THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4250 Game Theory, 2015-2016 Term 2
Mid-term Examination
Time allowed: 90 mins

Answer all questions.

1. (6 marks) Let $\oplus$ denotes the nim-sum.
   (a) Find $12 \oplus 23 \oplus 29$.
   (b) Find all winning moves of the game of nim from the position $(12, 23, 29)$.

2. (10 marks) There are two piles of chips on the table. Two players remove the chips from the table alternatively. In each turn, a player may either remove any positive number of chips from one of the piles, or move any positive number of chips from the first pile to the second pile. The player who removes the last chip wins. Let $g(x, y)$ be the Sprague-Grundy function of the game, where x is the number of chips in the first pile and y is the number of chips in the second pile.
   (a) Find $g(3, 1)$, $g(2, 2)$ and $g(3, 2)$.
   (b) Write down a guess of the set of P-positions.
   (c) Prove your assertion in (b).

3. (10 marks) Consider the following 3 games.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6\}$
Game 3:	At least half (In each turn, a player may remove at least half of the chips)

   Let g_1, g_2, g_3 be the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.
   (a) Write down the values of $g_1(6)$, $g_2(12)$ and $g_3(13)$
   (b) Find $g(6, 12, 13)$.
   (c) Find all winning moves of G from the position $(6, 12, 13)$.

4. (8 marks) Let

$$A = \begin{pmatrix} 4 & 2 & 1 & 3 & -2 \\ 2 & 1 & -1 & 4 & -3 \\ -2 & -1 & 4 & 0 & 5 \end{pmatrix}$$

   (a) Write down the reduced matrix obtained by deleting all dominated rows and columns of A.

   (b) Use the reduced matrix to solve the two-person zero sum game with game matrix A, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

5. (8 marks) Use simplex method to solve the game with the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -3 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

6. (8 marks)

   (a) Ronald and Cathy choose an integer from 1 to 5 simultaneously. Let i and j be the numbers chosen by Ronald and Cathy respectively. If $i < j$, Cathy pays 1 dollar to Ronald. If $i = j$, Cathy pays 2 dollars to Ronald. If $i > j$, then nothing happens. Find a maximin strategy for Ronald, a minimax strategy for Cathy and the value of the game.

   (b) Let n be a positive integer and $a > 1$ be a real number. Find a maximin strategy for the row player and a minimax strategy for the column player for the $n \times n$ game matrix

$$A = \begin{pmatrix} a+1 & a & \cdots & \cdots & a \\ 1 & a+1 & a & \cdots & a \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & a+1 & a \\ 1 & \cdots & \cdots & 1 & a+1 \end{pmatrix}$$

   and show that the value of A is

$$\frac{a^{n+1}-1}{a^n-1}.$$

End of paper