,2) \\ (2,-1) & (1,0) & (-1,-1) \end{bmatrix}$$

The payoff matrix for the row player is

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

By eliminating the dominated row and column, it reduces to

$$\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

and by using the calculation in the previous chapter $\big(v_1 = \max_{\mathbf{p}} \big(\min_j E(\mathbf{p}, j)\big)\big)$, we have $v_1 = 1/2$.

For the column payer, we consider the transpose

$$B^t = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

and again by eliminating the dominated row and column, B^t is reduced to

$$\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

and $v_2 = -1/4$.

We see that $v_1 \neq v_2$ which is different from the zero-sum game. We use another concept of "equilibrium" that has been defined in Chapter 1. For the two-person game, it reduces to

Definition 1.2 (Nash) *In the bimatrix,* $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ *is called an equilibrium pair if* for any $\mathbf{p} \in \mathcal{P}_1,\ \mathbf{q} \in \mathcal{P}_2$,

$$\mathbf{p}A\hat{\mathbf{q}}^t \leq \hat{\mathbf{p}}A\hat{\mathbf{q}}^t, \quad \hat{\mathbf{p}}B\mathbf{q}^t \leq \hat{\mathbf{p}}B\hat{\mathbf{q}}^t$$

The following is the fundamental theorem of noncooperative games.

Theorem 1.3 (Nash) *Every bimatrix has at least one equilibrium pair.*

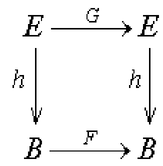
To prove the theorem, we need a well known theorem called the Brou wer fixed point theorem.

Theorem 1.4 *Let* $B = \{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq 1\}$ *be the closed unit ball. Then for any continuous function* $F : B \longrightarrow B$*, there is a fixed point in* B*, i.e.,* $F(\mathbf{x}) = \mathbf{x}$ *for some* $\mathbf{x} \in B$*.*

For our purpose, we need the fixed point theorem in a more general form.

Corollary 1.5 *Suppose* E *is homeomorphic to* B *(i.e. there exists* $h : E \longrightarrow B$ *such that* h^{-1} *exists and both* h *and* h^{-1} *are continuous). Then for any continuous* $G : E \longrightarrow E$*, there exists* $\mathbf{y} \in E$ *such that* $G(\mathbf{y}) = \mathbf{y}$*.*

Proof: Let $h : E \longrightarrow B$ be a homeomorphism. Let $F = hGh^{-1} : B \longrightarrow B$ (see the commutative diagram).



Then there exists x such that $F(\mathbf{x}) = \mathbf{x}$, i.e., $(hGh^{-1})(\mathbf{x}) = \mathbf{x}$. Let $\mathbf{y} = h^{-1}\mathbf{x}$, then $G(\mathbf{y}) = \mathbf{y}$.

$\square$

Proof of Nash's theorem: We use $\mathbf{x}$, $\mathbf{y}$ instead of $\mathbf{p}$, $\mathbf{q}$. Define

$$c_i = \max\{\mathbf{e}_i A\mathbf{y}^t - \mathbf{x}A\mathbf{y}^t,\ 0\}, \qquad 1 \le i \le m,$$

$$d_j = \max\{\mathbf{x}B\mathbf{e}_j^t - \mathbf{x}B\mathbf{y}^t,\ 0\}, \qquad 1 \le j \le n,$$

and let

$$x_i' = \frac{x_i + c_i}{1 + \sum_k c_k}, \quad y_j' = \frac{y_j + d_j}{1 + \sum_k d_k}$$

Define

$$T : \mathcal{P}_1 \times \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \times \mathcal{P}_2, \quad T(\mathbf{x}, \mathbf{y}) = (\mathbf{x}', \mathbf{y}')$$

Then T is continuous, $\mathcal{P}_1 \times \mathcal{P}_2$ is a bounded closed convex set and is homeomorphic to the closed unit ball of $\mathbb{R}^{m+n}$. Hence T has a fixed point $T(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$.

We claim that $(\mathbf{x}, \mathbf{y})$ *is a fixed point if and only if it is an equilibrium pair*:

$(\Leftarrow)$ Assume that $(\mathbf{x}, \mathbf{y})$ is an equilibrium pair, then

$$\mathbf{e}_i A\mathbf{y}^t \le \mathbf{x}A\mathbf{y}^t$$

By the definition of c_j, we have $c_j = 0$. Similarly

$$\mathbf{x}B\mathbf{e}_j^t \le \mathbf{x}B\mathbf{y}^t,$$

implies $d_j = 0$. Consequently $x'_i = x_i$, $y'_j = y_j$. i.e., $T(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$.

($\Rightarrow$) Assume that $(\mathbf{x}, \mathbf{y})$ is not an equilibrium pair, it means that there exists $\tilde{\mathbf{x}}$ such that either

$$\tilde{\mathbf{x}} A \mathbf{y}^t > \mathbf{x} A \mathbf{y}^t,$$

or there exists $\tilde{\mathbf{y}}$ such that

$$\mathbf{x} B \tilde{\mathbf{y}}^t > \mathbf{x} B \mathbf{y}^t$$

Without loss of generality, we consider the first case only (the second case is the same). Since $\tilde{\mathbf{x}} A \mathbf{y}^t$ is the probability average of $\mathbf{e}_i A \mathbf{y}^t$, one of the $\mathbf{e}_k A \mathbf{y}^t > \mathbf{x} A \mathbf{y}^t$. It follows from the definition of c_k that $c_k > 0$, hence $\sum_{k=1}^m c_k > 0$.

On the other hand consider $\mathbf{x} A \mathbf{y}^t$, being the probability average of $\mathbf{e}_i A \mathbf{y}^t$, $i = 1, \cdots, m$, there exists i such that $x_i > 0$ and $\mathbf{e}_i A \mathbf{y}^t \leq \mathbf{x} A \mathbf{y}^t$. For this i, we have $c_i = 0$. This implies that

$$x'_i = \frac{x_i}{1 + \sum_k c_k} < x_i$$

so that $\mathbf{x}' \neq \mathbf{x}$. This implies that $(\mathbf{x}, \mathbf{y})$ is not a fixed point of T. $\square$

It is easy to see that in the Prisoner's dilemma, the equilibrium pair is $(-5, -5)$, using the pure strategy "confess". But in general it is difficult to find the equilibrium pair. In the following we give a graphical method to compute the equilibrium pairs.

Example 1.4 (Battle of the buddies cont.). For the bimatrix

$$\begin{bmatrix} (5,1) & (0,0) \\ (0,0) & (1,5) \end{bmatrix}$$

Let $\mathbf{p} = [x, 1-x]$, $\mathbf{q} = [y, 1-y]$ be the two strategies.

$$\pi_1(\mathbf{p}, \mathbf{q}) = [x, 1-x] \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = 6xy - x - y + 1 = f(x, y),$$

$$\pi_2(\mathbf{p}, \mathbf{q}) = [x, 1-x] \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = 6xy - 5x - 5y + 5 = g(x, y)$$

Let

$$R_1 = \{(x, y) : f(x, y) \text{ attains maximum at } x \text{ for fixed } y\},$$

$$R_2 = \{(x, y) : g(x, y) \text{ attains maximum at } y \text{ for fixed } x\}$$

Then according to Definition 1.1, the intersection of R_1 and R_2 is an equilibrium pair.

Consider $f(x, y) = (6y - 1)x - y + 1$, it is easy to see that

$$R_1 = \begin{cases} x = 0 & \text{if } y < 1/6 \\ 0 \le x \le 1 & \text{if } y = 1/6 \\ x = 1 & \text{if } y > 1/6 \end{cases}$$

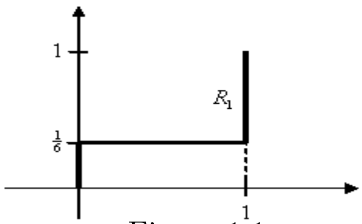


Figure 1.1

For $g(x, y) = (6x - 5)y - 5x + 5$,

$$R_2 = \begin{cases} y = 0 & \text{if } x < 5/6 \\ 0 \le y \le 1 & \text{if } x = 5/6 \\ y = 1 & \text{if } x > 5/6 \end{cases}$$

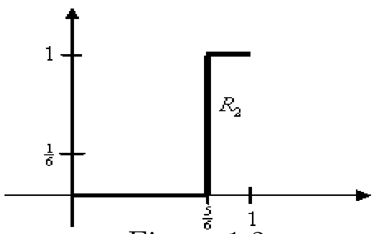


Figure 1.2

Hence the intersections of R_1 and R_2 give three equilibrium pairs.

(i) $\mathbf{p} = (5/6, 1/6)$, $\mathbf{q} = (1/6, 5/6)$; payoffs are 5/6, 5/6

(ii) $\mathbf{p} = (0, 1)$, $\mathbf{q} = (0, 1)$; payoffs are 1, 5

(iii) $\mathbf{p} = (1, 0)$, $\mathbf{q} = (1, 0)$; payoffs are 5, 1.

Note that (i) is also the solution of the maximin solution.

Example 1.5 Consider

$$\begin{bmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{bmatrix}$$

For this game

$$\begin{aligned} \pi_1(x,y) &= 4xy - x(1-y) + (1-x)(1-y) = (6y-2)x - y + 1 \\ \pi_2(x,y) &= -4xy - x(1-y) + (1-x)y = (-4x+1)y - x \end{aligned}$$

The only equilibrium pair is $\mathbf{p} = (1/4, 3/4)$, $\mathbf{q} = (1/3, 2/3)$. The expected payoff is $-2/3$ and the column player has expected payoff $-1/4$.

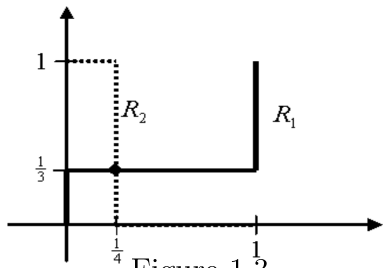


Figure 1.3

We will introduce more notations.

Definition 1.6 *For a two-person game,*

(i) *Let* $\mathbf{\Pi} = \{(\pi_1(\mathbf{p},\mathbf{q}), \pi_2(\mathbf{p},\mathbf{q})) : \mathbf{p} \in \mathcal{P}_1, \mathbf{q} \in \mathcal{P}_2\}$ *where* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are the sets of mixed strategies of the two players;* $\mathbf{\Pi}$ *is called the non-cooperative payoff region.*

(ii) *For* $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}', \mathbf{v}') \in \mathbf{\Pi}$, *We say that* $(\mathbf{u}, \mathbf{v})$ *dominates* $(\mathbf{u}', \mathbf{v}')$ *if*

$$\mathbf{u} \geq \mathbf{u}' \quad \textit{and} \quad \mathbf{v} \geq \mathbf{v}'$$

(iii) *If a payoff pair* $(\mathbf{u}, \mathbf{v})$ *is not dominated by another pair, it is called Pareto optimal.*

Definition 1.7 *Let* $(\mathbf{p}, \mathbf{q})$, $(\mathbf{r}, \mathbf{s})$ *be equilibrium pairs of mixed strategy, then*

(i) *they are said to be interchangeable if* $(\mathbf{p}, \mathbf{s})$ *and* $(\mathbf{r}, \mathbf{q})$ *are also equilibrium pairs.*

(ii) *they are said to be equivalent if* $\pi_i(\mathbf{p}, \mathbf{q}) = \pi_i(\mathbf{r}, \mathbf{s}), \quad i = 1, 2$

If the equilibrium pairs satisfy (i), (ii)*, then we say the game is solvable in the Nash sense.*

Example 1.6 (Prison's dilemma, cont.) Let

$$\begin{bmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{bmatrix}$$

Then for $\mathbf{p} = [x, 1-x]$, $\mathbf{q} = [y, 1-y]$ and

$$\begin{aligned} \pi_1(x,y) &= [x, 1-x] \begin{bmatrix} -5 & -1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = 4xy + x - 8y - 2, \\ \pi_2(x,y) &= 4xy + y - 8x - 2 \end{aligned}$$

Using the graphical method, we see that the intersection R_1 and R_2 is at $(1,1)$

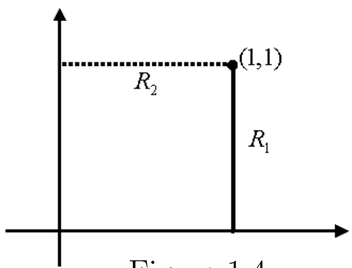


Figure 1.4

From the above expression of π_1 and π_2, we see that the payoff region is as follows.

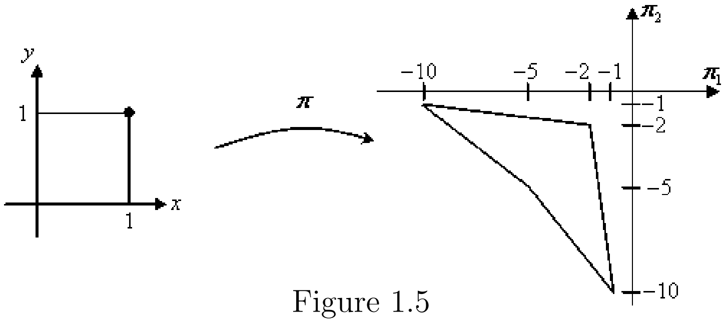


Figure 1.5

Since the game has only one equilibrium pair, it is solvable in the Nash sense, but the corresponding payoff pair $(-5,-5)$ is not Pareto optimal.

This example has the following features, which can be used in many practical cases

(i) Both players do well if they cooperate;

(ii) If one player plays the cooperative strategy and the other does not, then the defector will do well and the cooperator does badly;

(iii) Neither player trust the other.

Mathematically, we can set up the game matrix as

$$\begin{bmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{bmatrix}$$

where $c > a > d > b$ and $a > (b+c)/2$.

Example 1.7 (Battle of the buddies cont.) We recall that (Example 1.4)

$$\pi_1(x,y) = (6y-2)x - y + 1$$

$$\pi_2(x,y) = (6x-5)y - 5x + 5$$

The payoff region is as in Figure 1.6.

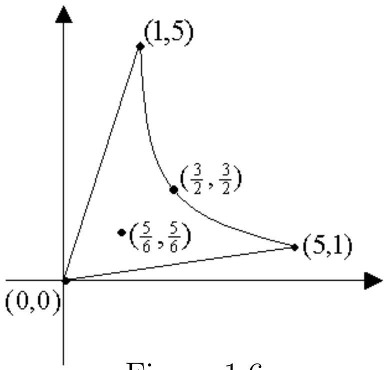


Figure 1.6

There are three equilibrium pairs $\big((1,0),(1,0)\big)$, $\big((0,1),(0,1)\big)$, $\big((5/6,1/6),(1/6,5/6)\big)$, with payoffs $(5,1)$, $(1,5)$ and $(5/6,5/6)$. The game is not Nash solvable.

By a direct calculation, the maximin value of the game is $3/2$ when they play with mixed probability $5/6$, $1/6$. The value is $(3/2,3/2)$ corresponding to the probability $1/2,1/2$ for both players. The other two equilibrium pairs have something to do with the "stubborness" of one player and the other giving in.

4.2 Cooperative Games

Definition 2.1 *Let (A, B) be an $m \times n$ payoff bimatrix. Let*

$$\mathcal{P} = \{P = [p_{ij}] : \; p_{ij} \geq 0, \sum_i \sum_j p_{ij} = 1\}$$

We call $P \in \mathcal{P}$ a joint strategy. The payoff for $P \in \mathcal{P}$ is

$$\pi_1(P) = \sum_i \sum_j p_{ij} a_{ij}, \qquad \pi_2(P) = \sum_i \sum_j p_{ij} b_{ij}$$

The cooperative region is

$$\mathcal{R} = \{\big(\pi_1(P), \pi_2(P)\big) : P \text{ is a joint strategy}\}$$

Proposition 2.1 *$\mathcal{R}$ is a bounded closed convex subset in $\mathbb{R}^{m+n}$. Moreover it is the convex hull of the points $\{(a_{ij}, b_{ij})\}_{i,j}$.*

Proof: Note that $\mathcal{P} \subseteq \mathbb{R}^{m \times n}$ is a bounded closed convex set (hence compct). The map $\boldsymbol{\pi} : \mathcal{P} \longrightarrow \mathbb{R}^2$ defined by $\boldsymbol{\pi}(P) = \big(\pi_1(P), \pi_2(P)\big)$ is affine $\big($i.e. $\boldsymbol{\pi}(\lambda P + (1-\lambda)Q) = \lambda\boldsymbol{\pi}(P) + (1-\lambda)\boldsymbol{\pi}(Q)$, where $0 \leq \lambda \leq 1$ $\big)$, hence it is continuous, the image $\boldsymbol{\pi}(\mathcal{P}) = \mathcal{R}$ is also bounded and closed (use continuous image of a compact set is compact) and is convex.

For the last statement, we note that $\mathcal{P}$ is the convex hull of the pure strategies $\{e_{ij}\}_{i,j}$, the image $\boldsymbol{\pi}(e_{ij}) = (a_{ij}, b_{ij})$. From this we conclude that $\mathcal{R}$ is the convex hull of the set of points $\{(a_{ij}, b_{ij})\}_{i,j}$. □

Consider Example 1.2, the battle of the buddies, the cooperative payoff region is as in Figure 2.1, which is larger than the non-cooperative case.

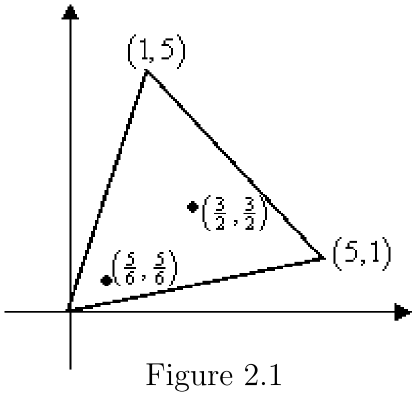


Figure 2.1

The players in a cooperative game have the chance to make agreement about which joint strategy to be adopted. There are two basic criterions.

- The payoff pair resulting from the joint strategy they agree on should be Pareto optimal.
- For each player, the game from the joint strategy should be as good as the maximin value.

Generally speaking, the more one player gets, the less the other player will be able to get. Now how much will one player be willing to give the other? How little will he be able to accept as a price of cooperation? We will set a minimum amount that a player will accept for himself, i.e., the amount he can obtain by unilateral action, whatever the other player does. The minimum value is the maximin values v_1 and v_2.

Definition 2.3 *The bargaining set for a two-person cooperative game is the set of Pareto optimal pairs* (u, v) *such that*

$$u \geq v_1, \quad v \geq v_2$$

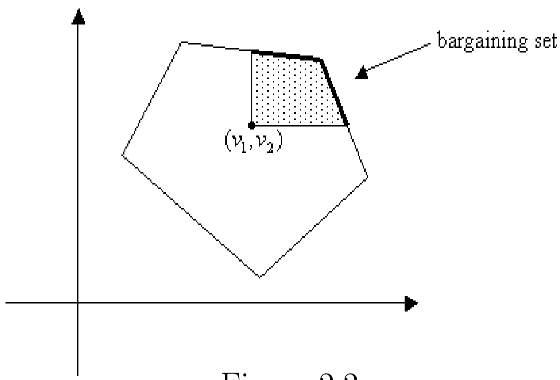


Figure 2.2

For the cooperative region in Figure 2.1, the bargaining set is the line segment joining $(1, 5)$ and $(5, 1)$. We look at another example.

Example 2.1 Suppose the bimatrix is

$$\begin{bmatrix} (2,0) & (-1,1) & (0,3) \\ (-2,-1) & (3,-1) & (0,2) \end{bmatrix}$$

Then the payoff region is the convex hull of the six payoff values. We can show that

$$v_1 = 0, \qquad v_2 = 2$$

The bargaining set is the bold face line segment as in Figure 2.3.

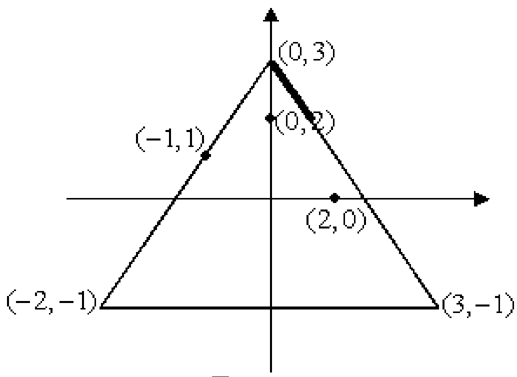


Figure 2.3

Nash established a fair method of which payoff pair in the bargaining should be agreed on. The idea is to show the existence of an "arbitration procedure" Ψ on the payoff region with respect to a "status quo" point $(u_1, v_2) \in \mathcal{P}$ which we take the maximin pair (v_1, v_2) usually.

Nash's axiom for bargaining: An *arbitration pair* is the pair $(u^*, v^*) = \Psi\big(\mathcal{R}, (u_0, v_0)\big)$ with respect to a bounded closed convex set $\mathcal{R}$ (payoff region) and a *status quo* pair $(u_0, v_0) \in \mathcal{R}$ such that the following axioms are satisfied:

(i) $u^* \geq u_0$, $v^* \geq v_0$;

(ii) (u^*, v^*) is Pareto optimal;

(iii) $(u^*, v^*) \in \mathcal{R}$;

(iv) If $\mathcal{R}'$ is another payoff region contained in $\mathcal{R}$ and contains (u_0, v_0), (u^*, v^*), then
$$\Psi\big(\mathcal{R}', (u_0, v_0)\big) = (u^*, v^*);$$

(v) Suppose $\mathcal{R}'$ is obtained by the linear transformation
$$u' = au + b, \quad v' = cv + d \quad \text{where} \quad a, c > 0,$$
then
$$\Psi\big(\mathcal{R}', (au_0 + b, cv_0 + d)\big) = (au^* + b, cv^* + d);$$

(vi) If $\mathcal{R}$ is symmetric (i.e., $(u, v) \in \mathcal{R}$ implies $(v, u) \in \mathcal{R}$) and $u_0 = v_0$, then $u^* = v^*$.

The main theorem for the cooperative game is

Theorem 2.4 (Nash) *There exists a unique arbitration procedure Ψ satisfies the above axioms.*

We will need two lemmas.

Lemma 2.5 *Let $\hat{\mathcal{R}}$ be the symmetric convex hull of $\mathcal{R}$ $\Big($i.e., $\hat{\mathcal{R}} = co\big(\mathcal{R} \cup \{(v, u) : (u, v) \in \mathcal{R}\big)\Big)$ If $u + v \leq k$ for all $(u, v) \in \mathcal{R}$, then $u + v \leq k$ for all $(u, v) \in \hat{\mathcal{R}}$.*

The proof of the above lemma follows directly from the definition of convex hull. (Please check as exercise.)

Lemma 2.6 *Let* $(u_0, v_0) \in \mathcal{R}$ *and suppose there exists* $(u, v) \in \mathcal{R}$ *such that* $u > u_0$, $v > v_0$. *Let*

$$g(u, v) = (u - u_0)(v - v_0), \quad u, v \in K$$

where $K = \{(u, v) \in \mathcal{R}\colon\ u \geq u_0,\ v \geq v_0\}$. *Then* $\max\limits_{(u,v)\in K} g(u, v)$ *is attained at a unique* (u^*, v^*).

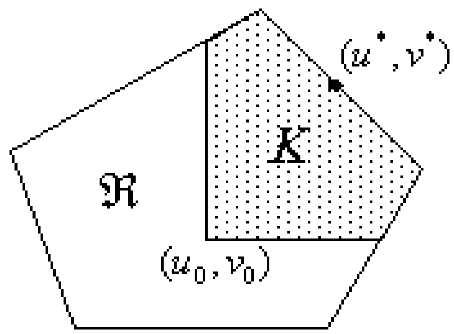


Figure 2.4

Proof: It follows from elementary analysis that there is at least one such (u^*, v^*). Suppose there exists $(\tilde{u}, \tilde{v}) \neq (u^*, v^*)$ and

$$g(u^*, v^*) = g(\tilde{u}, \tilde{v}) = \max_{u,v\in K} g(u, v) := M \tag{2.1}$$

Since the two points cannot dominate each other $\big($otherwise it will violate the maximality of g in (2,1)$\big)$, either

$$u^* > \tilde{u} \quad \text{and} \quad v^* < \tilde{v}$$

or

$$u^* < \tilde{u} \quad \text{and} \quad v^* > \tilde{v}$$

Let (u_1, v_1) be the mid-point of (u^*, v^*) and $(\tilde{u}, \tilde{v})$, then it is in K. By (2.1),

$$
\begin{aligned}
g(u_1, v_1) &= \left(\tfrac{u^*+\tilde{u}}{2} - u_0\right)\left(\tfrac{v^*+\tilde{v}}{2} - v_0\right) \\
&= \tfrac{1}{4}\big((u^* - u_0) + (\tilde{u} - u_0)\big)\big((v^* - v_0) + (\tilde{v} - v_0)\big) \\
&= \tfrac{1}{2}(u^* - u_0)(v^* - v_0) + \tfrac{1}{2}(\tilde{u} - u_0)(\tilde{v} - v_0) + \tfrac{1}{4}(u^* - \tilde{u})(\tilde{v} - v^*) \\
&> M
\end{aligned}
$$

This contradicts that M is the maximal value. □

Proof the Theorem 2.4: We first consider the case

$$U = \{(u, v) \in \mathcal{R} : u > u_0,\ v > v_0\} \neq \phi$$

(see Figure 2.4). Let $K = \overline{U}$.

Existence of Ψ: Let

$$g(u, v) = (u - u_0)(v - v_0), \quad (u, v) \in K$$

Then by Lemma 2.6, there exists a unique (u^*, v^*) such that g attains its maximum. Define

$$\Psi\big(\mathcal{R}, (u_0, v_0)\big) = (u^*, v^*) \tag{2.2}$$

We show that (2.2) satisfies all the six axioms:

(i) and (iii) are obvious.

(ii) Suppose it is not true, then there exists $(u, v) \in \mathcal{R}$ different from (u^*, v^*) and dominates (u^*, v^*). We hence have

$$(u - u_0) \geq (u^* - u_0), \quad (v - v_0) \geq (v^* - v_0)$$

and one of them is a strict inequality. Hence

$$g(u, v) > g(u^*, v^*)$$

and contradicts that g attains maximum at (u^*, v^*).

(iv) Let $K' = K \bigcap \mathcal{R}'$, it follows that the maximum of g on K' is not greater than the maximum of g on K. Since $(u^*, v^*) \in K'$ also, we have

$$\Psi\big(\mathcal{R}', (u_0, v_0)\big) = \Psi\big(\mathcal{R}, (u_0, v_0)\big)$$

(v) Note that for $(u', v') \in \mathcal{R}'$, $u' = au + b$, $v' = cv + d$ then

$$\big(u' - (au_0 + b)\big)\big(v' - (cv_0 + d)\big) = ac(u - u_0)(v - v_0)$$

and the maximum is attained at $(au^* + b, cu^* + d)$.

(iv) trivial.

Uniqueness: We show that for any $\tilde{\Psi}$ on $\mathcal{R}$ with respect to (u_0, v_0) and satisfies the six axioms, it will give the same (u^*, v^*). By applying a linear change of variables and by (v), we can assume without loss of generality $(u_0, v_0) = (0, 0)$, $(u^*, v^*) = (1, 1)$.

We claim that for $(u, v) \in K$, $u + v \leq 2$. For otherwise, $u + v > 2$, consider the line segment

$$t(u, v) + (1 - t)(1, 1) \in K, \quad 0 \leq t \leq 1$$

Let

$$h(t) = g\big(t(u, v) + (1 - t)(1, 1)\big) = \big(tu + (1 - t)\big)\big(tv + (1 - t)\big),$$

then

$$h'(t) = 2tuv + (1 - 2t)(u + v) - 2(1 - t)$$

so that $h'(0) = u + v - 2 > 0$. Since $h(0) = 1$, there exists t near 0 so that $h(t) > 1$. This contradicts the maximality that $g(1, 1) = 1$ and the claim follows.

Now let $\hat{\mathcal{R}}$ be the symmetric convex hull of $\mathcal{R}$, then $s + t \leq 2$ for all $(s, t) \in \hat{\mathcal{R}}$ (Lemma 2.5). Therefore for $(a, a) \in \hat{\mathcal{R}}$, $a \leq 1$. By axiom (vi) we have $\tilde{\Psi}\big(\hat{\mathcal{R}}, (0, 0)\big) = (1, 1)$ and by (iv) $\tilde{\Psi}\big(\mathcal{R}, (0, 0)\big) = (1, 1)$. Hence $\tilde{\Psi}\big(\mathcal{R}, (0, 0)\big) = \Psi\big(\mathcal{R}, (0, 0)\big)$.

The above completes the proof of existence and uniqueness of Ψ for the case $U \neq \phi$. It remains to consider the case $U = \phi$: there are three subcases (see Figure 2.5):

(i) there exists $(u_0, v) \in \mathcal{R}$ with $v > v_0$;

(ii) there exists $(u, u_0) \in \mathcal{R}$ with $u > u_0$;

(iii) Neither (i) nor (ii).

In case (i), we let $u^* = u_0$ and $v^* =$ the largest value of the $(u_0, v) \in \mathcal{R}$, and let $\Psi\big(\mathcal{R}, (u_0, v_0)\big) = (u_0^*, v_0^*)$. Similarly, we can define for (ii). For (iii) we let $(u^*, v^*) = (u_0, v_0)$. It is easy to show that all the conditions are satisfied.

□

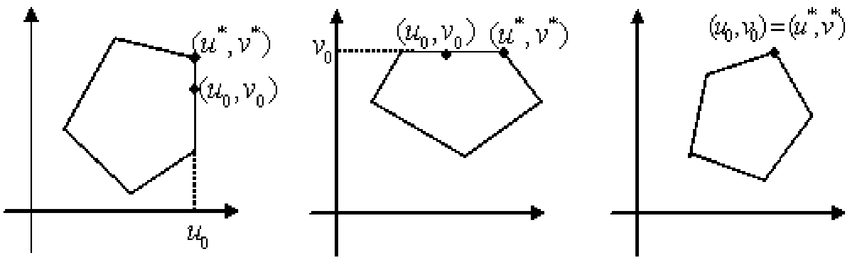


Figure 2.5

Consider Example 1.2, the battle of the buddies, the cooperative payoff region is given in Figure 2.1. The maximin pair is $(5/6, 5/6)$. The payoff region is symmetric (along the diagonal), hence the arbitration pair is (a, a) and is Pareto optimal $\big($axiom (ii)$\big)$, it follows that $(a, a) = (1, 1)$.

Example 2.2 Consider the cooperative game given by the bimatrix

$$\begin{bmatrix} (2,-1) & (-2,1) & (1,1) \\ (-1,2) & (0,2) & (1,-2) \end{bmatrix}$$

The maximin values are given by $(v_1, v_2) = (-2/5, 1)$. The payoff region is as in Figure 2.6.

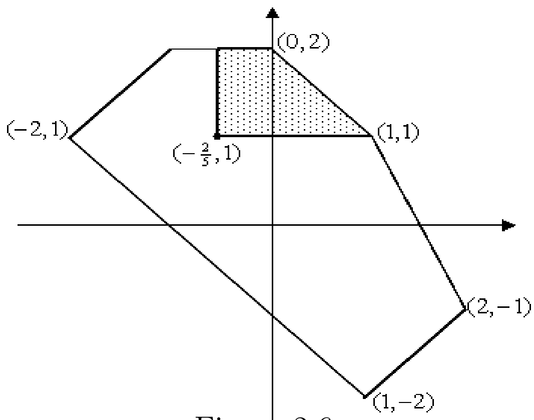


Figure 2.6

The arbitration pair is on the line $(0,2)$, $(1,1)$ which is $v = -u + 2$. Hence we need to optimize

$$g(u,v) = \big(u - (-2/5)\big)(v - 1) \quad \text{subject to} \quad v = -u + 2$$

It follows that

$$g(u,v) = (u + 2/5)(-u + 1) = -u^2 + 3u/5 + 2/5$$

and by calculus, we find that the maximum is attained at $(u,v) = (3/10, 7/10)$. This is the arbitration pair.

Example 2.3 Consider the cooperative game given by the matrix

$$\begin{bmatrix} (5,1) & (7,4) & (1,10) \\ (1,1) & (9,-2) & (5,1) \end{bmatrix}$$

The maximin pair is $(v_1, v_2) = (3,1)$. The payoff region is as indicated.

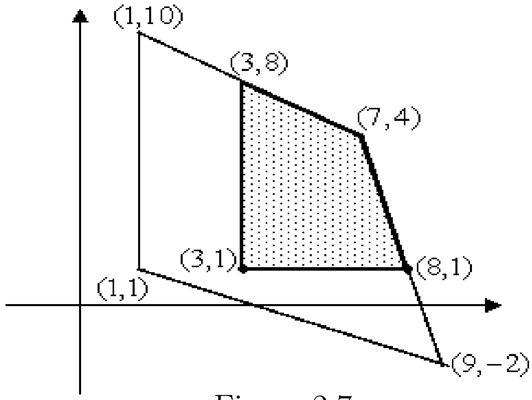


Figure 2.7

The Pareto optimal pairs are the two segments joining the three pair $(3, 8)$, $(7, 4)$ and $(8, 1)$. Consider

$$g(u, v) = (u - 3)(v - 1)$$

On the upper line segment $v = -u + 11$,

$$g(u, v) = (u - 3)(-u + 10) = -u^2 + 13u - 30$$

By calculus, g attains maximum at $u = 13/2$, $v = 9/2$ and $g(13/2, 9/2) = 49/7$.

On the lower line segment, $v = -3u + 25$,

$$g(u, v) = (u - 3)(-3u + 24) = -3u^2 + 33u - 72$$

Then g attains maximum at $u = 11/2$, $v = 17/2$ which is outside the bargaining set.

It follows that the arbitration point of the game is at $(13/2, 9/2)$.

Exercises

1. In Battle of the Buddies, suppose the row player plays the mixed strategy $(\frac{1}{4}, \frac{3}{4})$. What is the best way for the column player to play in response?

2. For the game Battle of the Buddies, verify that $\frac{5}{6}$ is the maximin value for each player.

3. For the game given by the bi-matrix

$$\begin{bmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{bmatrix},$$

   verify that the maximin value for the row player is $2/3$, and for the column player is -1.

4. For the bi-matrix
$$\begin{bmatrix} (2,-3) & (-1,3) \\ (0,1) & (1,-2) \end{bmatrix},$$
   compute the equilibrium pairs and the maximin values for both players.

5. For the bi-matrix
$$\begin{bmatrix} (2,-1) & (-1,1) \\ (0,2) & (1,-1) \end{bmatrix}$$
   find the maximin values and the equilibrium pairs of the mixed strategies.

6. For the game with bi-matrix

$$\begin{bmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{bmatrix},$$

   prove that $(1,0)$ is a Pareto optimal payoff pair.

7. Find the arbitration pair for the cooperative game described by the bi-matrix in Exercise 4.

8. Find the arbitration pair for the bi-matrix game

$$\begin{bmatrix} (-1,-1) & (4,0) \\ (0,4) & (-1,-1) \end{bmatrix}$$

9. Find the arbitration pair for the bi-matrix game

$$\begin{bmatrix} (-1,1) & (0,0) \\ (1,-1) & (0,1) \\ (-1,-1) & (1,1) \end{bmatrix}$$

Chapter 5

N-person Cooperative Games

For the N-person non-cooperative game, there is little difference between two-person and N-person game, and Nash's theorem still hold. For the N-person cooperative game, players not only make binding agreement of join strategies but also agree to pool their individual payoffs and redistribute the total in a specific way. Hence the emphasis is on the improvement of the payoff by coalition instead of the mixed strategies.

5.1 Coalition

Definition 1.1 *Let $\mathcal{A} = \{A_1, \cdots, A_N\}$ be N-players, a subset $\mathcal{S} \subseteq \mathcal{A}$ is called a coalition; the complement of $\mathcal{S}$, denoted by $\mathcal{S}^c$, is called the counter coalition.*

The pure strategies available to a coalitions $\mathcal{S}$ are the cartisian product of the X_i, the strategies of $A_i \in \mathcal{S}$. The payoff of a strategy in $\mathcal{S}$ is the sum of the payoffs of the strategies of each $A_i \in \mathcal{S}$.

Definition 1.2 *By the characteristic function ν of a N-person game, we mean a real valued function ν on the coalition $\mathcal{S} \subseteq \mathcal{A}$ such that $\nu(\mathcal{S})$ is the maximin value (with respective to $\mathcal{S}$ and $\mathcal{S}^c$ as in the two-person game).*

Example 1.1 Consider a 3-player game, A_1, A_2, A_3 are players with payoffs as follows:

strategies	payoffs
$(1,1,1)$	$(-2,1,2)$
$(1,1,2)$	$(1,1,-1)$
$(1,2,1)$	$(0,-1,2)$
$(1,2,2)$	$(-1,2,0)$
$(2,1,1)$	$(1,-1,1)$
$(2,1,2)$	$(0,0,1)$
$(2,2,1)$	$(1,0,0)$
$(2,2,2)$	$(1,2,-2)$

There are eight coalitions (including $\emptyset$). If we consider $\mathcal{S} = \{A_1, A_3\}$, $\mathcal{S}^c = \{A_2\}$. Then $\mathcal{S}$ has four strategies and $\mathcal{S}^c$ has two strategies. The payoff bimatrix is

	1	2
$(1,1)$	$(0,1)$	$(2,-1)$
$(1,2)$	$(0,1)$	$(-1,2)$
$(2,1)$	$(2,-1)$	$(1,0)$
$(2,2)$	$(1,0)$	$(-1,2)$

The maximin values can be computed from the matrix

$$\begin{bmatrix} 0 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & 0 & 2 \end{bmatrix}.$$

They are

$$\nu\big(\{A_1, A_3\}\big) = 4/3, \qquad \nu\big(\{A_2\}\big) = -1/3.$$

Similarly we have

$$\nu\big(\{A_1, A_2\}\big) = 1, \qquad \nu\big(\{A_3\}\big) = 0,$$

$$\nu\big(\{A_2, A_3\}\big) = 3/4, \qquad \nu\big(\{A_1\}\big) = 1/4,$$

$$\nu\big(\{A_1, A_2, A_3\}\big) = 1, \qquad \nu\big(\emptyset\big) = 0.$$

The coalition $\mathcal{A} = \{A_1, A_2, A_3\}$ is the "grand" coalition; $\nu(\mathcal{A})$ is the largest payoff the coalition that the grand coalition can achieve. Also from the above values, we can speculate about which coalitions are likely to form: A_2, A_3 may ask A_1 to form a coalition; A_1 will certainly demand more than 1/4. If he demands too much, then A_2 and A_3 will form a coalition themselves.

Example 1.2 (The Used Car Game). A_1 wishes to sell an old car which is worth nothing to him. A_2 values the car at \$500, A_3 values the car at \$700. They bid for the car and A_1 either accepts one of the bidding, or rejects both of them if the bidding is too low.

In this game, since there are all the variations of bidding, it is difficult to write down the "normal" form of the game. Instead, we represent it by the "characteristic function" $\nu(\mathcal{S})$ of the different coalitions.

$$\nu\big(\{A_1\}\big) = \nu\big(\{A_2\}\big) = \nu\big(\{A_3\}\big) = 0;$$

$$\nu\big(\{A_1, A_2\}\big) = 500, \quad \nu\big(\{A_1, A_3\}\big) = 700, \quad \nu\big(\{A_2, A_3\}\big) = 0;$$

$$\nu\big(\{A_1, A_2, A_3\}\big) = 700.$$

The reason behind these numbers are as follows: For A_1, he either
(i) accepts the higher bid, or
(ii) rejects both if the biddings are too low.
There exists a joint strategy for the counter coalition $\{A_2, A_3\}$ to bid 0. Hence the maximin value for A_1 is 0, i.e., $\nu\big(\{A_1\}\big) = 0$.

For A_2, A_3, the maximin values are $\nu\big(\{A_2\}\big)$, $\nu\big(\{A_3\}\big) = 0$ because the counter coalition can reject the bidding.

For the coalition $\{A_1, A_2\}$, the payoff can be \$500 independent of what A_3 does; similarly we can argue for $\{A_1, A_3\}$. The case $\nu(\{A_2, A_3\}) = 0$ is by considering the payoff as the value of the sum minus the money paid.

The grand coalition is the largest sum of the payoffs of a 3-tuple strategy, hence $\nu(\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}) = 700$.

Theorem 1.3 *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be disjoint coalitions, then*

$$\nu(\mathcal{S}_1 \cup \mathcal{S}_2) \geq \nu(\mathcal{S}_1) + \nu(\mathcal{S}_2), \qquad (superadditivity)$$

If $\mathcal{S}_1, \cdots \mathcal{S}_k$ are pairwise disjoint coalitions, then

$$\nu\big(\bigcup_{i=1}^{k} \mathcal{S}_i\big) \geq \sum_{i=1}^{k} \nu(\mathcal{S}_i).$$

In particular,

$$\nu(\mathcal{A}) \geq \sum_{i=1}^{N} \nu\big(\{\mathcal{A}_i\}\big).$$

Proof: Let $\mathbf{p}_1$, $\mathbf{p}_2$ be the mixed strategies for $\mathcal{S}_1$ and $\mathcal{S}_2$ to attain the maximin values $\nu(\mathcal{S}_1)$, $\nu(\mathcal{S}_2)$. For the coalition $\mathcal{S}_1 \cup \mathcal{S}_2$, since they are disjoint, we can use the same strategies on $\mathcal{S}_1 \cup \mathcal{S}_2$, and the payoff will be at least $\nu(\mathcal{S}_1) + \nu(\mathcal{S}_2)$. By taking maximum over all possible strategies on $\mathcal{S}_1 \cup \mathcal{S}_2$, we see that

$$\nu(\mathcal{S}_1 \cup \mathcal{S}_2) \geq \nu(\mathcal{S}_1) \cup \nu(\mathcal{S}_2).$$

The second inequality is proved by induction, and the last inequality follows by taking $\mathcal{S}_i = \{A_i\}$. □

For an N-person cooperative game, it is more convenient to consider the characteristic function $\nu(\mathcal{S})$ (as in Example 1.2) than the normal form (as in Example 1.1).

Definition 1.4 *A game in characteristic function form consists of a set $\mathcal{A} = \{A_1, \cdots, A_N\}$ of players together with a function ν defined on subsets $\mathcal{S} \subseteq \mathcal{A}$ such that $\nu(\emptyset) = 0$ and for any disjoint $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{A}$*

$$\nu(\mathcal{S}_1 \cup \mathcal{S}_2) \geq \nu(\mathcal{S}_1) \cup \nu(\mathcal{S}_2).$$

Definition 1.5 *A game in characteristic function form is essential if*

$$\nu(\mathcal{A}) > \sum_{i=1}^{N} \nu(\{A_i\});$$

otherwise $\big($*i.e.*, $\nu(\mathcal{A}) = \sum_{i=1}^{N} \nu(\{A_i\})\big)$, *it is called inessential.*

It is easy to see from the definition that in an inessential game,

$$\nu(\mathcal{A}) = \sum_{A_i \in \mathcal{S}} \nu(\{A_i\}).$$

Hence there is no reason for any other coalition, as cooperation does not result in a greater total payoff.

5.2 Imputations

In here we are concern with the distributions of the payoff. The amount going to the various players form an N-vector $\mathbf{x}$.

Definition 2.1 *An N-vector is said to be an imputation if*

(i) $x_i \geq \nu(\{A_i\})$ *for each player* A_i*;*

(ii) $\sum_{i=1}^{N} x_i = \nu(\mathcal{A})$.

We denote the set of imputations by $I(\nu)$.

Theorem 2.2 *If ν is inessential, then $I(\nu)$ is a singleton, namely*

$$\mathbf{x} = \big(\nu(A_1), \cdots, \nu(A_N)\big).$$

If ν is essential, then $I(\nu)$ is an infinite set.

Proof: If ν is inessential, then for any $\mathbf{x} \in I(\nu)$,

$$\nu(\mathcal{A}) = \sum_{i=1}^{N} \nu(\{A_i\}) \leq \sum_{i=1}^{N} x_i = \nu(\mathcal{A}).$$

This implies $x_i = \nu(\{A_i\})$, hence $I(\nu)$ is a singleton.

If ν is essential, let

$$\beta = \nu(\mathcal{A}) - \sum_{i=1}^{N} \nu(\{A_i\}) > 0$$

and let

$$x_i = \nu(\{A_i\}) + \alpha_i \quad \text{where} \quad \sum_{i=1}^{N} \alpha_i = \beta.$$

Then $\mathbf{x}$ is an imputation. Since there are infinitely many ways of choosing $\{\alpha_i\}_{i=1}^{N}$, there are infinitely many $\mathbf{x} \in I(\nu)$. $\square$

Definition 2.3 *Let* $\mathbf{x}, \mathbf{y} \in I(\nu)$*, we say that* $\mathbf{x}$ *dominates* $\mathbf{y}$ *through a coalition* $\mathcal{S}$*, denoted by* $\mathbf{x} \succ_{\mathcal{S}} \mathbf{y}$*, if*

(i) $x_i > y_i$ *for all* $A_i \in \mathcal{S}$*;*

(ii) $\sum_{A_i \in \mathcal{S}} x_i \leq \nu(\mathcal{S})$.

Example 2.1 (Example 1.1 cont.) We see that $(1/3, 1/3, 1/3)$ dominates $(1, 0, 0)$ through the coaliation $\{A_2, A_3\}$ and that $(1/4, 3/8, 3/8)$ dominates $(1/3, 1/3, 1/3)$ through the same coalition. Also, $(1/2, 1/2, 0)$ dominates $(1/3, 1/3, 1/3)$ through $\{A_1, A_2\}$

Definition 2.4 *The core* $C(\nu)$ *of* ν *consists of all imputations that are not dominated by any other imputations through any coalition.*

Theorem 2.5 *Let* $\mathbf{x} \in I(\nu)$*, then* $\mathbf{x} \in C(\nu)$ *if and only if*

$$\sum_{A_i \in \mathcal{S}} x_i \geq \nu(\mathcal{S}) \qquad \forall\ \mathcal{S} \subseteq \mathcal{A}.$$

Proof: ($\Leftarrow$) Suppose there exists $\mathbf{z} \in I(\nu)$ dominates $\mathbf{x}$ through a coalition $\mathcal{S}$, then

$$\sum_{A_i \in \mathcal{S}} z_i > \sum_{A_i \in \mathcal{S}} x_i \geq \nu(\mathcal{S}).$$

which contradicts Definition 2.3(ii). Thus $\mathbf{x} \in C(\nu)$.

($\Rightarrow$) Suppose $\mathbf{x} \in C(\nu)$. We assume in contrary that there exists $\mathcal{S}$ such that

$$\sum_{A_i \in \mathcal{S}} x_i < \nu(\mathcal{S}).$$

Then $\mathcal{S} \neq \mathcal{A}$ (by Definition 2.1). We also see that there exists $A_j \in \mathcal{S}^c$ such that

$$x_j > \nu(\{A_i\}).$$

(For otherwise, using the superadditivity,

$$\sum_{i=1}^{N} x_i < \nu(\mathcal{S}) + \sum_{A_i \in \mathcal{S}^c} x_i \le \nu(\mathcal{A}),$$

which is impossible by Definition 2.1.) Choose α so that

$$0 < \alpha \le x_j - \nu(\{A_i\}) \quad \textit{and} \quad \alpha \le \nu(\mathcal{S}) - \sum_{A_i \in \mathcal{S}} x_i.$$

Now let k denote the number of players in $\mathcal{S}$ and consider a new imputation $\mathbf{z}$:

$$z_i = \begin{cases} x_i + \alpha/k & \text{for } A_i \in \mathcal{S}; \\ x_j - \alpha & \text{for } i = j; \\ x_i & \text{for } A_i \in \mathcal{S}^c,\ i \ne j. \end{cases}$$

Then $\mathbf{z} \in I(\nu)$ dominates $\mathbf{x}$ through $\mathcal{S}$, and this contradicts the assumption of $\mathbf{x}$. Hence we conclude that

$$\sum_{A_i \in \mathcal{S}} x_i \ge \nu(\mathcal{S}).$$

□

In view of the definition and the theorem, we see that for $\mathbf{x} \in C(\nu)$, there is no group of players has reason to form a coalition and replace $\mathbf{x}$ with a different imputation. The "solution" of the game is to use the grand coalition and share the payoff in the way of $\mathbf{x}$.

Corollary 2.6 *Let* $\mathbf{x}$ *be an* N*-vector, then* $\mathbf{x} \in C(\nu)$ *if and only if*

$$\sum_{i=1}^{N} x_i = \nu(\mathcal{A}) \quad \textit{and} \quad \sum_{A_i \in \mathcal{S}} x_i \ge \nu(\mathcal{S}) \textit{ for every coalition } \mathcal{S}.$$

It follows that $C(\nu)$ is determined by the linear inequalities and is hence a bounded closed convex set.

Example 2.2 (Example 1.1 cont.) A vector $\mathbf{x} \in C(\nu)$ if and only if

$$x_1 + x_2 + x_3 = 1,$$

$$x_1 \geq 1/4, \quad x_2 \geq -1/3, \quad x_3 \geq 0,$$

$$x_1 + x_2 \geq 1, \quad x_1 + x_3 \geq 4/3, \quad x_2 + x_3 \geq 3/4.$$

It is easy to check that there is no solution. Hence $C(\nu) = \phi$.

Example 2.3 Consider a 3-player game whose characteristic function is

$$\nu(\{A_1\}) = -1/2, \quad \nu(\{A_2\}) = 0, \quad \nu(\{A_3\}) = -1/2,$$

$$\nu(\{A_1, A_2\}) = 1/4, \quad \nu(\{A_1, A_3\}) = 0, \quad \nu(\{A_2, A_3\}) = 1/2,$$

$$\nu(\{A_1, A_2, A_3\}) = 1.$$

It can be checked that the superadditivity holds. Hence $\mathbf{x} \in C(\nu)$ if and only if

$$x_1 \geq -1/2, \quad x_2 \geq 0, \quad x_3 \geq -1/2,$$

$$x_1 + x_2 \geq 1/4, \quad x_1 + x_3 \geq 0, \quad x_2 + x_3 \geq 1/2,$$

$$x_1 + x_2 + x_3 = 1.$$

The system has infinitely many solutions, e.g., $(1/3, 1/3, 1/3) \in C(\nu)$.

Example 2.4 (The Used Car Game cont.) A vector $\mathbf{x} = (x_1, x_2, x_3) \in C(\nu)$ if and only if

$$x_1 + x_2 + x_3 = 700, \quad x_i \geq 0,$$

$$x_1 + x_2 \geq 500, \quad x_1 + x_3 \geq 700, \quad x_2 + x_3 \geq 0.$$

It follows that

$$500 \leq x_1 \leq 700, \quad x_2 = 0, \quad x_3 = 700 - x_1.$$

The interpretation is that A_3 gets the car with a bid between \$500 to \$700; A_2 does not get the car, but his presence force the price up over \$500.

Since the game is cooperative, it is possible for A_2 and A_3 to compromise to bid low, say A_3 bids \$300 and A_2 bids \$0; if A_1 accepts the bid, then A_3 will pay A_2 \$200. The imputation in this arrangement is $(300, 200, 200)$, it is not in the core. If A_1 reject the bid from A_3, then the payoff is $(0, 0, 0)$, it is not an imputation.

Example 2.5 (Voting Game) In a small city, there is a Mayor and a city council with seven members. A bill can be passed to a law if

(i) The majority of council member passes and the Major signs it;

(ii) The council passes it but the Mayor vetos it, then the council votes again and at least six council members vote to override the veto.

The payoffs would be units of "power" gained by being on the winning side. For a coalition $\nu(\mathcal{S}) = 1$ if it is a "winning" coalition, $\nu(\mathcal{S}) = 0$ if it is a "losing" coalition.

It is clear that any one-person coalition is losing and the grand coalition is winning. Moreover $(x_M, x_1, \cdots, x_7)$ is an imputation if

$$x_M, x_1, \cdots, x_7 \geq 0 \quad \text{and} \quad x_M + x_1 + \cdots + x_7 = 1.$$

We claim that $C(\nu) = \emptyset$: For if otherwise, $(x_M, x_1, \cdots, x_7) \in C(\nu)$, then

$$x_M + x_1 + \cdots + x_7 = 1$$

and for each j,

$$\sum_{i \neq j} x_i \geq 1.$$

(because any six council member is a winning coalition). Hence x_M and all $x_i = 0$, it is a contradiction so that $C(\nu) = \emptyset$.

Let ν be the characteristic function of a game, we say that ν is *constant sum* if

$$\nu(\mathcal{S}) + \nu(\mathcal{S}^c) = \nu(\mathcal{A})$$

for every coalition $\mathcal{S}$; ν is called zero-sum if in addition $\nu(\mathcal{A}) = 0$.

Proposition 2.7 *Let π be the payoffs of an N-person game in normal form and*

$$\sum_{i=1}^{N} \pi_i(\sigma_1, \cdots, \sigma_N) = c$$

for all choice of strategies $(\sigma_1, \cdots, \sigma_N)$ (call it constant-sum game). Then its characteristic function ν is of constant sum.

Proof: We first consider the special case $c = 0$, it is easy to see that $\nu(\mathcal{A}) = 0$. For any coalition $\mathcal{S}$ and counter coalition $\mathcal{S}^c$, they form a zero-sum game. Hence $\nu(\mathcal{S}) = -\nu(\mathcal{S}^c)$ according to the minimax theorem. It follows that

$$\nu(\mathcal{S}) + \nu(\mathcal{S}^c) = 0 = \nu(\mathcal{A}).$$

For $c \neq 0$, we define the new game with payoff

$$\tau_i(\sigma_1, \cdots, \sigma_N) = \pi_i(\sigma_1, \cdots, \sigma_N) - c/N.$$

Then it is a zero-sum game. The corresponding characteristic function μ is given by

$$\mu(\mathcal{S}) = \nu(\mathcal{S}) - kc/N$$

where k is the number of players in $\mathcal{S}$, hence μ is zero sum (by the case $c = 0$ just proved). It follows from

$$\nu(\mathcal{S}) + \nu(\mathcal{S}^c) = \big(\mu(\mathcal{S}) + kc/N\big) + \big(\mu(\mathcal{S}^c) + (N-k)c/N\big) = c$$

that ν is constant sum. □

It follows from Theorem 2.2 and 2.5 that if ν is inessential, then $C(\nu)$ contains a unique imputation $\mathbf{x} = \big(\nu(A_1), \cdots, \nu(A_N)\big)$. In general it is not easy to determine $C(\nu)$. We have the following "negative" result.

Theorem 2.8 *If ν is both essential and constant-sum, then $C(\nu) = \emptyset$.*

Proof: Suppose there exists $\mathbf{x} \in C(\nu)$, then for such j,

$$x_j \geq \nu(\{A_j\}) \quad \text{and} \quad \sum_{i \neq j} x_i \geq \nu(\{A_j\}^c).$$

Hence

$$\nu(\mathcal{A}) = \sum_{i=1}^{N} x_i \geq \nu(\{A_i\}) + \nu(\{A_i\}^c) = \nu(\mathcal{A}).$$

(the last equality is by the constant sum property). This implies $\nu(\mathcal{A}) = \sum_{i=1}^{N} \nu(\{A_i\})$ and ν is hence inessential, a contradiction. $\square$

As is seen in the proof of Proposition 2.7, we can adjust constants to the game and might simplify the game. For this we define

Definition 2.9 *Two characteristic functions μ and ν are strategically equivalent if there exists $k > 0$ and $c_1, \cdots, c_N$ such that*

$$\mu(\mathcal{S}) = k\nu(\mathcal{S}) + \sum_{A_i \in \mathcal{S}} c_i \qquad \textit{for all coalition } \mathcal{S} \tag{2.1}$$

It is an easy exercise to prove the following.

Theorem 2.10 *Suppose μ, ν are strategically equivalent, then*

(i) they are both essential or both inessential;

(ii) $\mathbf{x} \in I(\nu)$ if and only if $\mathbf{y} = k\mathbf{x} + \mathbf{c} \in I(\mu)$;

(iii) $\mathbf{x} \in C(\nu)$ if and only if $\mathbf{y} = k\mathbf{x} + \mathbf{c} \in C(\mu)$.

Note that if ν is an essential characteristic function, then it can be reduced to a $(0, 1)$-form, i.e.,

$$\mu(\{A_i\}) = 0 \quad \text{and} \quad \mu(\mathcal{A}) = 1.$$

Indeed we can take

$$k = 1/\big(\nu(\mathcal{A}) - \sum_{i=1}^{N} \nu(\{A_i\})\big) > 0$$

and

$$c_i = -k\nu(\{A_i\}), \quad i = 1, \cdots, N,$$

then use the transform in Definition 2.6.

Example 2.6 (Example 1.1 cont.) We can use that above to reduced the characteristic function ν:

$$k = 1/\big(1 - \big(-\frac{1}{12}\big)\big) = 12/13$$

and let $c_i = -k\nu(\{A_i\})$, then

$$c_1 = -3/13, \quad c_2 = 1/13, \quad c_3 = 0.$$

It follows from (2.1) that

$$\begin{aligned} &\mu(\mathcal{A}) = 1; \quad \mu(\{A_1\}) = \mu(\{A_2\}) = \mu(\{A_3\}) = 0; \\ &\mu(\{A_1, A_2\}) = 1, \quad \mu(\{A_1, A_3\}) = 1, \quad \mu(\{A_2, A_3\}) = 1. \end{aligned} \tag{2.2}$$

We conclude that one of the two players will form the coalition. The prevailing imputation will be either $(1/2, 1/2, 0), (1/2, 0, 1/2), (0, 1/2, 1/2)$.

Note that for any three-player game, a constant sum, inessential characteristic function can be reduced to the form in (2.1) (Check this). For brevity we call the game in (2.2) the game THREE.

Example 2.7 (Example 2.2 cont.) By the same reduction, we obtain

$$\mu(\mathcal{A}) = 1; \qquad \mu(\{A_i\}) = 0, \quad i = 1, 2, 3;$$

$$\mu(\{A_1, A_2\}) = 3/8, \quad \mu(\{A_1, A_3\}) = 1/2, \quad \mu(\{A_2, A_3\}) = 1/2.$$

In this case the two-player coalitions seem weak and the grand coalition is likely to form. In fact we see that in the grand coalition, the imputation $(1/3, 1/3, 1/3) \in C(\nu)$.

5.3 Two solution concepts

The core $C(\nu)$ as a solution concept for the game is not very satisfactory. From the examples in the last section, we see that it might have no imputation in it ($C(\nu) = \emptyset$) or so many that we have no reasonable way to decide which are actually likely to occur. In the following we will consider two more concepts as solution for the coalitions.

Definition 3.1 *Let $X \subseteq I(\nu)$, we say that X is stable if*

(i) *for $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x}$ and $\mathbf{y}$ do not dominate each other in any coalition $\mathcal{S}$.*

(ii) *for $\mathbf{y} \notin X$, there exists a coalition $\mathcal{S}$ and $\mathbf{x} \in X$ such that $\mathbf{x}$ dominates $\mathbf{y}$ through $\mathcal{S}$.*

Example 3.1 Consider the game THREE with characteristic function defined by (2.1). Then the set

$$X = \{(0, 1/2, 1/2), (1/2, 0, 1/2), (1/2, 1/2, 0)\}$$

is a stable set for μ. Indeed condition (i) follows from checking Definition 2.2. To prove (ii), we observe that if $\mathbf{y} \notin X$, then there exists at least two coordinates, say, $y_1, y_2 \leq 1/2$. This implies that $\mathbf{y}$ is dominated by $(1/2, 1/2, 0)$ through $\mathcal{S} = \{A_1, A_2\}$.

Note that there is $\mathbf{x} \in I(\nu)$ dominates member of X, e.g., $(2/3, 1/3, 0)$ dominates $(1/2, 0, 1/2)$ through $\mathcal{S}_1 = \{A_1, A_2\}$. On the other hand $(2/3, 1/3, 0)$ is dominated by $(0, 1/2, 1/2)$ through $\mathcal{S}_2 = \{A_2, A_3\}$.

There are other stable sets. For example for $0 \leq c < 1/2$,

$$X_c = \{(c, x_2, x_3) : \; x_2, x_3 \geq 0, \quad x_2 + x_3 = 1 - c\}$$

is such a set. It means give A_1 a fixed amount c and A_2, A_3 negotiate among themselves to divide the rest. The reader should try to show that X_c is a stable set as an exercise.

The second solution concept is that of the Shapley value.

Definition 3.2 *Let ν be characteristic function and let $\mathcal{S}$ be a coalition. For $A_i \in \mathcal{S}$, we define*

$$\delta(A_i, \mathcal{S}) = \nu(\mathcal{S}) - \nu(\mathcal{S}\backslash\{A_i\}).$$

The Shapley value of A_i is defined as

$$\phi_i = \sum_{\{\mathcal{S}:\ A_i \in \mathcal{S}\}} \frac{(N-|\mathcal{S}|)!\ (|\mathcal{S}|-1)!}{N!} \delta(A_i, \mathcal{S}).$$

The vector $\boldsymbol{\phi} = (\phi_1, \cdots \phi_N)$ is called the Shapley vector of the game.

The term $\delta(A_i, \mathcal{S})$ measures the amount A_i contributed to the coalition $\mathcal{S}$. In the expression ϕ_i, the factorial expression is the "probability" of the $\mathcal{S}$ that contains A_i. Note that

$$\sum_{A_i \in \mathcal{S}} \frac{(N-|\mathcal{S}|)!\ (|\mathcal{S}|-1)!}{N!} = 1. \tag{3.1}$$

The Shapley value can be interpreted as the bargaining power of the players.

Theorem 3.3 *The Shapley vector is an imputation, i.e., $\boldsymbol{\phi} \in I(\nu)$.*

Proof: (i) We show that $\phi_i \geq \nu(\{A_i\})$: By the superadditivity, if $A_i \in \mathcal{S}$,

$$\delta(A_i, \mathcal{S}) = \nu(\mathcal{S}) - \nu(\mathcal{S}\backslash\{A_i\}) \geq \nu(\{A_i\}).$$

Hence by (3.1),

$$\sum_{A_i \in \mathcal{S}} \frac{(N-|\mathcal{S}|)!\ (|\mathcal{S}|-1)!}{N!} \delta(A_i, \mathcal{S}) \geq \nu(\{A_i\}).$$

(ii) We show that $\sum_{i=1}^N \phi_i = \nu(\mathcal{A})$. Note that

$$\begin{aligned}
&\textstyle\sum_{i=1}^N \phi_i \\
=\ &\textstyle\sum_{i=1}^N \sum_{\{\mathcal{S}:\ A_i \in \mathcal{S}\}} \frac{(N-|\mathcal{S}|)!\ (|\mathcal{S}|-1)!}{N!} \big(\nu(\mathcal{S}) - \nu(\mathcal{S}\backslash\{A_i\})\big) \\
=\ &\textstyle\sum_{\mathcal{S}} |\mathcal{S}|\ \frac{(N-|\mathcal{S}|)!\ (|\mathcal{S}|-1)!}{N!} \nu(\mathcal{S}) - \sum_{\mathcal{T} \neq \mathcal{A}} (N - |\mathcal{T}|)\ \frac{(N-1-|\mathcal{T}|)!\ |\mathcal{T}|!}{N!} \nu(\mathcal{T}) \\
=\ &\textstyle\sum_{\mathcal{S}}\ \frac{(N-|\mathcal{S}|)!\ |\mathcal{S}|!}{N!} \nu(\mathcal{S}) - \sum_{\mathcal{T} \neq \mathcal{A}}\ \frac{(N-|\mathcal{T}|)!\ |\mathcal{T}|!}{N!} \nu(\mathcal{T}) \\
=\ &N\ \frac{0!\ (N-1)!}{N!} \nu(\mathcal{A})\ =\ \nu(\mathcal{A}).
\end{aligned}$$

□

Example 3.2 (Example 1.1 cont.) To compute ϕ_1, we see that there are four coalitions containing A_1:

$$\{A_1\}, \quad \{A_1, A_2\}, \quad \{A_1, A_3\}, \quad \{A_1, A_2, A_3\}.$$

Hence using the values of ν in Example 1.1, we have

$$\delta(A_1, \{A_1\}) = 1/4 - 0 = 1/4, \qquad \delta(A_1, \{A_1, A_2\}) = 1 - (-1/3) = 4/3,$$

$$\delta(A_1, \{A_1, A_3\}) = 4/3 - 0 = 4/3, \quad \delta(A_1, \{A_1, A_2, A_3\}) = 1 - 3/4 = 1/4.$$

Then

$$\phi_1 = \frac{2!\,0!}{3!} \cdot \frac{1}{4} + \frac{1!\,1!}{3!} \cdot \frac{4}{3} + \frac{1!\,1!}{3!} \cdot \frac{4}{3} + \frac{0!\,2!}{3!} \cdot \frac{1}{4} = \frac{11}{18}.$$

By a similar calculation, we have

$$\phi_2 = 1/36, \quad \phi_3 = 13/36.$$

Note that ϕ_1 is largest, hence A_1 has the strongest bargaining power, followed by A_3, then A_2.

We also have a shorter way to calculate the Shapley vector, it is to transform the game to THREE as in Example 2.6 first. By symmetry, the Shapley vector is $(1/3, 1/3, 1/3)$. Then transform this vector back to obtain $(11/18, 1/36, 13/36)$.

Example 3.3 (The Used Car Game cont.) We can use the same calculation to show that

$$\phi_1 = 433.33, \quad \phi_2 = 83.33, \quad \phi_3 = 183.33.$$

Thus, A_3 gets the car for \$433.33, but he has to pay A_2 \$83.33 as a bribe for not bidding against him. A_1 is in the most powerful bargaining position.

Example 3.4 (The Voting Game cont.) We first compute ϕ_M. The nonzero terms in ϕ_M consists of coalitions that $\mathcal{S}\backslash\{M\}$ is a losing but $\mathcal{S}$ is winning. They are

(i) $\mathcal{S}$ contains the Mayor and four council members;

(ii) $\mathcal{S}$ contains the Mayor and five council members.

There are $\binom{7}{4} = 35$ ways in (i), hence the contribution to ϕ_M is

$$35 \cdot \frac{(8-5)!\,(5-1)!}{8!} = 7/56$$

There are $\binom{7}{5} = 21$ ways in (ii), hence the contribution to ϕ_M is

$$21 \cdot \frac{(8-6)!\,(6-1)!}{8!} = 7/56.$$

Adding these we have $\phi_M = 1/4$.

For the council members, the non-zero terms in ϕ_i corresponds to the coalitions are

(i) $\mathcal{S}$ contains the Mayor and four council members.

(ii) $\mathcal{S}$ contains six council members.

There are 20 sets in the first case and 6 in the second. Hence

$$\phi_i = 20\ \frac{(8-5)!\,(5-1)!}{8!} + 6\ \frac{(8-6)!\,(6-1)!}{8!} = 3/28.$$

The result says that the Mayor has more power than the council members.

Exercise

1. Show that the following (non-zero-sum) bi-matrix game is inessential:

$$\begin{bmatrix} (0,0) & (1,-2) \\ (-1,1) & (1,-1) \end{bmatrix}.$$

2. The 3-person game of Couples is played as follows. Each player choose one of the other two players; these choices are made simultaneously. If a couple forms (for example, if P_2 chooses P_3, P_3 chooses P_2), then each member of that couple receives a payoff of 1/2, while the person not in the couple receives -1. If no couple forms (for example, if P_1 chooses P_2, P_2 chooses P_3, and P_3 chooses P_1), then each receives a payoff of zero.

   Prove that the game Couples is zero-sum and essential.

3. Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be a set of players. Let a, b, c, d, e, f be non-negative numbers such that

$$\begin{aligned} a + d &\leq 1 \\ b + e &\leq 1 \\ c + f &\leq 1. \end{aligned}$$

   Define ν by

$$\nu(\emptyset) = 0, \quad \nu(\mathcal{P}) = 1,$$

$$\nu(\text{any 3-person coalition}) = 1,$$

$$\nu(\text{any 1 person coalition}) = 0,$$

$$\nu(\{P_1, P_2\}) = a, \quad \nu(\{P_1, P_3\}) = b, \quad \nu(\{P_1, P_4\}) = c,$$

$$\nu(\{P_3, P_4\}) = d, \quad \nu(\{P_2, P_4\}) = e, \quad \nu(\{P_2, P_3\}) = f.$$

   Prove that ν is a game in characteristic function form.

4. Let ν be a game in characteristic function form. (i) Prove that the set of all imputation is convex. (ii) Prove that the core is convex.

5. Prove that if λ is strategically equivalent to μ, and if μ is strategically equivalent to ν, then λ is strategically equivalent to ν.
6. Compute the $(0, 1)$-reduced form for the Used Car Game.
7. Prove that the set

$$\{(x, 0, 1 - x) : \ 0 \leq x \leq 700\}$$

is a stable set of imputation for the Used Car Game.
8. Find the Shapley values for the Used Car Game.