THE CHINESE UNIVERSITY OF HONG KONG
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MMAT5360 Game Theory

Contents

1 Two-person zero sum games

1.1 Game matrices

In a two-person zero sum game, two players, player I and player II, make their moves simultaneously. Then the payoffs to the players depend on the strategies used by the players. In this chapter, we study only **zero sum games** which means the sum of the payoffs to the players is always zero. We will also assume that the game has **perfect information** which means all players know how the outcomes depend on the strategies the players use.

Definition 1.1.1 (Strategic form of a two-person zero sum game)**.** *The* **strategic form** *of a two-person zero sum game is given by a triple* (X, Y, π) *where*

1. X *is the set of strategies of player* I*.*

2. Y *is the set of strategies of player* II*.*

3. $\pi : X \times Y \to \mathbb{R}$ *is the payoff function of player* I*.*

For $(x, y) \in X \times Y$, the value $\pi(x, y)$ is the payoff to player I when player I uses strategy x and player II uses strategy y. Note that the payoff to player II is equal to $-\pi(x, y)$ since the game is a zero sum game. The game has perfect information means that the function π is known to both players. We will always assume that the sets X and Y are finite. In this case we may assume $X = \{1, 2, \cdots, m\}$ and $Y = \{1, 2, \cdots, n\}$. Then the payoff function can be represented by an $m \times n$ matrix which is called the **game matrix** and we will denote it by $A = [a_{ij}]$. A two-person zero sum game is completely determined by its game matrix. When player I uses the i-th strategy and player II uses the j-th strategy, then the payoff to player I is the entry a_{ij} of A. The payoff to player II is then $-a_{ij}$. If a two-person zero sum game is represented by a game matrix, we will call player I the **row player** and player II the **column player**.

Given a game matrix A, we would like to know what the optimal strategies for the players are and what the payoffs to the players will be if both of them use their optimal strategies. The answer to this question is simple if A has a saddle point.

Definition 1.1.2 (Saddle point)**.** *We say that an entry* a_{kl} *is a* **saddle point** *of an* $m \times n$ *matrix* A *if*

1. $a_{kl} = \min_{j=1,2,\cdots,n} \{a_{kj}\}$

2. $a_{kl} = \max_{i=1,2,\cdots,m} \{a_{il}\}$

The first condition means that when the row player uses the k-th strategy, then the payoff to the row player is not less than a_{kl} no matter how the column player plays. The second condition means that when the column player uses the l-th strategy, then the payoff to the row player is not larger than a_{kl} no matter how the row player plays. Consequently we have

Theorem 1.1.3. *If A has a saddle point a_{kl}, then the row player may guarantee that his payoff is not less than a_{kl} by using the k-th strategy and the column player may guarantee that the payoff to the row player is not larger than a_{kl} by using the l-th strategy.*

Suppose A is a matrix which has a saddle point a_{kl}. The above theorem implies that the corresponding row and column constitute the optimal strategies for the players. To find the saddle points of a matrix, first write down the row minima of the rows and the column maxima of the columns. Then find the maximum of row minima which is called the **maximin**, and the minimum of the column maxima which is called the **minimax**. If the maximin is equal to the minimax, then the entry in the corresponding row and column is a saddle point. If the maximin and minimax are different, then the matrix has no saddle point.

Example 1.1.4.

$$\begin{array}{cc} & & \textit{min} \\ & \begin{pmatrix} 1 & 2 & 0 \\ 3 & 5 & 2 \\ 0 & -4 & -3 \\ -2 & 4 & 1 \end{pmatrix} & \begin{array}{c} 0 \\ 2 \\ -4 \\ -2 \end{array} \\ \textit{max} & \begin{array}{ccc} 3 & 5 & 2 \end{array} & \end{array}$$

Both the maximin and minimax are 2. Therefore the entry $a_{23} = 2$ is a saddle point. □

Example 1.1.5.

$$\begin{array}{cc} & \begin{pmatrix} 2 & -1 & 3 & 1 \\ -4 & 2 & 0 & 3 \\ 0 & 1 & -2 & 4 \end{pmatrix} \\ max & \begin{matrix} 2 & \;\;2 & \;\;3 & 4 \end{matrix} \end{array} \quad \begin{array}{c} min \\ -1 \\ -4 \\ -2 \\ \end{array}$$

The maximin is -1 *while the minimax is* 2 *which are not equal. Therefore the matrix has no saddle point.* □

Saddle point of a matrix may not be unique. However the values of saddle points are always the same.

Theorem 1.1.6. *The values of the saddle points of a matrix are the same. That is to say, if* a_{kl} *and* a_{pq} *are saddle points of a matrix, then* $a_{kl} = a_{pq}$*. Furthermore, we have* $a_{pq} = a_{pl} = a_{kq} = a_{kl}$*.*

Proof. We have

$$\begin{aligned} a_{kl} &\leq a_{kq} \quad (\text{since } a_{kl} \leq a_{kj} \text{ for any } j) \\ &\leq a_{pq} \quad (\text{since } a_{iq} \leq a_{pq} \text{ for any } i) \\ &\leq a_{pl} \quad (\text{since } a_{pq} \leq a_{pj} \text{ for any } j) \\ &\leq a_{kl} \quad (\text{since } a_{il} \leq a_{kl} \text{ for any } i) \end{aligned}$$

Therefore

$$a_{kl} = a_{kq} = a_{pq} = a_{pl}$$

□

We have seen that if A has a saddle point, then the two players have optimal strategies by using one of their strategies constantly (Theorem 1.1.3). If A has no saddle point, it is expected that the optimal ways for the players to play the game are not using one of the strategies constantly. Take the rock-paper-scissors game as an example.

Example 1.1.7 (Rock-paper-scissors)**.** *The* **rock-paper-scissors** *game has the game matrix*

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{matrix} R & \;\;P & \;\;S \end{matrix} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

Here we use the order rock(R), paper(P), scissors(S) to write down the game matrix. □

Everybody knows that the optimal strategy of playing the rock-paper-scissors game is not using any one of the gestures constantly. When one of the strategies of a player is used constantly, we say that it is a **pure strategy**. For games without saddle point like rock-paper-scissors game, mixed strategies instead of pure strategies should be used.

Definition 1.1.8 (Mixed strategy)**.** *A* **mixed strategy** *is a row vector* $\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$ *such that*

1. $x_i \geq 0$ *for any* $i = 1, 2, \cdots, m$
2. $\sum_{i=1}^{m} x_i = 1$

In other words, a vector is a mixed strategy if it is a **probability vector***. We will denote the set of probability* m *vectors by* $\mathcal{P}^m$*.*

When a mixed strategy $(x_1, x_2, \cdots, x_m)$ is used, the player uses his i-th strategy with a probability of x_i for $i = 1, 2, \cdots, m$. Mixed strategies are generalization of pure strategies. If one of the coordinates of a mixed strategy is 1 and all other coordinates are 0, then it is a pure strategy. So a pure strategy is also a mixed strategy. Suppose the row player and the column player use mixed strategies $\mathbf{x} \in \mathcal{P}^m$ and $\mathbf{y} \in \mathcal{P}^n$ respectively. Then the outcome of the game is not fixed because the payoffs to the players will then be random variables. We denote by $\pi(\mathbf{x}, \mathbf{y})$ the **expected payoff** to the row player when the row player uses mixed strategy $\mathbf{x}$ and the column player uses mixed strategy $\mathbf{y}$. We have the following simple formula for the expected payoff $\pi(\mathbf{x}, \mathbf{y})$ to the row player.

Theorem 1.1.9. *In a two-person zero sum game with* $m \times n$ *game matrix* A*, suppose the row player uses mixed strategies* $\mathbf{x}$ *and the column player uses mixed strategies* $\mathbf{y}$ *independently. Then the expected payoff to the row player is*

$$\pi(\mathbf{x}, \mathbf{y}) = \mathbf{x} A \mathbf{y}^T$$

where $\mathbf{y}^T$ *is the transpose of* $\mathbf{y}$*.*

Proof. The expected payoff to the row player is

$$\begin{aligned}
& E(\text{payoff to the row player}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy and } II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} P(I \text{ uses } i\text{-th strategy}) P(II \text{ uses } j\text{-th strategy}) \\
= & \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} a_{ij} x_i y_j \\
= & \mathbf{x} A \mathbf{y}^T
\end{aligned}$$

□

Let A be an $m \times n$ game matrix. For $\mathbf{x} \in \mathcal{P}^m$, the vector

$$\mathbf{x}A \in \mathbb{R}^n$$

has the following interpretation. The j-th coordinate, $j = 1, 2, \cdots, n$, of the vector is the expected payoff to the row player if the row player uses mixed strategy $\mathbf{x}$ and the column player uses the j-th strategy constantly. In this case a rational column player would use the l-th strategy, $1 \le l \le n$, such that the l-th coordinate of the vector $\mathbf{x}A$ is the least coordinate among all coordinates of $\mathbf{x}A$. (Note that the column player wants the expected payoff to the row player as small as possible since the game is a zero sum game.)

On the other hand, for $\mathbf{y} \in \mathcal{P}^n$, the i-th coordinate, $i = 1, 2, \cdots, m$, of the column vector

$$A\mathbf{y}^T \in \mathbb{R}^m$$

is the expected payoff to the row player if the row player uses his i-th strategy constantly and the column player uses the mixed strategy $\mathbf{y}$. In this case a rational row player would use the k-th strategy, $1 \le k \le m$, such that the k-th coordinate of $A\mathbf{y}^T$ is the largest coordinate among all coordinates of $A\mathbf{y}^T$.

When a game matrix does not have a saddle point, both players do not have optimal pure strategies. However there always exists optimal mixed strategies for the players by the following minimax theorem due to von Neumann.

Theorem 1.1.10 (Minimax theorem)**.** *Let A be an $m \times n$ matrix. Then there exists real number $\nu \in \mathbb{R}$, mixed strategy for the row player $\mathbf{p} \in \mathbb{R}^m$, and mixed strategy for the column player $\mathbf{q} \in \mathbb{R}^n$ such that*

1. $\mathbf{p}A\mathbf{y}^T \geq \nu$, *for any* $\mathbf{y} \in \mathcal{P}^n$
2. $\mathbf{x}A\mathbf{q}^T \leq \nu$, *for any* $\mathbf{x} \in \mathcal{P}^m$
3. $\mathbf{p}A\mathbf{q}^T = \nu$

In the above theorem, the real number $\nu = \nu(A)$ is called the **value**, or the **security level**, of the game matrix A. The strategy $\mathbf{p}$ is called a **maximin strategy** for the row player and the strategy $\mathbf{q}$ is called a **minimax strategy** for the column player. The value ν of a matrix is unique. However maximin strategy and minimax strategy are in general not unique.

The maximin strategy $\mathbf{p}$ and the minimax strategy $\mathbf{q}$ are the optimal strategies for the row player and the column player respectively. It is because the row player may guarantee that his payoff is at least ν no matter how the column player plays by using the maximin strategy $\mathbf{p}$. This is also the reason why the value ν is called the security level. Similarly, the column player may guarantee that the payoff to the row player is at most ν, and thus his payoff is at least $-\nu$, no matter how the row player plays by using the minimax strategy $\mathbf{q}$. We will prove the minimax theorem in Section 2.4.

1.2 2×2 games

In this section, we study 2×2 game matrices closely and see how one can solve them, that means finding the maximin strategies for the row player, minimax strategies for the column player and the values of the game. First we look at a simple example.

Example 1.2.1 (Modified rock-paper-scissors)**.** *The rules of the modified rock-paper-scissors are the same as the ordinary rock-paper-scissors except that the row player can only show the gesture rock(R) or paper(P) but not scissors while the column player can only show the gesture scissors(S) or rock(R) but not paper. The game matrix of the game is*

$$\begin{array}{cc} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} R \\ P \end{array} & \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \end{array}$$

It is easy to see that the game matrix does not have a saddle point. Thus there is no pure maximin or minimax strategy. To solve the game, suppose *the row player uses mixed strategy* $\mathbf{x} = (x, 1-x)$*. Consider*

$$\mathbf{x}A = (x, 1-x)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (x-(1-x), 1-x) = (2x-1, 1-x)$$

This shows that when the row player uses mixed strategy $\mathbf{x} = (x, 1-x)$*, then his payoff is* $2x-1$ *if the column player uses his 1st strategy scissors(S) and is* $1-x$ *if the column player uses his 2nd strategy rock(R). Now we solve the equation* $2x-1 = 1-x$ *and get* $x = \frac{2}{3}$*. One may see that if* $0 \le x < \frac{2}{3}$*, then* $2x-1 < x-1$ *and a rational column player would use his 1st strategy scissors(S) and the payoff to the row player would be* $2x-1 < \frac{1}{3}$*. On the other hand, if* $\frac{2}{3} < x \le 1$*, then* $2x-1 > 1-x$ *and a rational column player would use his 2nd strategy rock(R) and the payoff to the row player would be* $1-x < \frac{1}{3}$*. Now if* $x = \frac{2}{3}$*, that is if the row player uses the mixed strategy* $(\frac{2}{3}, \frac{1}{3})$*, then he may guarantee that his payoff is* $1-x = 2x-1 = \frac{1}{3}$ *no matter how the column player plays. This is the largest payoff he may guarantee and therefore the mixed strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *is the maximin strategy for the row player. Similarly, suppose the column player uses mixed strategy* $\mathbf{y} = (y, 1-y)$*. Consider*

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} y \\ -y+(1-y) \end{pmatrix}\begin{pmatrix} y \\ 1-2y \end{pmatrix}$$

If $0 \le y < \frac{1}{3}$*, then* $y < 1-2y$ *and a rational row player would use his 2nd strategy paper(P) and his payoff would be* $1-2y > \frac{1}{3}$*. If* $\frac{1}{3} < y \le 1$*, then* $y > 1-2y$ *and a rational row player would use his 1st strategy rock(R) and his payoff would be* $y > \frac{1}{3}$*. If* $y = \frac{1}{3}$*, then the payoff to the row player is always* $\frac{1}{3}$ *no matter how he plays. Therefore* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *is the minimax strategy for the column player and he may guarantee that the payoff to the row player is* $\frac{1}{3}$ *no matter how the row player plays. Moreover the value of the game is* $\nu = \frac{1}{3}$*. We summarize the above discussion in the following statements.*

1. *The row player may use his maximin strategy* $\mathbf{p} = (\frac{2}{3}, \frac{1}{3})$ *to guarantee that his payoff is* $\nu = \frac{1}{3}$ *no matter how the column player plays.*

2. *The column player may use his minimax strategy* $\mathbf{q} = (\frac{1}{3}, \frac{2}{3})$ *to guarantee that the payoff to the row player is* $\nu = \frac{1}{3}$ *no matter how the row player plays.* □

Now we give the complete solutions to 2×2 games.

Theorem 1.2.2. *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 game matrix. Suppose A has no saddle point. Then

1. *The value of the game is*

$$\nu = \frac{ad - bc}{a - b - c + d}$$

2. *The maximin strategy for the row player is*

$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$

3. *The minimax strategy for the column player is*

$$\mathbf{q} = \left(\frac{d - b}{a - b - c + d}, \frac{a - c}{a - b - c + d} \right)$$

Proof. Suppose the row player uses mixed strategy $\mathbf{x} = (x, 1 - x)$. Consider

$$\mathbf{x}A = (x, 1-x) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+c(1-x), bx+d(1-x)) = ((a-c)x+c, (b-d)x+d)$$

Now the payoff to the row player that he can guarantee is

$$\min\{(a - c)x + c, (b - d)x + d\}$$

Since A has no saddle point, we have $a - c$ and $b - d$ are of different sign and the maximum of the above minimum is obtained when

$$\begin{aligned} (a - c)x + c &= (b - d)x + d \\ \Rightarrow \qquad x &= \frac{d - c}{a - b - c + d} \end{aligned}$$

Note that x and $1 - x = \frac{a-b}{a-b-c+d}$ must be of the same sign because A has no saddle point. We must have $0 < x < 1$ and we conclude that the maximin strategy for the row player is

$$\mathbf{p} = \left(\frac{d - c}{a - b - c + d}, \frac{a - b}{a - b - c + d} \right)$$

Similarly suppose the column player uses mixed strategy $\mathbf{y} = (y, 1-y)$. Consider

$$A\mathbf{y}^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} aq + b(1-y) \\ cq + d(1-y) \end{pmatrix} = \begin{pmatrix} (a-b)y + b \\ (c-d)y + d \end{pmatrix}$$

The column player may guarantee that the payoff to the row player is at most

$$\max\{(a-b)y + b, (c-d)y + d\}$$

The above quantity attains it minimum when

$$\begin{aligned} (a-b)y + b &= (c-d)y + d \\ \Rightarrow \qquad y &= \frac{d-b}{a-b-c+d} \end{aligned}$$

and the minimax strategy for the column player is

$$\mathbf{q} = \left(\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d} \right)$$

By calculating

$$\mathbf{p}A = \left(\frac{ad-bc}{a-b-c+d}, \frac{ad-bc}{a-b-c+d} \right) \text{ and } A\mathbf{q}^T = \begin{pmatrix} \frac{ad-bc}{a-b-c+d} \\ \frac{ad-bc}{a-b-c+d} \end{pmatrix}$$

we see that the maximum payoff that the row player may guarantee to himself and the minimum payoff to the row player that the column player may guarantee are both $\frac{ad-bc}{a-b-c+d}$. In fact the minimax theorem (Theorem 1.1.10) says that these two values must be equal. We conclude that the value of A is $\nu = \frac{ad-bc}{a-b-c+d}$. □

Note that the above formulas work only when A has no saddle point. If A has a saddle point, the vectors $\mathbf{p}$ and $\mathbf{q}$ obtained using the formulas may not be probability vectors.

Example 1.2.3. *Consider the modified rock-paper-scissors game (Example 1.2.1) with game matrix*

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

The game matrix has no saddle point. By Theorem 1.2.2, the value of the game is

$$\nu = \frac{ad - bc}{a - b - c + d} = \frac{1 \times 1 - 0 \times (-1)}{1 - 0 - (-1) + 1} = \frac{1}{3}$$

the maximin strategy for the row player is

$$\begin{aligned}\mathbf{p} &= \left(\frac{d-c}{a-b-c+d}, \frac{a-b}{a-b-c+d}\right) \\ &= \left(\frac{1-(-1)}{1-0-(-1)+1}, \frac{1-0}{1-0-(-1)+1}\right) \\ &= \left(\frac{2}{3}, \frac{1}{3}\right)\end{aligned}$$

and the minimax strategy for the column player is

$$\begin{aligned}\mathbf{q} &= \left(\frac{d-b}{a-b-c+d}, \frac{a-c}{a-b-c+d}\right) \\ &= \left(\frac{1-0}{1-0-(-1)+1}, \frac{1-(-1)}{1-0-(-1)+1}\right) \\ &= \left(\frac{1}{3}, \frac{2}{3}\right)\end{aligned}$$

□

Example 1.2.4. *In a game, each of the two players Andy and Bobby calls out a number simultaneously. Andy may call out either* 1 *or* −2 *while Bobby may call out either* 1 *or* −3*. Then Bobby pays p dollars to Andy where p is the product of the two numbers (Andy pays Bobby $-p$ dollars when p is negative). The game matrix of the game is*

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}$$

The value of the game is

$$\nu = \frac{1 \times 6 - (-2) \times (-3)}{1 - (-3) - (-2) + 6} = 0$$

the maximin strategy for Andy is

$$\mathbf{p} = \left(\frac{6-(-2)}{1-(-3)-(-2)+6}, \frac{1-(-3)}{1-(-3)-(-2)+6}\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

and the minimax strategy for Bobby is

$$\mathbf{q} = \left(\frac{6-(-3)}{1-(-3)-(-2)+6}, \frac{1-(-2)}{1-(-3)-(-2)+6} \right) = \left(\frac{3}{4}, \frac{1}{4} \right)$$

□

We say that a two-person zero sum game is **fair** if its value is zero. The game in Example 1.2.4 is a fair game.

1.3 Games reducible to 2×2 games

To solve an $m \times n$ game matrix for $m, n > 2$ without saddle point, we may first remove the dominated rows or columns. A row dominates another if all its entries are larger than or equal to the corresponding entries of the other. Similarly, a column dominates another if all its entries are smaller than or equal to the corresponding entries of the other.

Definition 1.3.1. *Let $A = [a_{ij}]$ be an $m \times n$ game matrix.*

1. *We say that the k-th row is dominated by the r-th row if $a_{kj} \leq a_{rj}$ for any $j = 1, 2, \cdots, n$.*

2. *We say that the l-th column is dominated the s-th column if $a_{il} \geq a_{is}$ for any $i = 1, 2, \cdots, m$.*

We say that a row (column) is a **dominated row (column)** *if it is dominated by another row (column).*

If the k-th row of A is dominated by the r-th row, then for the row player, playing the r-th strategy is at least as good as playing the k-th strategy. Thus the k-th row can be ignored in finding the maximin strategy for the row player. Similarly the column player may ignore a dominated column when finding his minimax strategy.

Theorem 1.3.2. *Let A be an $m \times n$ game matrix. Suppose A has a dominated row or dominated column. Let A' be the matrix obtained by deleting a dominated row or dominated column from A. Then*

1. *The value of A' is equal to the value of A.*

2. *The players of A have maximin/minimax strategies which never use dominated row/column.*

More precisely, if the k-th row is a dominated row of A, A' is the $(m-1)\times n$ matrix obtained by deleting the k-th row from A, and $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ is a maximin strategy for the row player of A', then $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$ is a maximin strategy for the row player of A. Similarly, if the l-th column is a dominated row of A, A' is the $m \times (n-1)$ matrix obtained by deleting the l-th column from A, and $\mathbf{q}' = (q_1, \cdots, q_{l-1}, q_{l+1} \cdots, q_n) \in \mathcal{P}^{n-1}$ is a minimax strategy of A', then $\mathbf{q} = (q_1, \cdots, q_{l-1}, 0, q_{l+1}, \cdots, q_n) \in \mathcal{P}^n$ is a minimax strategy of A.

Proof. Suppose the k-th row of A is dominated by the r-th row and A' is obtained by deleting the k-th row from A. Let ν' be the value of A' and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy of A'. For any mixed strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$, define $\mathbf{x}' = (x'_1, \cdots, x'_{k-1}, x'_{k+1}, \cdots, x'_m) \in \mathcal{P}^{m-1}$ by

$$x'_i = \begin{cases} x_i & \text{if } i \neq r \\ x_k + x_r & \text{if } i = r \end{cases}$$

and we have

$$\mathbf{x}A\mathbf{q}^T \leq \mathbf{x}'A'\mathbf{q}^T \leq \nu'$$

Here the first inequality holds because the k-th is dominated by the r-th row and the second inequality holds because $\mathbf{q}$ is a minimax strategy of A'. Thus the value of A is less than or equal to ν'. On the other hand, let $\mathbf{p}' = (p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_m) \in \mathcal{P}^{m-1}$ be a maximin strategy of A' and let $\mathbf{p} = (p_1, \cdots, p_{k-1}, 0, p_{k+1}, \cdots, p_m) \in \mathcal{P}^m$. Then we have

$$\mathbf{p}A\mathbf{y}^T = \mathbf{p}'A'\mathbf{y}^T \geq \nu'$$

for any $\mathbf{y} \in \mathcal{P}^n$. It follows that the value of A is ν' and $\mathbf{p}$ is a maximin strategy of A. The proof of the second statement is similar. □

The removal of dominated rows or columns does not change the value of a game. The above theorem only says that there is at least one optimal strategy with zero probability at the dominated rows and columns. There may be other optimal strategies which have positive probability at the dominated rows or columns. However any optimal strategy must have zero probability at strictly dominated rows and columns. Here a row is strictly dominated by another row if all its entries are strictly smaller than the corresponding entries of the other. Similarly a column is strictly dominated by another column if all its entries are strictly larger than the corresponding entries of the other.

Example 1.3.3. *To solve the game matrix*

$$A = \begin{pmatrix} 3 & -1 & 4 \\ 2 & -3 & 1 \\ -2 & 4 & 0 \end{pmatrix}$$

we may delete the second row since it is dominated by the first row and get the reduced matrix

$$A' = \begin{pmatrix} 3 & -1 & 4 \\ -2 & 4 & 0 \end{pmatrix}$$

Then we may delete the third column since is dominated by the first column. Hence the matrix A *is reduced to the* 2×2 *matrix*

$$A'' = \begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$$

The value of this 2×2 *matrix is* 0.7*. The maximin and minimax strategies are* $(0.6, 0.4)$ *and* $(0.5, 0.5)$ *respectively. Therefore the value of* A *is* 0.7*, a maximin strategy for the row player is* $(0.6, 0, 0.4)$ *and a minimax strategy for the column player is* $(0.5, 0.5, 0)$*. Note that we need to insert the zeros to the dominated rows and columns when writing down the maximin and minimax strategies for the players.* □

1.4 $2 \times n$ and $m \times 2$ games

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

be a $2 \times n$ matrix. We are going to explain how to solve the game with game matrix A if there is no dominated row or column. Suppose the row player uses strategy $\mathbf{x} = (x, 1-x)$ for $0 \leq x \leq 1$. The payoff to the row player is given by

$$\begin{aligned} \mathbf{x}A &= (x, 1-x) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} \\ &= (a_{11}x + a_{21}(1-x), a_{12}x + a_{22}(1-x), \cdots, a_{1n}x + a_{2n}(1-x)) \end{aligned}$$

Now we need to find the value of x so that the minimum

$$\min_{1\leq j\leq n}\{a_{1j}x + a_{2j}(1-x)\}$$

of the coordinates of $\mathbf{x}A$ attains its maximum. We may use graphical method to achieve this goal.

Step 1.
For each $1 \leq j \leq n$, draw the graph of

$$v = a_{1j}x + a_{2j}(1-x), \text{ for } 0 \leq x \leq 1$$

The graph shows the payoff to the row player if the column player uses the j-th strategy.

Step 2.
Draw the graph of

$$v = \min_{1\leq j\leq n}\{a_{1j}x + a_{2j}(1-x)\}$$

This is called the **lower envelope** of the graph.

Step 3.
Suppose (p, ν) is a maximum point of the lower envelope. Then ν is the value of the game and $\mathbf{p} = (p, 1-p)$ is a maximin strategy for the row player.

Step 4.
The solutions for $\mathbf{y} \in \mathcal{P}^n$ to the equation

$$A\mathbf{y}^T = \nu\mathbf{1}^T$$

where $\mathbf{1} = (1, 1)$, give the minimax strategy for the column player.

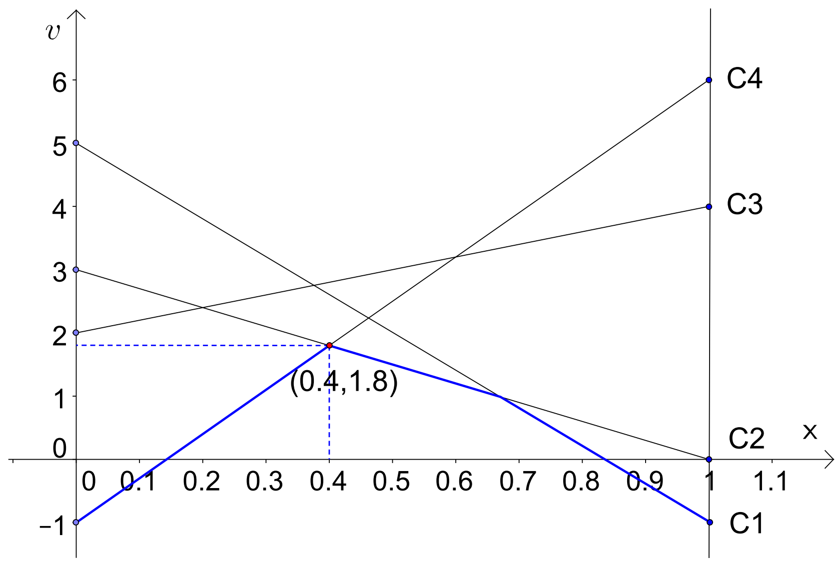
Example 1.4.1. *Solve the* 2×4 *game matrix*

$$A = \begin{pmatrix} -1 & 0 & 4 & 6 \\ 5 & 3 & 2 & -1 \end{pmatrix}$$

Solution.

Step 1. Draw the graph of

$$\begin{cases} C1: \ v = -x + 5(1-x) \\ C2: \ v = 3(1-x) \\ C3: \ v = 4x + 2(1-x) \\ C4: \ v = 6x - (1-x) \end{cases}$$



Step 2. Draw the lower envelope (blue polygonal curve).

Step 3. The maximum point of the lower envelope is the intersection point of $C2$ and $C4$. By solving

$$\begin{cases} C2: \ v = 3(1-x) \\ C4: \ v = 6x - (1-x) \end{cases}$$

we obtain the maximum point $(p, \nu) = (0.4, 1.8)$ of the lower envelope.

Step 4. Find the minimax strategies for the column player by solving

$$\begin{pmatrix} 0 & 6 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} y_2 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1.8 \\ 1.8 \end{pmatrix}$$

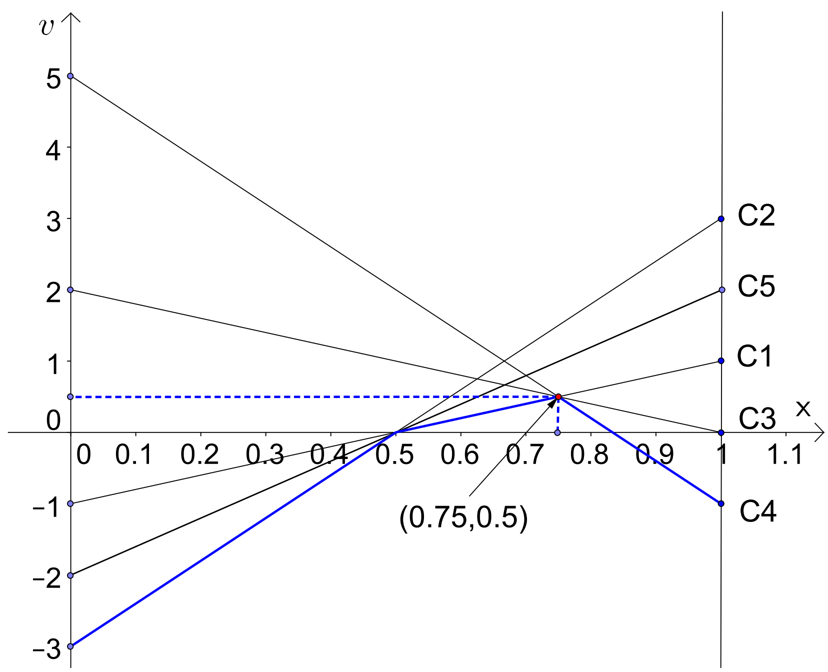
and get $y_2 = 0.7$ and $y_4 = 0.3$.

Therefore the value of the game is $\nu = 1.8$. The maximin strategy for the row player is $\mathbf{p} = (0.4, 0.6)$ and the minimax strategy for the column player is $\mathbf{q} = (0, 0.7, 0, 0.3)$. □

Example 1.4.2. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ -1 & -3 & 2 & 5 & -2 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



By solving

$$\begin{cases} C1: & v = x - (1-x) \\ C3: & v = 2(1-x) \\ C4: & v = -x + 5(1-x) \end{cases}$$

we see that the maximum point of the lower envelope is $(p, \nu) = (0.75, 0.5)$. Thus the maximin strategy for the row player is $(0.75, 0.25)$ and the value of the game is $\nu = 0.5$. To find minimax strategies for the column player, we solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix}$$

Note that we have added the equation $y_1 + y_3 + y_4 = 1$ to exclude the solutions which are not probability vectors. (Explain why we didn't do it in Example 1.4.1.) Using row operation, we obtain the row echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0.5 \\ -1 & 2 & 5 & 0.5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0.5 \\ 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The non-negative solution to the system of equations is

$$(y_1, y_3, y_4) = (0.5 + t, 0.5 - 2t, t) \text{ for } 0 \le t \le 0.25$$

Therefore the column player has minimax strategies

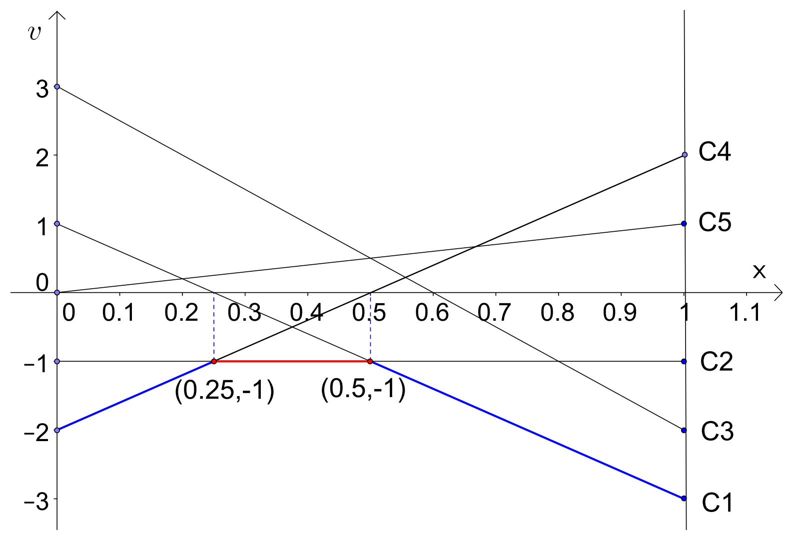
$$\mathbf{q} = (0.5 + t, 0, 0.5 - 2t, t, 0) \text{ for } 0 \le t \le 0.25$$

In particular, $(0.5, 0, 0.5, 0, 0)$ and $(0.75, 0, 0, 0.25, 0)$ are minimax strategies for the column player. □

Example 1.4.3. *Solve the* 2×5 *game matrix*

$$A = \begin{pmatrix} -3 & -1 & -2 & 2 & 1 \\ 1 & -1 & 3 & -2 & 0 \end{pmatrix}$$

Solution. The lower envelope is shown in the following figure.



The maximum points of the lower envelope are points lying on the line segment joining $(0.25, -1)$ and $(0.5, -1)$. Thus the value of the game is $\nu = -1$. The maximin strategies for the row player are

$$\mathbf{p} = (p, 1-p) \text{ for } 0.25 \leq p \leq 0.5$$

and the minimax strategy for the column player is

$$\mathbf{q} = (0, 1, 0, 0, 0)$$

□

Next we consider $m \times 2$ games. There are two methods to solve such games.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy for the column player. Draw the upper envelope

$$v = \max_{1 \leq i \leq m} \{a_{i1}y + a_{i2}(1-y)\}$$

Suppose the minimum point of the upper envelope is (q, ν). Then the value of the game is ν and the minimax strategy for the column player is $\mathbf{q} = (q, 1-q)$. Moreover the maximum strategies for the row player are the solutions for $\mathbf{x} \in \mathcal{P}^m$ to the equation

$$\mathbf{x}A = \nu\mathbf{1} = (\nu, \nu)$$

Method 2.
Solve the game with $2 \times m$ game matrix $-A^T$. Then

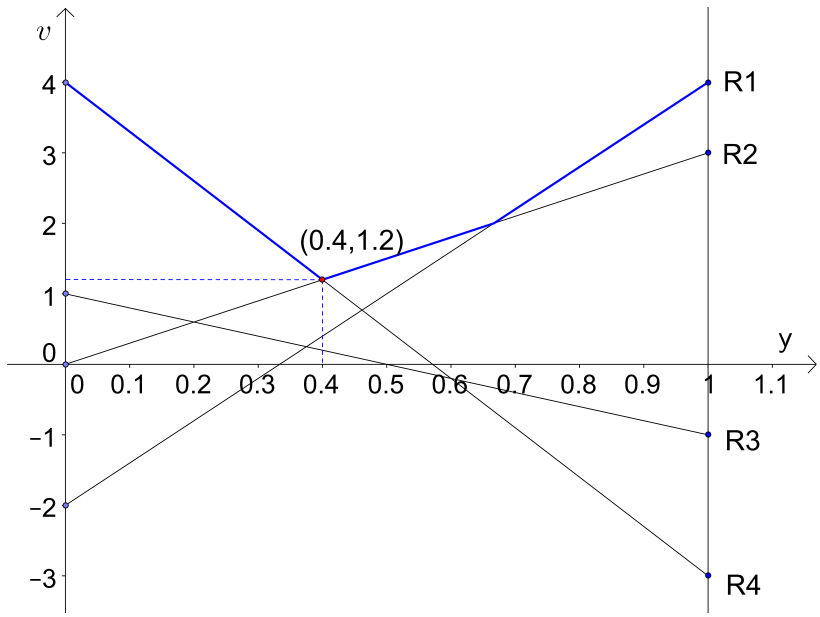
value of $A = -$ value of $-A^T$
maximin strategy of $A =$ minimax strategy of $-A^T$
minimax strategy of $A =$ maximin strategy of $-A^T$

Example 1.4.4. *Solve the* 4×2 *game matrix*

$$A = \begin{pmatrix} 4 & -2 \\ 3 & 0 \\ -1 & 1 \\ -3 & 4 \end{pmatrix}$$

Solution.

Method 1.
Let $\mathbf{y} = (y, 1-y)$, $0 \leq y \leq 1$, be the strategy of the column player. The upper envelope is



Solving

$$\begin{cases} R2: & v = 3(1-y) \\ R4: & v = -3y + 4(1-y) \end{cases}$$

the minimum point of the upper envelope is $(q, \nu) = (0.4, 1.2)$. Now the row player would only use the 2nd and 4th strategy and we solve

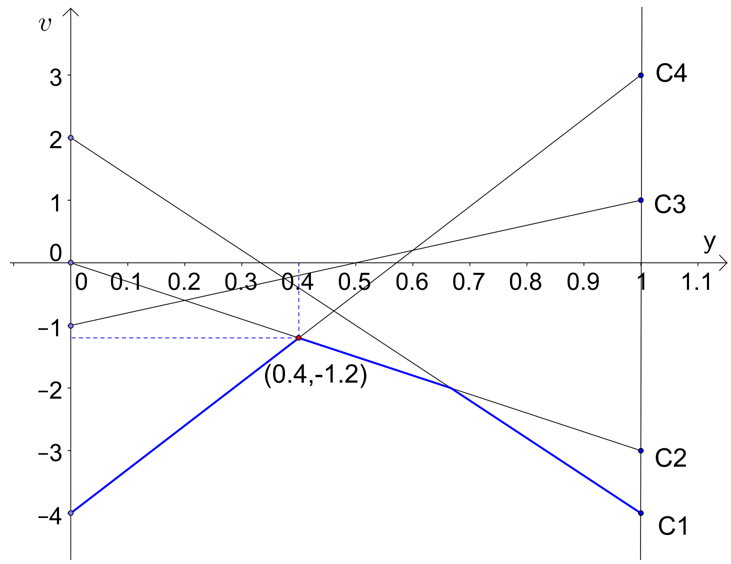
$$(x_2, x_4) \begin{pmatrix} 3 & 0 \\ -3 & 4 \end{pmatrix} = (1.2, 1.2)$$

which gives $(x_2, x_4) = (0, 7, 0.3)$. Therefore the value of the game is $\nu = 1.2$, the maximin strategy for the row player is $\mathbf{p} = (0, 0.7, 0, 0.3)$ and the minimax strategy for the column player is $\mathbf{q} = (0.4, 0.6)$.

Method 2.
Consider

$$-A^T = \begin{pmatrix} -4 & -3 & 1 & 3 \\ 2 & 0 & -1 & -4 \end{pmatrix}$$

Draw the lower envelope



We see that the value of $-A^T$ is -1.2 and the maximin strategy of $-A^T$ is $(0.4, 0.6)$. Solving

$$\begin{pmatrix} -3 & 3 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1.2 \\ -1.2 \end{pmatrix}$$

We get $x_2 = 0.7$ and $x_4 = 0.3$. Thus the minimax strategy of $-A^T$ is $(0, 0.7, 0, 0.3)$. Therefore

value of $A = -$ value of $-A^T = 1.2$
maximin strategy of $A =$ minimax strategy of $-A^T = (0, 0.7, 0, 0.3)$
minimax strategy of $A =$ maximin strategy of $-A^T = (0.4, 0.6)$

□

Theorem 1.4.5 (Principle of indifference)**.** *Let A be an $m \times n$ game matrix. Suppose ν is the value of A, $\mathbf{p} = (p_1, \cdots, p_m)$ be a maximin strategy for the row player and $\mathbf{q} = (q_1, \cdots, q_n)$ be a minimax strategy for the column player. For any $k = 1, 2, \cdots, m$, if $p_k > 0$, then $\sum_{j=1}^{n} a_{kj} q_j = \nu$. In particular, when the column player uses his minimax strategy $\mathbf{q}$, then the payoff to the row* *player are indifferent among all his k-th strategies with* $p_k > 0$. *Similarly, for any* $l = 1, 2, \cdots, n$, *if* $q_l > 0$, *then* $\sum_{i=1}^{m} a_{il} p_i = \nu$. *In particular, when the row player uses his maximin strategy* $\mathbf{p}$, *then the payoff to the row player are indifferent among all the l-th strategies of the column player with* $q_l > 0$.

Proof. For each $k = 1, 2, \cdots, m$, we have

$$\sum_{j=1}^{n} a_{kj} q_j \leq \nu$$

since $\mathbf{q}$ is a minimax strategy for the column player. On the other hand,

$$\nu = \mathbf{p} A \mathbf{q}^T = \sum_{k=1}^{m} p_k \left(\sum_{j=1}^{n} a_{kj} q_j \right) \leq \sum_{k=1}^{m} p_k \nu = \nu$$

Thus we have

$$p_k \sum_{j=1}^{n} a_{kj} q_j = p_k \nu$$

for any $k = 1, 2, \cdots, m$. Therefore

$$\sum_{j=1}^{n} a_{kj} q_j = \nu$$

whenever $p_k > 0$. The proof of the second statement is similar. □

Exercise 1

1. Find the values of the following game matrices by finding their saddle points

   (a) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -4 & 5 & -3 & -3 \\ 0 & 1 & 3 & -1 \\ -3 & -1 & 2 & -5 \\ 2 & -4 & 0 & -2 \end{pmatrix}$

2. Solve the following game matrix, that is, find the value of the game, a maximin strategy for the row player and a minimax strategy for the column.

(a) $\begin{pmatrix} 1 & 7 \\ 2 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & -3 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -3 \\ -3 & 5 \\ 2 & -1 \\ 4 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & -2 & 4 \\ 3 & -3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 1 & -2 & 6 \\ -1 & 0 & 1 & -2 \\ 3 & 2 & 5 & 4 \end{pmatrix}$

3. Raymond holds a black 2 and a red 9. Calvin holds a red 3 and a black 8. Each of them chooses one of the cards from his hand and then two players show the chosen cards simultaneously. If the chosen cards are of the same colour, Raymond wins and Calvin wins if the cards are of different colours. The loser pays the winner an amount equal to the number on the winner's card. Write down the game matrix, find the value of the game and the optimal strategies of the players.

4. Alex and Becky point fingers to each other, with either one finger or two fingers. If they match with one finger, Becky pays Alex 3 dollars. If they match with two fingers, Becky pays Alex 11 dollars. If they don't match, Alex pays Becky 1 dollar.

   (a) Find the optimal strategies for Alex and Becky.

   (b) Suppose Alex pays Becky k dollars as a compensation before the game. Find the value of k to make the game fair.

5. Player I and II choose integers i and j respectively where $1 \leq i, j \leq 7$. Player II pays Player I one dollar if $|i - j| = 1$. Otherwise there is no payoff. Write down the game matrix of the game, find the value of the game and the optimal strategies for the players.

6. Use the principle of indifference to solve the game with game matrix

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7. In the Mendelsohn game, two players choose an integer from 1 to 5 simultaneously. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1 dollar from its opponent. The player that chooses a number two or more larger than his opponent loses 2 dollars to its opponent. Find the game matrix and solve the game.

8. Aaron puts a chip in either his left hand or right hand. Ben guesses where the chip is. If Ben guesses the left hand, he receives \$2 from Aaron if he is correct and pays \$4 to Aaron if he is wrong. If Ben guesses the right hand, he receives \$1 from Aaron if he is correct and pays \$3 to Aaron if he is wrong.

   (a) Write down the payoff matrix of Aaron. (Use order of strategies: Left, Right.)

   (b) Find the maximin strategy for Aaron, the minimax strategy for Ben and the value of the game.

9. Let
$$A = \begin{pmatrix} -3 & 1 \\ c & -2 \end{pmatrix}$$
where c is a real number.

   (a) Find the range of values of c such that A has a saddle point.

   (b) Suppose the zero sum game with game matrix A is a fair game.

      (i) Find the value of c.

      (ii) Find the maximin strategy for the row player and the minimax strategy for the column player.

10. Prove that if A is a skewed symmetric matrix, that is, $A^T = -A$, then the value of A is zero.

11. Let n be a positive integer and $\mathbf{1} = (1, 1, \cdots, 1) \in \mathbb{R}^n$. Prove the following statements.

   (a) If A is an $n \times n$ symmetric matrix, that is $A^T = A$, and there exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T = v\mathbf{1}^T$ where $v \in \mathbb{R}$ is a real number, then v is the value of A.

(b) There exists an $n \times n$ matrix A, a probability vector $\mathbf{y} \in \mathcal{P}^n$ and a real number v such that $A\mathbf{y}^T = v\mathbf{1}^T$ but v is not the value of A.

12. Let n be a positive integer and

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

be an $n \times n$ diagonal matrix where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(a) Suppose $\lambda_1 \leq 0$ and $\lambda_n > 0$. Find the value of the zero sum game with game matrix D.

(b) Suppose $\lambda_1 > 0$. Solve the zero sum game with game matrix D.

13. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(a) Find a vector $\mathbf{x} = (1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and a real number a such that

$$A\mathbf{x}^T = (0, 0, 0, 0, a)^T$$

(b) Find a vector $\mathbf{y} = (1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5$ and a real number b such that

$$A\mathbf{y}^T = (1, 1, 1, 1, b)^T$$

(c) Find the maximin strategy, the minimax strategy and the value of A. (Hint: Find real numbers $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{q} = \alpha\mathbf{x} + \beta\mathbf{y}$ satisfies $A\mathbf{q}^T = v\mathbf{1}^T$ for some $v \in \mathbb{R}$.)

14. For positive integer k, define

$$A_k = \begin{pmatrix} 4k-3 & -(4k-2) \\ -(4k-1) & 4k \end{pmatrix}.$$

(a) Solve A_k, that is, find the maximin strategy, minimax strategy and value of A_k in terms of k.

(b) Let $r_1, r_2, \cdots, r_n > 0$ be positive real numbers. Using the principle of indifference, or otherwise, find, in terms of $r_1, r_2, \cdots, r_n$, the value of

$$D = \begin{pmatrix} \frac{1}{r_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{r_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{r_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{r_n} \end{pmatrix}.$$

(c) Find, with proof, the value of the matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{25} \end{pmatrix}.$$

2 Linear programming and maximin theorem

2.1 Linear programming

In this chapter we study two-person zero sum game with $m \times n$ game matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Suppose the row player uses strategy $\mathbf{x} = (x_1, \cdots, x_m) \in \mathcal{P}^m$. Then the column player would use his j-th strategy such that

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m$$

is minimum among $j = 1, 2, \cdots, n$. Thus the payoff to the row player that he can guarantee is

$$\min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

Hence if the above expression attains its maximum at $\mathbf{x} = \mathbf{p} \in \mathcal{P}^m$, then $\mathbf{p}$ is a maximin strategy for the row player. Moreover, the value of the game is

$$\nu = \max_{\mathbf{x} \in \mathcal{P}^m} \min_{j=1,2,\cdots,n} \{a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m\}$$

By introducing a new variable v, we can restate the **maximin problem**, that is finding a maximin strategy, as the following linear programming problem

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

Similarly, to find a minimax strategy for the column player, we need to solve the following **minimax** problem

$$\begin{array}{rl} \text{min} & v \\ \text{subject to} & a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n \leq v \\ & a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n \leq v \\ & \vdots \\ & a_{m1}q_1 + a_{m2}q_2 + \cdots + a_{mn}q_n \leq v \\ & q_1 + q_2 + \cdots + q_n = 1 \\ & q_1, q_2, \cdots, q_n \geq 0 \end{array}$$

To solve the maximin and minimax problems, first we transform them to a pair of primal and dual problems.

Definition 2.1.1 (Primal and dual problems)**.** *A linear programming problem in the following form is called a* **primal problem***.*

$$\begin{array}{rl} \text{max} & f(y_1, \cdots, y_n) = \sum_{j=1}^{n} c_j y_j + d \\ \textit{subject to} & \sum_{j=1}^{n} a_{ij} y_j \leq b_i, \ i = 1, 2, \cdots, m \\ & y_1, y_2, \cdots, y_n \geq 0 \end{array}$$

The **dual problem** *associated to the above primal problem is*

$$\begin{array}{rl} \text{min} & g(x_1, \cdots, x_m) = \sum_{i=1}^{m} b_i x_i + d \\ \textit{subject to} & \sum_{i=1}^{m} a_{ij} x_i \geq c_j, \ j = 1, 2, \cdots, n \\ & x_1, x_2, \cdots, x_m \geq 0 \end{array}$$

Here $x_1, \cdots, x_m, y_1, \cdots, y_n$ *are variables, and* $a_{ij}, b_i, c_j, d,\ i = 1, 2, \cdots, m,$ $j = 1, 2, \cdots, n,$ *are constants. The linear functions* f *and* g *are called* **objective functions***. The primal problem and the dual problem can be written in the following matrix forms*

Primal problem	max *subject to* 	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$ $\mathbf{y} \geq \mathbf{0}$
Dual problem	min *subject to* 	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$ $\mathbf{x} \geq \mathbf{0}$

Here $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ *are variable vectors,* A *is an* $m \times n$ *constant matrix,* $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$ *are constant vectors and* $d \in \mathbb{R}$ *is a real constant. The inequality* $\mathbf{u} \leq \mathbf{v}$ *for vectors* $\mathbf{u}, \mathbf{v}$ *means each of the coordinates of* $\mathbf{v} - \mathbf{u}$ *is non-negative.*

For primal and dual problems, we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$. In other words, all variables are non-negative. From now on, we will not write down the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$ for primal and dual problems and it is understood that all variables are non-negative.

Definition 2.1.2. *Suppose we have a pair of primal and dual problems.*

1. *We say that a vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **feasible** *if it satisfies the constraints of the problem. We say that the primal problem (or the dual problem) is feasible there exists a feasible vector for the problem.*

2. *We say that the primal problem, (or the dual problem), is* **bounded** *if the objective function is bounded above, (or below) on the set of feasible vectors.*

3. *We say that a feasible vector* $\mathbf{x} \in \mathbb{R}^m$ *in the dual problem, (or* $\mathbf{y} \in \mathbb{R}^n$ *in the primal problem), is* **optimal** *if the objective function* f *(or* g*) attains its maximum (or minimax) at* $\mathbf{x}$ *(or* $\mathbf{y}$*) on the set of feasible vectors.*

Theorem 2.1.3. *Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *are feasible vectors in the dual and primal problems respectively. Then*

$$f(\mathbf{y}) \leq g(\mathbf{x})$$

Proof. We have

$$\begin{aligned} f(\mathbf{y}) &= \mathbf{c}\mathbf{y}^T + d \\ &\leq \mathbf{x}A\mathbf{y}^T + d \quad \text{(since } \mathbf{x} \text{ is feasible and } \mathbf{y} \geq \mathbf{0}) \\ &\leq \mathbf{x}\mathbf{b}^T + d \quad \text{(since } \mathbf{y} \text{ is feasible and } \mathbf{x} \geq \mathbf{0}) \\ &= g(\mathbf{x}) \end{aligned}$$

□

The theorem above has a simple and important consequence that the primal problem is bounded if the dual problem associated with it has a feasible vector, and vice verse.

Theorem 2.1.4. *Suppose we have a pair of primal and dual problems.*

1. *If the primal problem is feasible, then the dual problem is bounded.*
2. *If the dual problem is feasible, then the primal problem is bounded.*
3. *If both problems are feasible, then both problems are solvable, that is, there exists optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problems respectively. Moreover we have* $f(\mathbf{p}) \leq g(\mathbf{q})$*.*

Proof. For the first statement, suppose the primal problem has a feasible vector $\mathbf{q}$. Then for any feasible vector $\mathbf{x}$ of the dual problem, we have $g(\mathbf{x}) \geq f(\mathbf{q})$ by Theorem 2.1.3. Hence the dual problem is bounded. The proof of the second statement is similar. For the third statement, suppose both problems are feasible. Then both problems are bounded by the first two statements. Observe that the set of feasible vectors is closed. It follows that the optimal values of the objective functions f and g are attainable. Therefore there exists optimal vectors $\mathbf{p}$ and $\mathbf{q}$ for the dual and primal problems respectively and $f(\mathbf{q}) \leq g(\mathbf{p})$ by Theorem 2.1.3. □

Furthermore we have the following important theorem in linear programming concerning the solutions to the primal and dual problems.

Theorem 2.1.5. *Suppose both the dual problem and the primal problem are feasible. Then there exist optimal vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *for the dual and primal problem respectively, and we have*

$$f(\mathbf{q}) = g(\mathbf{p})$$

Proof. We have proved the solvability of the problems. The equality $f(\mathbf{q}) = g(\mathbf{p})$ can be proved using minimax theorem and we omit the proof here. □

2.2 Transforming maximin problem to dual problem

To find a maximin strategy for the row player of a two-person zero sum game, we have seen in the previous section that we need to solve the following maximin problem.

$$\begin{array}{rl} \max & v \\ \text{subject to} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq v \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq v \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq v \\ & p_1 + p_2 + \cdots + p_m = 1 \\ & p_1, p_2, \cdots, p_m \geq 0 \end{array}$$

which can be written into following matrix form

$$\begin{array}{rl} \max & v \\ \text{subject to} & \mathbf{p}A \geq v\mathbf{1} \\ & \mathbf{p}\mathbf{1}^T = 1 \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$. We solve the above maximin problem in the following two steps.

1. Transform the maximin problem to a dual problem.

2. Use simplex method to solve the dual problem.

In this section, we are going to discuss how to transform a maximin problem to a dual problem. Note that the maximin problem is neither a primal nor dual problem because there is a constraint $p_1 + p_2 + \cdots + p_m = 1$ which is not allowed and we do not have the constraint $v \geq 0$. To transform the maximin problem into a dual problem, first we add a constant k to each entry of A so that the value of the game matrix is positive. Secondly, we let

$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

Then to maximize v is the same as minimizing

$$x_1 + x_2 + \cdots + x_m = \frac{p_1 + p_2 + \cdots + p_m}{v} = \frac{1}{v}$$

Moreover for each $j = 1, 2, \cdots, n$, the constraint

$$a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{mj}p_m \geq v$$

is equivalent to

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{mj}x_m \geq 1$$

and the maximin problem would become a dual problem. We summarize the above procedures as follows.

1. First, add a constant k to each entry of A so that every entry of A is positive. (This is done to make sure that the value of the game matrix is positive.)

2. Let
$$x_i = \frac{p_i}{v}, \text{ for } i = 1, 2, \cdots, m$$

3. Write down the dual problem
$$\begin{array}{rl} \min & g(x_1, x_2, \cdots, x_m) = x_1 + x_2 + \cdots + x_m \\ \text{subject to} & a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq 1 \\ & a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq 1 \\ & \vdots \\ & a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq 1 \end{array}$$
(Note that we always have the constraints $x_1, x_2, \cdots, x_m \geq 0$) or in matrix form
$$\begin{array}{rl} \min & g(\mathbf{x}) = \mathbf{x}\mathbf{1}^T \\ \text{subject to} & \mathbf{x}A \geq \mathbf{1} \end{array}$$
where $\mathbf{1} = (1, 1 \cdots, 1) \in \mathbb{R}^m$.

4. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ is an optimal vector of the dual problem and
$$d = g(\mathbf{x}) = x_1 + x_2 + \cdots + x_m$$
is the minimum value. Then
$$\mathbf{p} = \frac{\mathbf{x}}{d} = \left(\frac{x_1}{d}, \frac{x_2}{d}, \cdots, \frac{x_m}{d}\right)$$
is a maximin strategy for the row player and the value of the game matrix A is
$$v = \frac{1}{d} - k$$

To find the minimax strategy for the column player, we need to solve the following minimax problem.

$$\begin{array}{rl} \min & v \\ \text{subject to} & A\mathbf{q}^T \leq v\mathbf{1}^T \\ & \mathbf{1}\mathbf{q}^T = 1 \\ & \mathbf{q} \geq \mathbf{0} \end{array}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$. If we assume that $v > 0$, the above optimization problem can be transformed to the following primal problem by taking $y_j = \frac{q_j}{v}$ for $j = 1, 2, \cdots, n$.

$$\begin{array}{rl} \max & f(\mathbf{y}) = \mathbf{1}\mathbf{y}^T \\ \text{subject to} & A\mathbf{y} \leq \mathbf{1}^T \end{array}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. (Note that we always have the constraint $\mathbf{y} \geq \mathbf{0}$ for primal problem.) Suppose $\mathbf{y}$ is an optimal vector for the above primal problem. Then $\mathbf{q} = \frac{\mathbf{y}}{d}$ is a minimax strategy for the column player.

2.3 Simplex method

We have seen that a pair of maximin and minimax problems can be transformed to a pair of dual and primal problems. In this section, we will show how to use simplex method to solve the dual and primal problems simultaneously. Recall that the primal and dual problems are optimization problems of the following forms. Please be reminded that we always have the constraints $\mathbf{x}, \mathbf{y} \geq \mathbf{0}$.

Primal problem	max subject to	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T \leq \mathbf{b}^T$
Dual problem	min subject to	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A \geq \mathbf{c}$

We describe the **simplex method** as follows.

Step 1. Introduce new variables $x_{m+1}, \cdots, x_{m+m}, y_{n+1}, \cdots, y_{n+m}$ which are called **slack variables** and set up the tableau

	y_1	$\cdots$	y_n	-1	
x_1	a_{11}	$\cdots$	a_{1n}	b_1	$= -y_{n+1}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	b_m	$= -y_{n+m}$
-1	c_1	$\cdots$	c_n	$-d$	$= f$
	$\shortparallel$	$\cdots$	$\shortparallel$	$\shortparallel$	
	x_{m+1}	$\cdots$	x_{m+n}	g	

Step 2.

(i) If $c_1, c_2, \cdots, c_n \le 0$, then the solution to the problems are

Primal problem	maximum value of $f = d$ $y_1 = y_2 = \cdots = y_n = 0$, $y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m$
Dual problem	minimum value of $g = d$ $x_1 = x_2 = \cdots = x_m = 0$, $x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_m$

(ii) Otherwise go to step 3.

Step 3. Choose $l = 1, 2, \cdots, n$ such that $c_l > 0$.

(i) If $a_{il} \le 0$ for all $i = 1, 2, \cdots, m$, then the problems are unbounded (because y_l can be arbitrarily large) and there is no solution.

(ii) Otherwise choose $k = 1, 2, \cdots, m$, such that

$$\frac{b_k}{a_{kl}} = \min_{a_{il}>0} \left\{ \frac{b_i}{a_{il}} \right\}$$

Step 4. Pivot on the entry a_{kl} and swap the variables at the pivot row with the variables at the pivot column. The **pivoting operation** is performed as follows.

	y_l	y_j	
x_k	a^*	b	$= -y_{n+k}$
x_i	c	d	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_{m+l}	x_{m+j}	

$\longrightarrow$

	y_{n+k}	y_j	
x_{m+l}	$\frac{1}{a}$	$\frac{b}{a}$	$= -y_l$
x_i	$-\frac{c}{a}$	$d - \frac{bc}{a}$	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_k	x_{m+j}	

Step 5. Go to Step 2.

To understand how the simplex method works, we introduce basic forms of linear programming problem.

Definition 2.3.1 (Basic form)**.** *A* **basic form** *of a pair of primal and dual problems is a problem of the form*

Primal basic form	max *subject to*	$f(\mathbf{y}) = \mathbf{c}\mathbf{y}^T + d$ $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ $\mathbf{y} \geq \mathbf{0}$
Dual basic form	min *subject to*	$g(\mathbf{x}) = \mathbf{x}\mathbf{b}^T + d$ $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ $\mathbf{x} \geq \mathbf{0}$

where $\mathbf{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ *and* $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$*. The pair of basic forms can be represented by the tableau*

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

The variables at the rightmost column and at the bottom row are called **basic variables***. The other variables at the leftmost columns and at the top row are called* **independent/non-basic variables***.*

A pair of primal and dual problems may be expressed in basic form in many different ways. The pivot operation changes one basic form of the pair of primal and dual problems to another basic form of the same pair of problems, and swaps one basic variable with one independent variable.

Theorem 2.3.2. *The basic forms before and after a pivot operation are equivalent.*

Proof. The tableau before the pivot operation

	y_l	y_j	
x_k	a^*	b	$= -y_{n+k}$
x_i	c	d	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_{m+l}	x_{m+j}	

is equivalent to the system of equations

$$\begin{cases} ax_k + cx_i = x_{m+l} \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} ay_l + by_j = -y_{n+k} \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} -x_{m+l} + cx_i = -ax_k \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} y_{n+k} + by_j = -ay_l \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ bx_k + dx_i = x_{m+j} \end{cases} \text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ cy_l + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ b\left(\frac{1}{a}x_{m+l} - \frac{c}{a}x_i\right) + dx_i = x_{m+j} \end{cases}$$

$$\text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ c\left(\frac{1}{a}y_{n+k} + \frac{b}{a}y_j\right) + dy_j = -y_{n+i} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{a}x_{m+l} - \frac{c}{a}x_i = x_k \\ \frac{b}{a}x_{m+l} + \left(d - \frac{bc}{a}\right)x_i = x_{m+j} \end{cases}$$

$$\text{and} \begin{cases} \frac{1}{a}y_{n+k} + \frac{b}{a}y_j = -y_l \\ -\frac{c}{a}y_{n+k} + \left(d - \frac{bc}{a}\right)y_j = -y_{n+i} \end{cases}$$

which is equivalent to the tableau

	y_{n+k}	y_j	
x_{m+l}	$\frac{1}{a}$	$\frac{b}{a}$	$= -y_l$
x_i	$-\frac{c}{a}$	$d - \frac{bc}{a}$	$= -y_{n+i}$
	$\shortparallel$	$\shortparallel$	
	x_k	x_{m+j}	

The above calculation shows that the constraints before and after a pivot operation are equivalent, and the values of the objective functions f and g for any given $x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}$ and $y_1, \cdots, y_n, y_{n+1}, \cdots, y_{m+n}$ satisfying the constraints remain unchanged. □

For each pair of basic forms, there associates a pair of basic solutions which will be defined below. Note that the basic solutions are not really solutions to the primal and dual problems because basic solutions are not necessarily feasible.

Definition 2.3.3 (Basic solution). *Suppose we have a pair of basic forms represented by the tableau*

$$
\begin{array}{c|ccc|c|l}
 & y_1 & \cdots & y_n & -1 & \\
\hline
x_1 & a_{11} & \cdots & a_{1n} & b_1 & = -y_{n+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \quad\vdots \\
x_m & a_{m1} & \cdots & a_{mn} & b_m & = -y_{n+m} \\
\hline
-1 & c_1 & \cdots & c_n & -d & = f \\
 & \shortparallel & \cdots & \shortparallel & \shortparallel & \\
 & x_{m+1} & \cdots & x_{m+n} & g &
\end{array}
$$

The **basic solution** *to the basic form is*

$$
\begin{aligned}
&x_1 = x_2 = \cdots = x_m = 0, x_{m+1} = -c_1, x_{m+2} = -c_2, \cdots, x_{m+n} = -c_n \\
&y_1 = y_2 = \cdots = y_n = 0, y_{n+1} = b_1, y_{n+2} = b_2, \cdots, y_{n+m} = b_m
\end{aligned}
$$

The basic solutions are obtained by setting the independent variables, that is the variables at the top and at the left, to be 0 and then solving for the basic variables, that is the variables at the bottom and at the right, by the constraints.

The basic solutions always satisfy the equalities in the constraints, but they may not be feasible since some variables may have negative values. However if both the dual and primal basic solutions are feasible, then they must be optimal.

Theorem 2.3.4. *Suppose we have a pair of basic forms.*

1. *The basic solution to the primal basic form is feasible if and only if* $b_1, b_2, \cdots, b_m \geq 0$.

2. *The basic solution to the dual basic form is feasible if and only if* $c_1, c_2, \cdots, c_n \le 0$.

3. *The pair of basic solutions are optimal if* $b_1, \cdots, b_m \ge 0$ *and* $c_1, \cdots, c_n \le 0$.

Proof. Observe that the basic solutions always satisfy the equalities $\mathbf{x}A - \mathbf{c} = (x_{m+1}, \cdots, x_{m+n})$ and $A\mathbf{y}^T - \mathbf{b}^T = -(y_{n+1}, \cdots, y_{n+m})^T$ of the constraints.

1. The basic solution to the primal basic form is $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$. Thus it is feasible if and only if all the variables are non-negative which is equivalent to $b_1, b_2, \cdots, b_m \ge 0$.

2. The basic solution to the dual basic form is $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$. Thus it is is feasible if and only if all the variables are non-negative which is equivalent to $c_1, c_2, \cdots, c_n \le 0$.

3. Suppose $b_1, b_2, \cdots, b_m \ge 0$ and $c_1, c_2, \cdots, c_n \le 0$. For any feasible vectors $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n})$ of the dual basic form and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m})$ of the primal basic form, we have

$$\begin{aligned} f(y_1, \cdots, y_n) &= (c_1, \cdots, c_n)(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)A(y_1, \cdots, y_n)^T + d \\ &\le (x_1, \cdots, x_m)(b_1, \cdots, b_m)^T + d \\ &= g(x_1, \cdots, x_m) \end{aligned}$$

   On the other hand, the basic solutions $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+n}) = (0, \cdots, 0, -c_1, \cdots, -c_n)$ and $(y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}) = (0, \cdots, 0, b_1, \cdots, b_m)$ are feasible and

$$f(0, \cdots, 0) = d = g(0, \cdots, 0)$$

   Therefore f attains its maximin and g attains its minimum at the basic solutions.

□

In practice, we do not write down the basic variables. We would swap the variables at the left and at the top when preforming pivot operation. One may find the basic and independent variables by referring to the following table.

	Left	Top
x_i	x_i is independent variable y_{n+i} is basic variable	x_i is basic variable y_{n+i} is independent variable
y_j	y_j is basic variable x_{m+j} is independent variable	y_j is independent variable x_{m+j} is basic variable

In other words, when we write down a tableau of the form

$$\begin{array}{c|cc|c} & x_i & y_l & -1 \\ \hline y_j & & & b_i \\ x_k & & A & b_k \\ \hline -1 & c_j & c_l & -d \end{array}$$

the basic solution associated with it is

$$\begin{array}{l} x_i = -c_j, x_k = 0, x_{m+j} = 0, x_{m+l} = -c_l \\ y_j = b_i, y_l = 0, y_{n+i} = 0, y_{n+k} = b_k \end{array}$$

and the genuine tableau is

$$\begin{array}{c|cc|l} & y_{n+i} & y_l & -1 \\ \hline x_{m+j} & & & b_i = -y_j \\ x_k & & A & b_k = -y_{n+k} \\ \hline -1 & c_j & c_l & -d \\ & \shortparallel & \shortparallel & \\ & x_i & x_{m+l} & \end{array}$$

Example 2.3.5. *Solve the following primal problem.*

$$\begin{array}{rl} \max & f = 6y_1 + 4y_2 + 5y_3 + 150 \\ \textit{subject to} & 2y_1 + y_2 + y_3 \leq 180 \\ & y_1 + 2y_2 + 3y_3 \leq 300 \\ & 2y_1 + 2y_2 + y_3 \leq 240 \end{array}$$

Solution. Set up the tableau and perform pivot operations successively. The pivoting entries are marked with asterisks.

	y_1	y_2	y_3	-1
x_1	2^*	1	1	180
x_2	1	2	3	300
x_3	2	2	1	240
-1	6	4	5	-150

$\longrightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	90
x_2	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	210
x_3	-1	1^*	0	60
-1	-3	1	2	-690

$\longrightarrow$

	x_1	x_3	y_3	-1
y_1	1	$-\frac{1}{2}$	$\frac{1}{2}$	60
x_2	1	$-\frac{3}{2}$	$\frac{5}{2}^*$	120
y_2	-1	1	0	60
-1	-2	-1	2	-750

$\longrightarrow$

	x_1	x_3	x_2	-1
y_1	$\frac{4}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	36
y_3	$\frac{2}{5}$	$-\frac{3}{5}$	$\frac{2}{5}$	48
y_2	-1	1^*	0	60
-1	$-\frac{14}{5}$	$\frac{1}{5}$	$-\frac{4}{5}$	-846

$\longrightarrow$

	x_1	y_2	x_2	-1
y_1	$\frac{3}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	48
y_3	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	84
x_3	-1	1	0	60
-1	$-\frac{13}{5}$	$-\frac{1}{5}$	$-\frac{4}{5}$	-858

The independent variables are y_2, y_4, y_5 and the basic variables are y_1, y_3, y_6. The basic solution is

$$y_2 = y_4 = y_5 = 0, y_1 = 48, y_3 = 84, y_6 = 60$$

Thus an optimal vector for the primal problem is

$$(y_1, y_2, y_3) = (48, 0, 84)$$

The maximum value of f is 858.

We may also write down an optimal solution to the dual problem. The dual problem is

$$\begin{array}{rl} \min & g = 180x_1 + 300x_2 + 240x_3 + 150 \\ \text{subject to} & 2x_1 + x_2 + 2x_3 \geq 6 \\ & x_1 + 2x_2 + 2x_3 \geq 4 \\ & x_1 + 3x_2 + x_3 \geq 5 \end{array}$$

From the last tableau, the independent variables are x_3, x_4, x_6 and the basic variables are x_1, x_2, x_5. The basic solution is

$$x_3 = x_4 = x_6 = 0, x_1 = \frac{13}{5}, x_2 = \frac{4}{5}, x_5 = \frac{1}{5}$$

Therefore an optimal vector for the dual problem is

$$(x_1, x_2, x_3) = \left(\frac{13}{5}, \frac{4}{5}, 0\right)$$

The minimum value of g is 858 which is equal to the maximum value of f.□

To use simplex method solving a game matrix, first we add a constant k to every entry so that the entries are all non-negative and there is no zero column. This is done to make sure that the value of the new matrix is positive. Then we take $\mathbf{b} = (1, \cdots, 1) \in \mathbb{R}^m$, $\mathbf{c} = (1, \cdots, 1) \in \mathbb{R}^n$ to set up the initial tableau

	y_1	$\cdots$	y_n	
x_1	a_{11}	$\cdots$	a_{1n}	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	1
	1	$\cdots$	1	0

and apply the simplex algorithm. Then the value of the game matrix is

$$\nu = \frac{1}{d} - k$$

where d is the maximum value of f or the minimum value of g, and k is the constant which is added to the game matrix at the beginning. A maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}\mathbf{x} = \frac{1}{d}(x_1, x_2, \cdots, x_m)$$

and a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}\mathbf{y} = \frac{1}{d}(y_1, x_2, \cdots, y_n)$$

To avoid making mistakes, one may check that the following conditions must be satisfied in every step.

1. The rightmost number in each row is always non-negative. This is guaranteed by the choice of the pivoting entry.

2. The value of the number in the lower right corner is always equal to the sum of those entries in the lower row which associate with x_i's at the top row (and similarly equal to the sum of those entries at the rightmost column associate with y_j's at the leftmost column.)

3. The value of the number in the lower right corner never increases.

Finally, one may also check that the result should satisfy the following two conditions.

1. Every entry of $\mathbf{p}A$ is larger than or equal to ν.

2. Every entry of $A\mathbf{q}^T$ is less than or equal to ν.

Example 2.3.6. *Solve the two-person zero sum game with game matrix*

$$\begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

Solution. Add $k = 1$ to each of the entries, we obtain the matrix

$$\begin{pmatrix} 0 & 6 & 4 & 3 \\ 7 & 0 & 1 & 5 \end{pmatrix}$$

Applying simplex algorithm, we have

	y_1	y_2	y_3	y_4	-1
x_1	0	6	4	3	1
x_2	7^*	0	1	5	1
-1	1	1	1	1	0

$\longrightarrow$

	x_2	y_2	y_3	y_4	-1
x_1	0	6^*	4	3	1
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	1	$\frac{6}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$

$\longrightarrow$

	x_2	x_1	y_3	y_4	-1
y_2	0	$\frac{1}{6}$	$\frac{2}{3}^*$	$\frac{1}{2}$	$\frac{1}{6}$
y_1	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{1}{7}$
-1	$-\frac{1}{7}$	$-\frac{1}{6}$	$\frac{4}{21}$	$-\frac{3}{14}$	$-\frac{13}{42}$

$\longrightarrow$

	x_2	x_1	y_2	y_4	-1
y_3	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{1}{4}$
y_1	$-\frac{1}{7}$	$-\frac{1}{28}$	$-\frac{3}{14}$	$\frac{17}{28}$	$\frac{3}{28}$
-1	$-\frac{1}{7}$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{5}{14}$

The independent variables are $x_3, x_5, y_2, y_4, y_5, y_6$ and the basic variables are $x_1, x_2, x_4, x_6, y_1, y_3$. The basic solution is

$$x_3 = x_5 = 0, x_1 = \frac{3}{14}, x_2 = \frac{1}{7}, x_4 = \frac{2}{7}, x_6 = \frac{5}{14}$$
$$y_2 = y_4 = y_5 = y_6 = 0, y_1 = \frac{3}{28}, y_3 = \frac{1}{4}$$

The optimal value is $d = \dfrac{5}{14}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2) = \frac{14}{5}\left(\frac{3}{14}, \frac{1}{7}\right) = \left(\frac{3}{5}, \frac{2}{5}\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3, y_4) = \frac{14}{5}\left(\frac{3}{28}, 0, \frac{1}{4}, 0\right) = \left(\frac{3}{10}, 0, \frac{7}{10}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{14}{5} - 1 = \frac{9}{5}$$

□

Example 2.3.7. *Solve the two-person zero sum game with game matrix*

$$A = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

Solution. Add 2 to each of the entries, we obtain the matrix

$$\begin{pmatrix} 4 & 1 & 8 \\ 2 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$

Applying simplex method, we have

	y_1	y_2	y_3	-1
x_1	4^*	1	8	1
x_2	2	3	1	1
x_3	0	4	3	1
-1	1	1	1	0

$\longrightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{4}$	$\frac{1}{4}$	2	$\frac{1}{4}$
x_2	$-\frac{1}{2}$	$\frac{5}{2}^*$	-3	$\frac{1}{2}$
x_3	0	4	3	1
-1	$-\frac{1}{4}$	$\frac{3}{4}$	-1	$-\frac{1}{4}$

$\longrightarrow$

	x_1	x_2	y_3	-1
y_1	$\frac{3}{10}$	$-\frac{1}{10}$	$\frac{23}{10}$	$\frac{1}{5}$
y_2	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{6}{5}$	$\frac{1}{5}$
x_3	$\frac{4}{5}$	$-\frac{8}{5}$	$\frac{39}{5}$	$\frac{1}{5}$
-1	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{2}{5}$

The independent variables are $x_3, x_4, x_5, y_3, y_4, y_5$ and the basic variables are $x_1, x_2, x_6, y_1, y_2, y_6$. The basic solution is

$$x_3 = x_4 = x_5 = 0, x_1 = \frac{1}{10}, x_2 = \frac{3}{10}, x_6 = \frac{1}{10}$$
$$y_3 = y_4 = y_5 = 0, y_1 = \frac{1}{5}, y_2 = \frac{1}{5}, y_6 = \frac{1}{5}$$

The optimal value is $d = \frac{2}{5}$. Therefore a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{5}{2}\left(\frac{1}{10}, \frac{3}{10}, 0\right) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$$

A minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{5}{2}\left(\frac{1}{5}, \frac{1}{5}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

The value of the game is

$$\nu = \frac{1}{d} - k = \frac{5}{2} - 1 = \frac{1}{2}$$

One may check the result by the following calculations

$$\mathbf{p}A = \left(\frac{1}{4}, \frac{3}{4}, 0\right)\begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix} = \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right)$$
$$A\mathbf{q}^T = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

One sees that the row player may guarantee that his payoff is at least $\frac{1}{2}$ by using $\mathbf{p} = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$ and the column player may guarantee that the payoff to the row player is at most $\frac{1}{2}$ by using $\mathbf{q} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. □

2.4 Minimax theorem

In this section, we prove the minimax theorem (Theorem 1.1.10). The theorem was first published by John von Neumann in 1928. Another way to state the minimax theorem is that the row value and the column value of a matrix are always the same.

Definition 2.4.1 (Row and column values)**.** *Let A be an $m \times n$ matrix.*

1. *The* **row value** *of A is defined*[1] *by*

$$\nu_r(A) = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T$$

2. *The* **column value** *of A is defined by*

$$\nu_c(A) = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T$$

The row value $\nu_r(A)$ of a game matrix A is the largest payoff of the row player that he may guarantee himself. The column value $\nu_c(A)$ of A is the least payoff that the column player may guarantee that the row player cannot surpass. The strategies for the players to achieve these goals are called maximin and minimax strategies.

Definition 2.4.2 (Maximin and minimax strategies)**.** *Let A be an $m \times n$ matrix.*

1. *A* **maximin strategy** *is a strategy $\mathbf{p} \in \mathcal{P}^m$ for the row player such that*

$$\min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu_r(A)$$

2. *A* **minimax strategy** *is a strategy $\mathbf{q} \in \mathcal{P}^n$ for the column player such that*

$$\max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T = \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T = \nu_c(A)$$

It can be seen readily that we always have $\nu_r(A) \le \nu_c(A)$ for any matrix A and we give a rigorous proof here.

---

[1] Note that since the payoff function $\pi(\mathbf{x},\mathbf{y}) = \mathbf{x}A\mathbf{y}^T$ is continuous and the sets $\mathcal{P}^m, \mathcal{P}^n$ are compact, that is closed and bounded, the payoff function attains its maximum and minimum by extreme value theorem.

Theorem 2.4.3. *For any $m \times n$ matrix A, we have*

$$\nu_r(A) \leq \nu_c(A)$$

Proof. Let $\mathbf{p} \in \mathcal{P}^m$ be a maximin strategy for the row player and $\mathbf{q} \in \mathcal{P}^n$ be a minimax strategy for the column player. Then we have

$$\begin{aligned} \nu_r(A) &= \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \\ &= \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{p}A\mathbf{y}^T \\ &\leq \mathbf{p}A\mathbf{q}^T \\ &\leq \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{q}^T \\ &= \min_{\mathbf{y}\in\mathcal{P}^n} \max_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}A\mathbf{y}^T \\ &= \nu_c(A) \end{aligned}$$

□

Before we prove the minimax theorem, let's study some properties of convex sets.

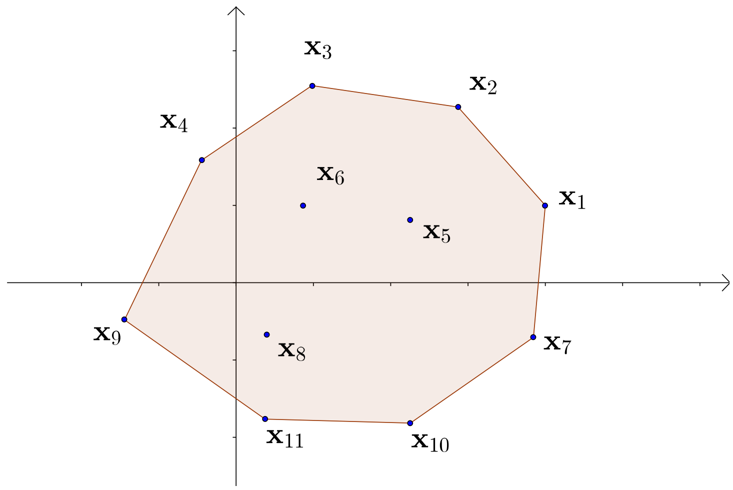
Definition 2.4.4 (Convex set)**.** *A set $C \subset \mathbb{R}^n$ is said to be* **convex** *if*

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C \text{ for any } \mathbf{x}, \mathbf{y} \in C,\ 0 \leq \lambda \leq 1$$

Geometrically, a set $C \subset \mathbb{R}^n$ is convex if the line segment joining any two points in C is contained in C. It is easy to see from the definition that intersection of convex sets is convex.

Definition 2.4.5 (Convex hull)**.** *The* **convex hull** *of a set $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$ of vectors in $\mathbb{R}^n$ is defined by*

$$\begin{aligned} &\mathrm{Conv}(\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}) \\ = \ &\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{x}_i \text{ with } \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^k \lambda_i = 1\} \end{aligned}$$



The convex hull of a set of vectors can also be defined as the smallest convex set which contains all vectors in the set.

To prove the minimax theorem, we prove a lemma concerning properties of convex sets. Recall that the standard inner product and the norm on $\mathbb{R}^n$ are defined as follows. For any $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$,

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$
2. $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

The following lemma says that we can always use a plane to separate the origin and a closed convex set C not containing the origin. It is a special case of the **hyperplane separation theorem**[2].

Lemma 2.4.6. *Let $C \subset \mathbb{R}^n$ be a closed convex set with $\mathbf{0} \notin C$. Then there exists $\mathbf{z} \in C$ such that*

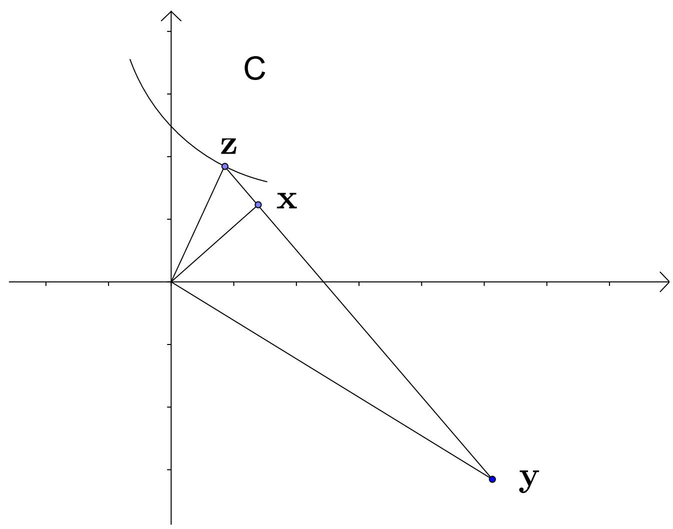
$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \text{ for any } \mathbf{y} \in C$$

[2] The hyperplane separation theorem says that we can always use a hyperplane to separate two given sets which are closed and convex, and at least one of them is bounded.

Proof. Since C is closed, there exists $\mathbf{z} \in C$ such that

$$\|\mathbf{z}\| = \min_{\mathbf{y} \in C} \|\mathbf{y}\|$$

We are going to prove that $\langle \mathbf{z}, \mathbf{y} \rangle > 0$ for any $\mathbf{y} \in C$ by contradiction. Suppose there exists $\mathbf{y} \in C$ such that $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$. Let $\mathbf{x} \in \mathbb{R}^n$ be a point which lies on the straight line passing through $\mathbf{z}$, $\mathbf{y}$, and is orthogonal to $\mathbf{z} - \mathbf{y}$. The point $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$, that is lying between $\mathbf{z}$ and $\mathbf{y}$, because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$.



Since $\mathbf{z}, \mathbf{y} \in C$ and C is convex, we have $\mathbf{x} \in C$. (The expression for $\mathbf{x}$ is not important in the proof but let's include here for reference

$$\mathbf{x} = \frac{\langle \mathbf{y} - \mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{z} + \frac{\langle \mathbf{z} - \mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y} - \mathbf{z}\|^2} \mathbf{y}$$

Note that $\frac{\langle \mathbf{y}-\mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2}, \frac{\langle \mathbf{z}-\mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} \geq 0$ because $\langle \mathbf{z}, \mathbf{y} \rangle \leq 0$ and $\frac{\langle \mathbf{y}-\mathbf{z}, \mathbf{y} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} + \frac{\langle \mathbf{z}-\mathbf{y}, \mathbf{z} \rangle}{\|\mathbf{y}-\mathbf{z}\|^2} = 1$ which shows that $\mathbf{x}$ lies on the line segment joining $\mathbf{z}$, $\mathbf{y}$.)

Moreover, we have

$$\begin{aligned} \|\mathbf{z}\|^2 &= \|\mathbf{x} + (\mathbf{z} - \mathbf{x})\|^2 \\ &= \|\mathbf{x}\|^2 + \|(\mathbf{z} - \mathbf{x})\|^2 \text{ (since } \mathbf{x} \perp \mathbf{z} - \mathbf{x}) \\ &> \|\mathbf{x}\|^2 \end{aligned}$$

which contradicts that $\mathbf{z}$ is a point in C closest to the origin $\mathbf{0}$. $\square$

The following theorem says that for any matrix A, we have either $\nu_r(A) > 0$ or $\nu_c(A) \leq 0$. The key of the proof is to consider the convex hull C generated by the column vectors of A and the standard basis for $\mathbb{R}^m$, and study the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Theorem 2.4.7. *Let A be an $m \times n$ matrix. Then one of the following statements holds.*

1. *There exists probability vector $\mathbf{x} \in \mathcal{P}^m$ such that $\mathbf{x}A > \mathbf{0}$, that is all coordinates of $\mathbf{x}A$ are positive. In this case, $\nu_r(A) > 0$.*

2. *There exists probability vector $\mathbf{y} \in \mathcal{P}^n$ such that $A\mathbf{y}^T \leq \mathbf{0}$, that is all coordinates of $A\mathbf{y}^T$ are non-positive. In this case, $\nu_c(A) \leq 0$.*

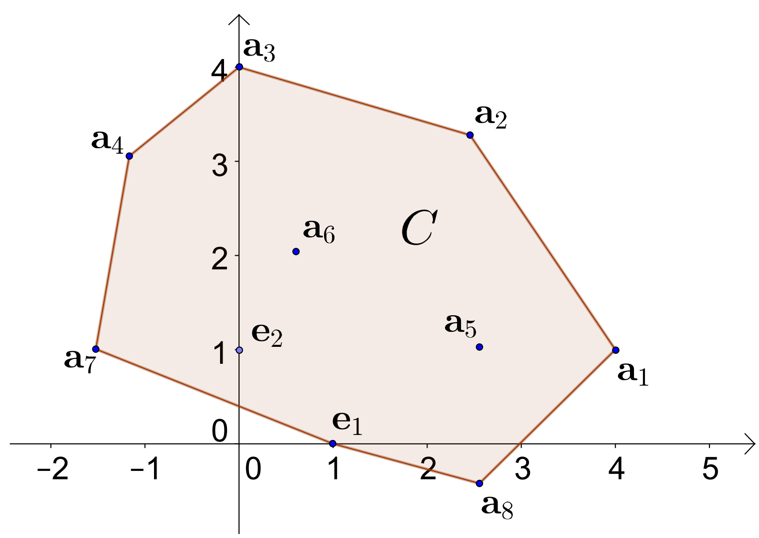
Proof. For $j = 1, 2, \cdots, n$, let

$$\mathbf{a}_j = (a_{1j}, a_{2j}, \cdots, a_{mj}) \in \mathbb{R}^m$$

In other words, $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$ are the column vectors of A and we may write $A = [\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T]$. Let

$$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

be the convex hull of $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$.



We are going to prove that the two statements in the theorem correspond to the two cases $\mathbf{0} \notin C$ and $\mathbf{0} \in C$.

Case 1. Suppose $\mathbf{0} \notin C$. Then by Lemma 2.4.6, there exists $\mathbf{z} = (z_1, z_2, \cdots, z_m) \in \mathbb{R}^m$ such that

$$\langle \mathbf{z}, \mathbf{y} \rangle > 0 \text{ for any } \mathbf{y} \in C$$

In particular, we have

$$\langle \mathbf{z}, \mathbf{e}_i \rangle = z_i > 0 \text{ for any } i = 1, 2, \cdots, m$$

Then we may take

$$\mathbf{x} = \frac{\mathbf{z}}{z_1 + z_2 + \cdots + z_m} \in \mathcal{P}^m$$

and we have

$$\langle \mathbf{x}, \mathbf{a}_j \rangle = \frac{\langle \mathbf{z}, \mathbf{a}_j \rangle}{z_1 + z_2 + \cdots + z_m} > 0 \text{ for any } j = 1, 2, \cdots, n$$

which means $\mathbf{x}A > \mathbf{0}$. Let $\alpha > 0$ be the smallest coordinate of the vector $\mathbf{x}A$ and we have

$$\nu_r(A) \geq \min_{\mathbf{y} \in \mathcal{P}^n} \mathbf{x}A\mathbf{y}^T \geq \alpha > 0$$

Case 2. Suppose $\mathbf{0} \in C$. Then there exists $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ with $\lambda_i \geq 0$ for all i, and $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$ such that

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_n \mathbf{a}_n + \lambda_{n+1} \mathbf{e}_1 + \lambda_{n+2} \mathbf{e}_2 + \cdots + \lambda_{n+m} \mathbf{e}_m = \mathbf{0}$$

which implies

$$\begin{aligned} & A \left(\lambda_1, \lambda_2, \cdots, \lambda_n\right)^T \\ = \;& \lambda_1 \mathbf{a}_1^T + \lambda_2 \mathbf{a}_2^T + \cdots + \lambda_n \mathbf{a}_n^T \\ = \;& -(\lambda_{n+1} \mathbf{e}_1^T + \lambda_{n+1} \mathbf{e}_2^T + \cdots + \lambda_{n+m} \mathbf{e}_m^T) \\ = \;& -\left(\lambda_{n+1}, \lambda_{n+2}, \cdots, \lambda_{n+m}\right)^T \end{aligned}$$

Since $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ are linearly independent, at least one of $\lambda_1, \lambda_2, \cdots, \lambda_n$ is positive for otherwise all $\lambda_1, \lambda_2, \cdots, \lambda_{m+n}$ are zero which contradicts $\lambda_1 + \lambda_2 + \cdots + \lambda_{m+n} = 1$. Then we may take

$$\mathbf{y} = \frac{(\lambda_1, \lambda_2, \cdots, \lambda_n)}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \in \mathcal{P}^n$$

and we have

$$A\mathbf{y}^T = -\frac{1}{\lambda_1 + \lambda_2 + \cdots + \lambda_n} \begin{pmatrix} \lambda_{n+1} \\ \vdots \\ \lambda_{n+m} \end{pmatrix} \leq \mathbf{0}$$

which implies

$$v_c(A) \leq \max_{\mathbf{x} \in \mathcal{P}^m} \mathbf{x} A \mathbf{y}^T \leq 0$$

□

Now we give the proof of the minimax theorem (Theorem 1.1.10) which can be stated in the following form.

Theorem 2.4.8 (Minimax theorem)**.** *For any matrix A, the row value and columns value of A are equal. In other words, we have*

$$\nu_r(A) = \nu_c(A)$$

Proof. It has been proved that $\nu_r(A) \leq \nu_c(A)$ for any matrix A (Theorem 2.4.3). We are going to prove that $\nu_c(A) \leq \nu_r(A)$ by contradiction. Suppose there exists matrix A such that $\nu_r(A) < \nu_c(A)$. Let k be a real number such that $\nu_r(A) < k < \nu_c(A)$. Let A' be the matrix obtained by subtracting every entry of A by k. Then $\nu_r(A') = \nu_r(A) - k < 0$ and $\nu_c(A') = \nu_c(A) - k > 0$ which is impossible by applying Theorem 2.4.7 to A'. The contradiction shows that $\nu_c(A) \leq \nu_r(A)$ for any matrix A and the proof of the minimax theorem is complete. □

Exercise 2

1. Solve the following primal problems. Then write down the dual problems and the solutions to the dual problems.

   (a)
$$\begin{array}{rl} \max & f = 3y_1 + 5y_2 + 4y_3 + 12 \\ \text{subject to} & 3y_1 + 2y_2 + 2y_3 \leq 15 \\ & 4y_2 + 5y_3 \leq 24 \end{array}$$

   (b)
$$\begin{array}{rl} \max & f = 2y_1 + 4y_2 + 3y_3 + y_4 \\ \text{subject to} & 3y_1 + y_2 + y_3 + 4y_4 \leq 12 \\ & y_1 - 3y_2 + 2y_3 + 3y_4 \leq 7 \\ & 2y_1 + y_2 + 3y_3 - y_4 \leq 10 \end{array}$$

2. Solve the zero sum games with the following game matrices, that is find the value of the game, a maximin strategy for the row player and a minimax strategy for the column player.

   (a) $\begin{pmatrix} 2 & -3 & 3 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 & -5 \\ -1 & -2 & 6 \\ -2 & -1 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & 0 & -2 \\ -1 & -3 & 3 \\ -2 & 2 & 0 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & -1 \\ 1 & -2 & 2 \\ -1 & 1 & -2 \end{pmatrix}$

   (f) $\begin{pmatrix} -3 & 2 & 0 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}$

3. Prove that if C_1 and C_2 are convex sets in $\mathbb{R}^n$, then the following sets are also convex.

   (a) $C_1 \cap C_2$

   (b) $C_1 + C_2 = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in C_1, \mathbf{x}_2 \in C_2\}$

4. Let A be an $m \times n$ matrix. Prove that the set of maximin strategies for the row player of A is convex.

5. Let C be a convex set in $\mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \in C$. Let $\mathbf{z} \in \mathbb{R}^n$ be a point on the straight line joining $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{z}$ is orthogonal to $\mathbf{x} - \mathbf{y}$.

   (a) Find $\mathbf{z}$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

   (b) Suppose $\langle \mathbf{x}, \mathbf{y} \rangle < 0$. Prove that $\mathbf{z} \in C$.

6. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1^T, \mathbf{a}_2^T, \cdots, \mathbf{a}_n^T$. Let $\nu_c(A)$ be the column value of A and let

$$C = \text{Conv}(\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\})$$

   where $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$. Prove that if $\nu_c(A) \leq 0$, then $\mathbf{0} \in C$.

3 Bimatrix games

In this chapter, we study bimatrix game. A bimatrix game is a two-person game with perfect information. In a bimatrix game, two players, player I and player II, choose their strategies simultaneously. Then the payoffs to the players depend on the strategies used by the players. Unlike zero sum game, we have no assumption on the sum of payoffs to the players. We will first study non-cooperative games where the solutions are the Nash equilibria. Then we will study Nash's bargaining model and threat solution in cooperative game with nontransferable and transferable utilities respectively.

3.1 Nash equilibrium

A bimatrix game can be represented by two matrices, hence its name.

Definition 3.1.1 (Bimatrix game)**.** *The normal form of a* **bimatrix game** *is given by a pair of $m \times n$ matrices (A, B). The matrices A and B are payoff matrices for the row player (player I) and the column player (player II) respectively. Suppose the row player uses strategy $\mathbf{x} \in \mathcal{P}^m$ and the column player uses strategy $\mathbf{y} \in \mathcal{P}^n$. Then the payoff to the row player and column player are given by the payoff functions*

$$\begin{aligned} \pi(\mathbf{x}, \mathbf{y}) &= \mathbf{x}A\mathbf{y}^T \\ \rho(\mathbf{x}, \mathbf{y}) &= \mathbf{x}B\mathbf{y}^T \end{aligned}$$

respectively.

Definition 3.1.2. *The* **safety level***, or* **security level***, of the row player is*

$$\mu = \max_{\mathbf{x}\in\mathcal{P}^m} \min_{\mathbf{y}\in\mathcal{P}^n} \mathbf{x}A\mathbf{y}^T = \nu(A)$$

where $\nu(A)$ denotes the value of the matrix A when A is considered as the game matrix of a two-person zero sum game. The safety level of the column player is

$$\nu = \max_{\mathbf{y}\in\mathcal{P}^n} \min_{\mathbf{x}\in\mathcal{P}^m} \mathbf{x}B\mathbf{y}^T = \nu(B^T)$$

where $\nu(B^T)$ is the value of the transpose B^T of B.

Note that the value of a matrix is defined to be the maximum payoff that the row payoff may guarantee himself. The safety level of the column player of the bimatrix game (A, B) is the value ν_{B^T} of the transpose B^T of B, not the value of B.

Definition 3.1.3 (Nash equilibrium). *Let (A, B) be a game bimatrix. We say that a pair of strategies $(\mathbf{p}, \mathbf{q})$ is an* **equilibrium pair**, *or* **mixed Nash equilibrium**, *or just* **Nash equilibrium**, *for (A, B) if*

$$\mathbf{x}A\mathbf{q}^T \le \mathbf{p}A\mathbf{q}^T \text{ for any } \mathbf{x} \in \mathcal{P}^m$$

and

$$\mathbf{p}B\mathbf{y}^T \le \mathbf{p}B\mathbf{q}^T \text{ for any } \mathbf{y} \in \mathcal{P}^n$$

Example 3.1.4 (Prisoner dilemma). *Let*

$$(A, B) = \begin{pmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{pmatrix}$$

which represents a version of the famous **prisoner dilemma**. *The strategy pair $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$ is a Nash equilibrium. The Nash equilibrium is unique in this example.* □

Example 3.1.5 (Dating game). *Consider*

$$(A, B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}.$$

It is an example of a **dating game**. There are two obvious Nash equilibria, which are pure Nash equilibria, namely $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$ and $((0,1),(0,1))$. The game has one more mixed Nash equilibrium (non-pure Nash equilibrium which is harder to find out. To see what it is, suppose the row player uses strategy $\mathbf{x} = (x, 1-x)$, where $0 \le x \le 1$. Then

$$\mathbf{x}B = (x, 1-x)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (2x, 3-3x)$$

It means that the payoff to the column player is $2x$, and $3-3x$ if the column player constantly uses his 1st, and 2nd strategies respectively. Setting $2x = 3-3x$, we have $x = 0.6$ and

$$(0.6, 0.4)B = (0.6, 0.4)\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = (1.2, 1.2)$$

Thus if the row player uses mixed strategy $(0.6, 0.4)$, then the payoff to the column player is always 1.2 no matter how the column player plays. Similarly suppose the column player uses $\mathbf{y} = (y, 1-y)$, $0 \le y \le 1$. Then

$$A\mathbf{y}^T = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 4y \\ 1-y \end{pmatrix}$$

It means that the payoff to the row player is $4y$, and $1-y$ if the row player constantly uses his 1st, and 2nd strategies respectively. Setting $4y = 1-y$, we have $y = 0.2$. Then

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix}$$

Thus if the column player uses mixed strategy $(0.2, 0.8)$, then the payoff to the row player is always 0.8 no matter how the row player plays. Therefore the strategy pair $(\mathbf{p}, \mathbf{q}) = ((0.6, 0.4), (0.2, 0.8))$ is a Nash equilibrium. In conclusion, the dating game has three Nash equilibria and we list them in the following table.

Nash equilibrium and the corresponding payoff pair

Row player's strategy $\mathbf{p}$	Column player's strategy $\mathbf{q}$	Payoff pair (π, ρ)
$(1, 0)$	$(1, 0)$	$(4, 2)$
$(0, 1)$	$(0, 1)$	$(1, 3)$
$(0.6, 0.4)$	$(0.2, 0.8)$	$(0.8, 1.2)$

□

Note that in the third Nash equilibrium of the above example, the strategy for the row player $\mathbf{p} = (0.6, 0.4)$ is the minimax strategy for the column player of B^T, not the maximin strategy for the row player of A. That means what the row player should do is to fix the payoff to its opponent (the column player) to be 1.2 instead of guaranteeing the payoff to himself to be 0.8. Similarly, the strategy for the column player $\mathbf{q} = (0.2, 0.8)$ in this Nash equilibrium is the minimax strategy for the column player of A. So the column player should use a strategy to fix the row player's payoff instead of guaranteeing his own payoff.

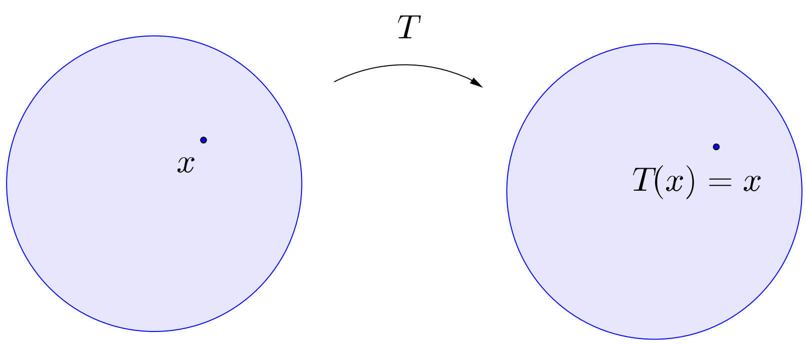
3.2 Nash's theorem

One of the most fundamental works in game theory is the following theorem of Nash which greatly extended the minimax theorem (Theorem 1.1.10). The theorem says that Nash equilibrium always exists in a non-cooperative game with finitely many players.

Theorem 3.2.1 (Nash's theorem)**.** *Every finite*[3] *game with finite number of players has at least one Nash equilibrium.*

Nash invoked the following celebrated theorem in topology to prove his theorem.

Theorem 3.2.2 (Brouwer's fixed-point theorem)**.** *Let X be a topological space which is homeomorphic to the closed unit ball $D^n = \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| \leq 1\}$. Then any continuous map $T : X \to X$ has at least one fixed-point, that is, there exists $x \in X$ such that $T(x) = x$.*

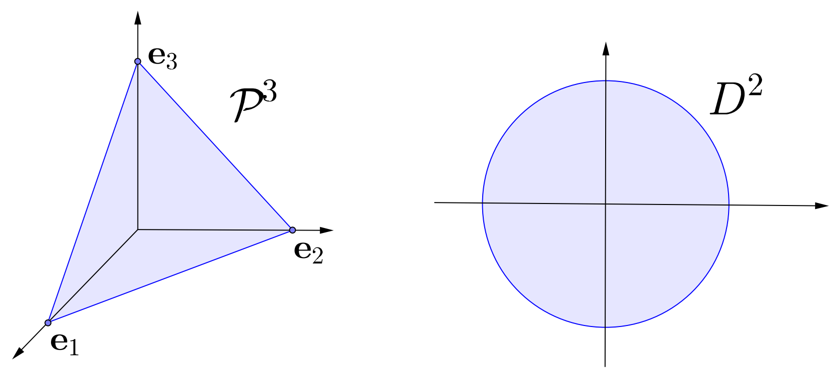


Remarks:

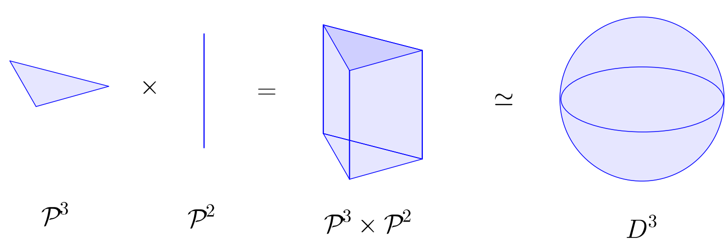
1. Two topological space X and Y are homeomorphic if there exists bijective map $\varphi : X \to Y$ such that both φ and its inverse φ^{-1} are continuous.

2. The set $\mathcal{P}^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_1, \cdots, x_n \geq 0 \text{ and } x_1 + x_2 + \cdots + x_n = 1\}$ of probability vectors in $\mathbb{R}^n$ is homeomorphic to D^{n-1}.

[3] A game is finite if the number of strategies of each player is finite.



Moreover $\mathcal{P}^m \times \mathcal{P}^n$ is homeomorphic to D^{m+n-2}.



The proof of the Brouwer's fixed-point theorem is out of the propose and scope of this notes. Now we give the proof of Nash's theorem assuming the Brouwer's fixed-point theorem.

Proof of Nash's theorem. For simplicity, we consider two-person game only. The proof for the general case is similar. Let (A, B) be the game bimatrix of a two-person game. Define $T : \mathcal{P}^m \times \mathcal{P}^n \to \mathcal{P}^m \times \mathcal{P}^n$ by

$$T(\mathbf{x}, \mathbf{y}) = (\mathbf{u}, \mathbf{v}) = ((u_1, u_2, \cdots, u_m), (v_1, v_2, \cdots, v_n))$$

where for $k = 1, 2, \cdots, m$ and $l = 1, 2, \cdots, n$,

$$u_k = \frac{x_k + c_k}{1 + \sum\limits_{i=1}^{m} c_i} \text{ and } v_l = \frac{y_l + d_l}{1 + \sum\limits_{j=1}^{n} d_j}$$

and

$$\begin{aligned} c_k &= \max\{\pi(\mathbf{e}_k, \mathbf{y}) - \pi(\mathbf{x}, \mathbf{y}) = \mathbf{e}_k A \mathbf{y}^T - \mathbf{x} A \mathbf{y}^T, 0\} \\ d_l &= \max\{\rho(\mathbf{x}, \mathbf{e}_l) - \rho(\mathbf{x}, \mathbf{y}) = \mathbf{x} B \mathbf{e}_l^T - \mathbf{x} B \mathbf{y}^T, 0\} \end{aligned}$$

Here $\mathbf{e}_k$, $\mathbf{e}_l$ are vectors in the standard bases in $\mathbb{R}^m$, $\mathbb{R}^n$ respectively. By definition, c_k is the increase of payoff of the first player if the first player changes his strategy from $\mathbf{x}$ to $\mathbf{e}_k$ while the strategy of the second player remains at $\mathbf{y}$. However, if there is no increase, then we set $c_k = 0$. The numbers d_k are similarly defined. Note that $\mathbf{u} \in \mathcal{P}^m$ and $\mathbf{v} \in \mathcal{P}^n$ because

$$c_k, d_l \geq 0$$

and

$$\begin{aligned} \sum_{k=1}^{m} \left(\frac{x_k + c_k}{1 + \sum\limits_{i=1}^{m} c_i} \right) &= \frac{\sum\limits_{k=1}^{m} x_k + \sum\limits_{k=1}^{m} c_k}{1 + \sum\limits_{i=1}^{m} c_i} = \frac{1 + \sum\limits_{k=1}^{m} c_k}{1 + \sum\limits_{i=1}^{m} c_i} = 1 \\ \sum_{l=1}^{n} \left(\frac{y_l + d_l}{1 + \sum\limits_{j=1}^{n} d_j} \right) &= \frac{\sum\limits_{l=1}^{n} y_l + \sum\limits_{l=1}^{n} d_l}{1 + \sum\limits_{j=1}^{n} d_j} = \frac{1 + \sum\limits_{l=1}^{n} d_l}{1 + \sum\limits_{j=1}^{n} d_j} = 1 \end{aligned}$$

Now T is a continuous map from $\mathcal{P}^m \times \mathcal{P}^n$ to $\mathcal{P}^m \times \mathcal{P}^n$. By Brouwer's fixed-point theorem (Theorem 3.2.2), there exists $(\mathbf{p}, \mathbf{q}) \in \mathcal{P}^m \times \mathcal{P}^n$ such that

$$T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$$

The proof of Nash's theorem is complete if we can prove that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose on the contrary that $(\mathbf{p}, \mathbf{q})$ is not a Nash equilibrium. Then either there exists $\mathbf{r} \in \mathcal{P}^m$ such that $\mathbf{r} A \mathbf{q}^T > \mathbf{p} A \mathbf{q}^T$ or there exists $\mathbf{s} \in \mathcal{P}^n$ such that $\mathbf{p} B \mathbf{s}^T > \mathbf{p} B \mathbf{q}^T$. Without loss of generality, we consider the former case. Write $\mathbf{r} = (r_1, r_2, \cdots, r_m)$. Since

$$\mathbf{p} A \mathbf{q}^T < \mathbf{r} A \mathbf{q}^T = \sum_{k=1}^{m} r_k \mathbf{e}_k A \mathbf{q}^T$$

and $\mathbf{r}$ is a probability vector, we see that there exists $1 \leq k \leq m$ such that

$$\mathbf{p}A\mathbf{q}^T < \mathbf{e}_k A\mathbf{q}^T$$

It follows that

$$c_k = \max\{\mathbf{e}_k A\mathbf{q}^T - \mathbf{p}A\mathbf{q}^T, 0\} > 0$$

and thus $\sum_{i=1}^{m} c_i > 0$. On the other hand, since

$$\mathbf{p}A\mathbf{q}^T = \sum_{i=1}^{m} p_i \mathbf{e}_i A\mathbf{q}^T$$

and $\mathbf{p}$ is a probability vector, there exists $1 \leq r \leq m$ such that $p_r > 0$ and

$$\mathbf{e}_r A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$$

which implies, by the definition of c_r, that $c_r = 0$. Hence we have

$$\frac{p_r + c_r}{1 + \sum_{i=1}^{m} c_i} = \frac{p_r}{1 + \sum_{i=1}^{m} c_i} \leq \frac{p_r}{1 + c_k} < p_r$$

which contradicts that $(\mathbf{p}, \mathbf{q})$ is a fixed-point of T. Therefore $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium and the proof of Nash's theorem is complete. □

We have seen in the proof of Nash's theorem that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium if it is a fixed-point of T. As a matter of fact, the converse of this statement is also true. For if $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, then $\mathbf{e}_i A\mathbf{q}^T \leq \mathbf{p}A\mathbf{q}^T$ for any $1 \leq i \leq m$. Thus $c_i = 0$ for any $1 \leq i \leq m$. Similarly $d_j = 0$ for any $1 \leq j \leq n$. Therefore $T(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q})$.

To find Nash equilibria of a 2×2 game bimatrix (A, B), we may let $\mathbf{x} = (x, 1-x)$, $\mathbf{y} = (y, 1-y)$ and consider the payoff functions

$$\begin{aligned} \pi(x, y) &= \pi(\mathbf{x}, \mathbf{y}) = \mathbf{x}A\mathbf{y}^T \\ \rho(x, y) &= \rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}B\mathbf{y}^T \end{aligned}$$

Define

$$\begin{aligned} P &= \{(x, y) : \pi(x, y) \text{ attains its maximum at } x \text{ for fixed } y.\} \\ Q &= \{(x, y) : \rho(x, y) \text{ attains its maximum at } y \text{ for fixed } x.\} \end{aligned}$$

Then $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium if and only if $(x, y) \in P \cap Q$.

Example 3.2.3 (Prisoner dilemma). *Consider the prisoner dilemma (Example 3.1.4) with bimatrix*

$$(A,B) = \begin{pmatrix} (-5,-5) & (-1,-10) \\ (-10,-1) & (-2,-2) \end{pmatrix}$$

The payoff to the row player is given by

$$\begin{aligned} \pi(x,y) &= (x, 1-x)\begin{pmatrix} -5 & -1 \\ -10 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x)\begin{pmatrix} -4y-1 \\ -8y-2 \end{pmatrix} \end{aligned}$$

Since $-8y-2 < -4y-1$ for any $0 \le y \le 1$, we have

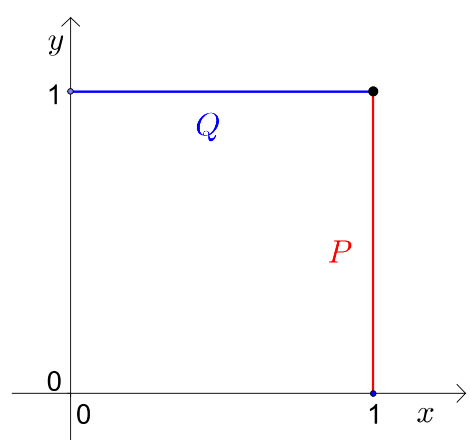
$$P = \{(1,y) : 0 \le y \le 1\}$$

On the other hand,

$$\begin{aligned} \rho(x,y) &= (x, 1-x)\begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (-4x-1, -8x-2)\begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Since $-8x-2 < -4x-1$ for any $0 \le x \le 1$, we have

$$Q = \{(x,1) : 0 \le p \le 1\}$$



Now

$$P \cap Q = \{(1,1)\}$$

Therefore the game has a unique Nash equilibrium $(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0))$.□

Example 3.2.4 (Dating game)**.** *Consider the dating game (Example 3.1.5) with bimatrix*

$$(A, B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

We have

$$\begin{aligned} \pi(x,y) &= (x, 1-x) \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (x, 1-x) \begin{pmatrix} 4y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 4y < 1-y & \text{if} \quad 0 \leq y < \frac{1}{5} \\ 4y = 1-y & \text{if} \quad y = \frac{1}{5} \\ 4y > 1-y & \text{if} \quad \frac{1}{5} < y \leq 1 \end{cases}$$

Thus

$$P = \left\{(x,y) : \left(x = 0 \wedge 0 \leq y < \frac{1}{5}\right) \vee \left(0 \leq x \leq 1 \wedge y = \frac{1}{5}\right) \vee \left(x = 1 \wedge \frac{1}{5} < y \leq 1\right)\right\}$$

On the other hand,

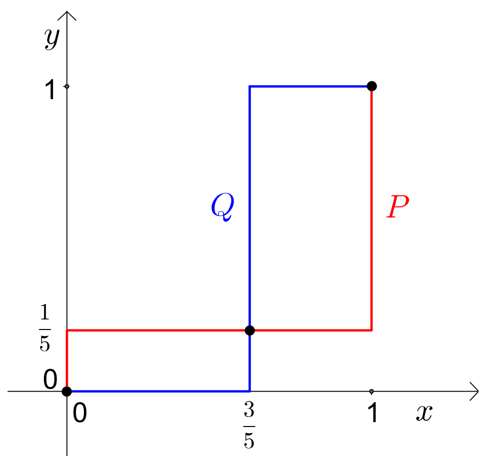
$$\begin{aligned} \rho(x,y) &= (x, 1-x) \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} \\ &= (2x, 3-3x) \begin{pmatrix} y \\ 1-y \end{pmatrix} \end{aligned}$$

Now

$$\begin{cases} 2x < 3-3x & \text{if} \quad 0 \leq x < \frac{3}{5} \\ 2x = 3-3x & \text{if} \quad x = \frac{3}{5} \\ 2x > 3-3x & \text{if} \quad \frac{3}{5} < x \leq 1 \end{cases}$$

Thus

$$Q = \left\{ (x,y) : \left(0 \le x < \frac{3}{5} \wedge y = 0 \right) \vee \left(x = \frac{3}{5} \wedge 0 \le y \le 1 \right) \vee \left(\frac{3}{5} < x \le 1 \wedge y = 1 \right) \right\}$$



Now

$$P \cap Q = \left\{ (0,0), (1,1), \left(\frac{3}{5}, \frac{1}{5} \right) \right\}$$

Therefore the game has three Nash equilibria

$$(\mathbf{p}, \mathbf{q}) = ((1,0),(1,0)), ((0,1),(0,1)), \left(\frac{3}{5}, \frac{2}{5} \right), \left(\left(\frac{1}{5}, \frac{4}{5} \right) \right).$$

We list the associated payoff pairs in the following table.

$\mathbf{p}$	$\mathbf{q}$	(π, ρ)
$(1,0)$	$(1,0)$	$(4,2)$
$(0,1)$	$(0,1)$	$(1,3)$
$\left(\frac{3}{5}, \frac{2}{5} \right)$	$\left(\frac{1}{5}, \frac{4}{5} \right)$	$\left(\frac{4}{5}, \frac{6}{5} \right)$

□

Definition 3.2.5. *Let (A, B) be a game bimatrix.*

1. *We say that two Nash equilibria* $(\mathbf{p}, \mathbf{q})$ *and* $(\mathbf{p}', \mathbf{q}')$ *are* **interchangeable** *if* $(\mathbf{p}', \mathbf{q})$ *and* $(\mathbf{p}, \mathbf{q}')$ *are also Nash equilibria.*

2. *We say that two Nash equilibria* $(\mathbf{p}, \mathbf{q})$ *and* $(\mathbf{p}', \mathbf{q}')$ *are equivalent if*

$$\pi((\mathbf{p}, \mathbf{q}), \rho(\mathbf{p}, \mathbf{q})) = \pi((\mathbf{p}', \mathbf{q}'), \rho(\mathbf{p}', \mathbf{q}'))$$

3. *We say that a bimatrix game* (A, B) *is* **solvable in the Nash sense** *if any two Nash equilibria are interchangeable and equivalent.*

For the prisoner dilemma (Example 3.2.3), there is only one Nash equilibrium. Thus the prisoner dilemma is solvable in the Nash sense. For the dating game (Example 3.2.4), there are three Nash equilibria which are not interchangeable. So the dating game is not solvable in the Nash sense.

Example 3.2.6. *Solve the game bimatrix*

$$(A, B) = \begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$$

Solution. Consider

$$A\mathbf{y}^T = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} -4y+5 \\ y+3 \end{pmatrix}$$

Now

$$\begin{cases} -4y+5 > y+3 & \text{if} \quad 0 \le y < \dfrac{2}{5} \\ -4y+5 = y+3 & \text{if} \quad y = \dfrac{2}{5} \\ -4y+5 < y+3 & \text{if} \quad \dfrac{2}{5} < y \le 1 \end{cases}$$

We see that

$$P = \left\{(x, y) : \left(x = 0 \wedge \frac{2}{5} < y \le 1\right) \vee \left(0 \le x \le 1 \wedge y = \frac{2}{5}\right) \vee \left(x = 1 \wedge 0 \le y < \frac{2}{5}\right)\right\}$$

On the other hand

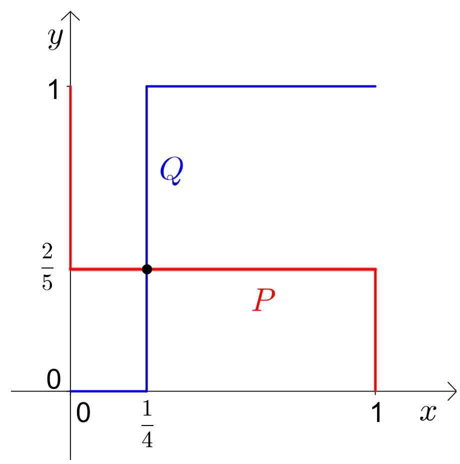
$$\mathbf{x}B = (x, 1-x) \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = (2x+2, -2x+3)$$

and

$$\begin{cases} 2x+2 < -2x+3 & \text{if} \quad 0 \le x < \dfrac{1}{4} \\ 2x+2 = -2x+3 & \text{if} \quad x = \dfrac{1}{4} \\ 2x+2 > -2x+3 & \text{if} \quad \dfrac{1}{4} < x \le 1 \end{cases}$$

We see that

$$Q = \left\{(x,y) : \left(0 \le x < \frac{1}{4} \wedge y = 0\right) \vee \left(x = \frac{1}{4} \wedge 0 \le y \le 1\right) \vee \left(\frac{1}{4} < x \le 1 \wedge y = 1\right)\right\}$$



Now

$$P \cap Q = \left\{\left(\frac{1}{4}, \frac{2}{5}\right)\right\}$$

Therefore the game has Nash equilibrium

$$(\mathbf{p}, \mathbf{q}) = \left(\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{2}{5}, \frac{3}{5}\right)\right)$$

and is solvable in the Nash sense since the Nash equilibrium is unique. □

3.3 Nash bargaining model

A bimatrix game can be played as a cooperative game with **non-transferable utility**. This means the players may make agreements on what strategies they are going to use. However they are not allowed to share the payoffs they obtained in the game. In such a game, players may use joint strategies.

Definition 3.3.1. *Let (A, B) be an $m \times n$ bimatrix of a two-person game.*

1. *A* **joint strategy** *of (A, B) is an $m \times n$ matrix*

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{pmatrix}$$

which satisfies

(i) $p_{ij} \geq 0$ *for any* $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$

(ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = 1$

In other words, P is a joint strategy if it is a **probability matrix**. *The set of all $m \times n$ probability matrices is denoted by*

$$\mathcal{P}^{m\times n} = \{P = [p_{ij}] : p_{ij} \geq 0 \text{ and } \sum p_{ij} = 1\}$$

In particular, if $\mathbf{p} = (p_1, \cdots, p_m) \in \mathcal{P}^m$ and $\mathbf{q} = (q_1, \cdots, q_n) \in \mathcal{P}^n$, then

$$\mathbf{p}^T\mathbf{q} = \begin{pmatrix} p_1q_1 & \cdots & p_1q_n \\ \vdots & \ddots & \vdots \\ p_mq_1 & \cdots & p_mq_n \end{pmatrix} \in \mathcal{P}^{m\times n}$$

is a joint strategy. In this case, the row player uses strategy $\mathbf{p}$ and the column player uses strategy $\mathbf{q}$ independently. Not all joint strategies are of this form. For example

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

cannot be expressed as the form $\mathbf{p}^T\mathbf{q}$. When this joint strategy is used, the players may flip a coin and both use their first strategies if a 'head' is obtained and both use their second strategies if a 'tail' is obtained.

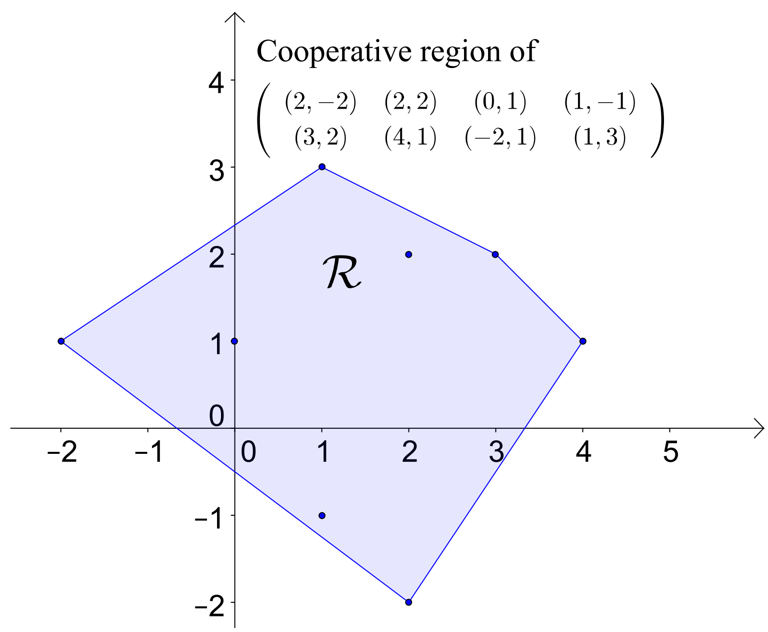
2. *For joint strategy* $P = [p_{ij}] \in \mathcal{P}^{m\times n}$*, the payoff* u *to the row player and the payoff* v *to the column player are given by the payoff pair*

$$\begin{aligned}(u(P), v(P)) &= \left(\sum_{i,j} a_{ij}p_{ij}, \sum_{i,j} b_{ij}p_{ij}\right) \\ &= \sum_{i,j} p_{ij}(a_{ij}, b_{ij})\end{aligned}$$

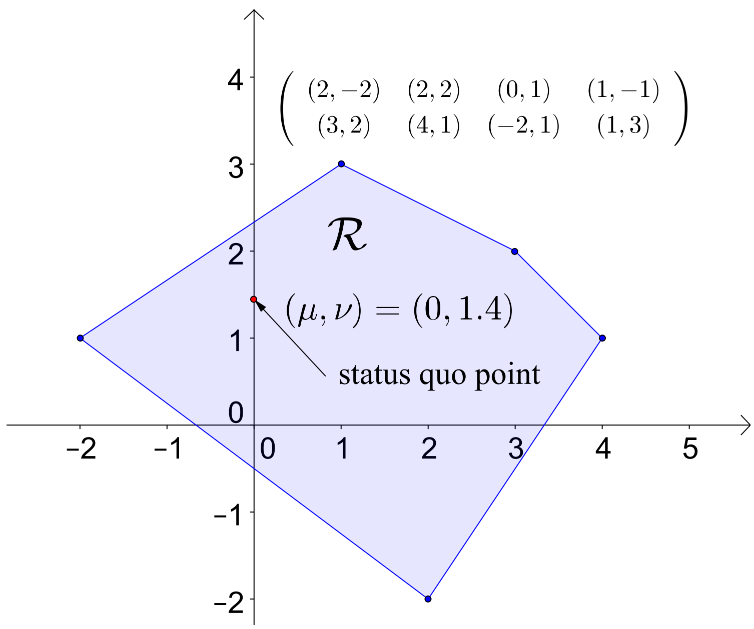
3. *The* **cooperative region** *of* (A, B) *is the set of all feasible payoff pairs*

$$\begin{aligned}\mathcal{R} &= \{(u(P), v(P)) \in \mathbb{R}^2 : P \in \mathcal{P}^{m\times n}\} \\ &= \left\{(u, v) \in \mathbb{R}^2 : (u, v) = \sum_{i,j} p_{ij}(a_{ij}, b_{ij}) \text{ for some } [p_{ij}] \in \mathcal{P}^{m\times n}\right\}\end{aligned}$$

In other words, the cooperative region $\mathcal{R}$ *is the convex hull of the set of points* $\{(a_{ij}, b_{ij}) : 1 \leq i \leq m, 1 \leq j \leq n\}$ *in* $\mathbb{R}^2$*. Note that* $\mathcal{R}$ *is a closed convex polygon in* $\mathbb{R}^2$*.*



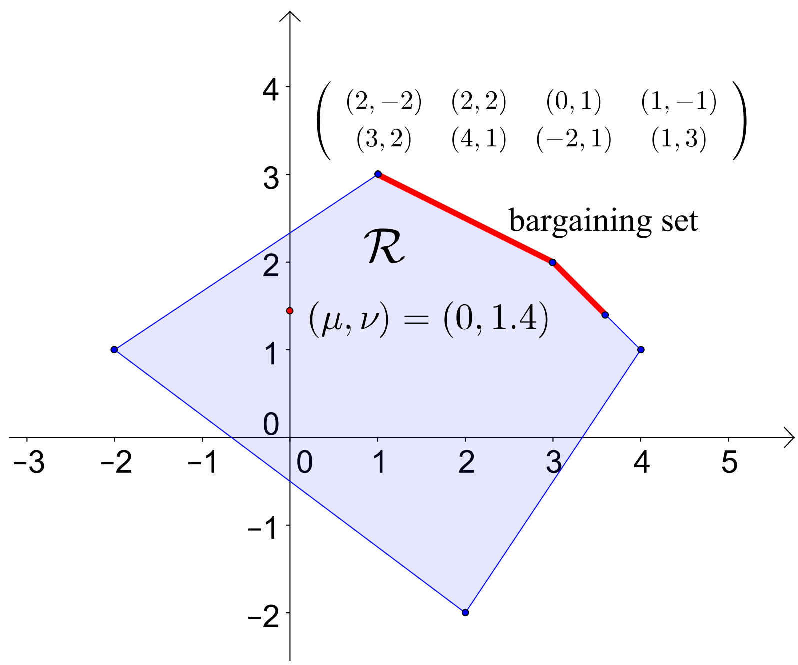
4. *The* **status quo point** *is the payoff pair* (μ, ν) *for the players associated to the solution of the game when* (A, B) *is considered as a non-cooperative game. In other words, the status quo point is the payoffs that the players may expect if the negotiations break down. Unless otherwise specified, we will take* $(\mu, \nu) = (\nu(A), \nu(B^T))$ *to be the status quo point where* $v(A)$ *and* $\nu(B^T)$ *are the values of* A *and the transpose* B^T *of* B *respectively.*



5. *We say that a payoff pair* (u, v) *is* **Pareto optimal** *if* $u' \geq u$, $v' \geq v$ *and* $(u', v') \in \mathcal{R}$ *implies* $(u', v') = (u, v)$ *where* $\mathcal{R}$ *is the cooperative region.*

6. *The* **bargaining set** *of* (A, B) *is the set of Pareto optimal payoff pairs* $(u, v) \in \mathcal{R}$ *such that* $u \geq \mu$ *and* $v \geq \nu$ *where* (μ, ν) *is the status quo point. In other words, the bargaining set is*

$$\{(u, v) \in \mathcal{R} : u \geq \mu, v \geq \nu \text{ and } (u, v) \text{ is Pareto optimal}\}$$



When the status quo point is not Pareto optimal, the two players of the game would have a tendency to cooperate. The **bargaining problem** is a problem to understand how the players should cooperate in this situation. Nash proposed that the solution to the bargaining problem is a function, called the arbitration function, depending only on the cooperative region $\mathcal{R}$ and the status quo point $(\mu, \nu) \in \mathcal{R}$, which satisfies certain properties called Nash bargaining axioms.

Definition 3.3.2 (Nash bargaining axioms)**.** *An* **arbitration function** *is a function $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$ defined for a closed and bounded convex set $\mathcal{R} \subset \mathbb{R}^2$ (cooperative region) and a point $(\mu, \nu) \in \mathcal{R}$ (status quo point) such that the following* **Nash bargaining axioms** *are satisfied.*

1. *(Individual rationality) $\alpha \geq \mu$ and $\beta \geq \nu$.*

2. *(Pareto optimality) For any $(u, v) \in \mathcal{R}$, if $u \geq \alpha$ and $v \geq \nu$, then $(u, v) = (\alpha, \beta)$.*

3. *(Feasibility) $(\alpha, \beta) \in \mathcal{R}$.*

4. *(Independence of irrelevant alternatives) If $\mathcal{R}' \subset \mathcal{R}$, $(\mu,\nu) \in \mathcal{R}'$ and $(\alpha,\beta) = A(\mathcal{R},(\mu,\nu)) \in \mathcal{R}'$, then $A(\mathcal{R}',(\mu,\nu)) = (\alpha,\beta) = A(\mathcal{R},(\mu,\nu))$.*

5. *(Invariant under linear transformation) Let $a,b,c,d \in \mathbb{R}$ be any real numbers with $a,c > 0$. Let $\mathcal{R}' = \{(au+b, cv+d) : (u,v) \in \mathcal{R}\}$ and $(\mu',\nu') = (a\mu+b, c\nu+d)$. Then $A(\mathcal{R}',(\mu',\nu')) = (a\alpha+b, c\beta+d)$.*

6. *(Symmetry) Suppose $\mathcal{R}$ is symmetry, that is $(u,v) \in \mathcal{R}$ implies $(v,u) \in \mathcal{R}$, and $\mu = \nu$. Then $\alpha = \beta$.*

Theorem 3.3.3 (Nash bargaining solution)**.** *There exists a unique arbitration function $A(\mathcal{R},(\mu,\nu))$ for closed and bounded convex set $\mathcal{R}$ and $(\mu,\nu) \in \mathcal{R}$ which satisfies the Nash bargaining axioms.*

Before proving Theorem 3.3.3, first we prove a lemma.

Lemma 3.3.4. *Let $\mathcal{R} \subset \mathbb{R}^2$ be any closed and bounded convex set and $(\mu,\nu) \in \mathcal{R}$. Let*

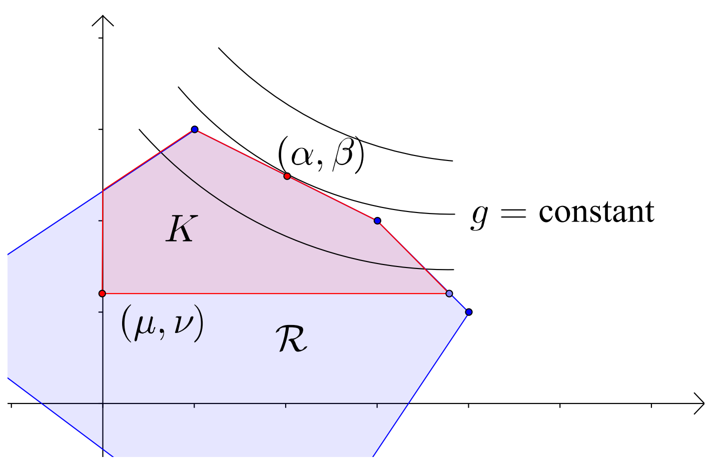
$$K = \{(u,v) \in \mathcal{R} : u \geq \mu, v \geq \nu\}$$

Let $g : K \to \mathbb{R}$ be the function defined by

$$g(u,v) = (u-\mu)(v-\nu) \text{ for } (u,v) \in K$$

Suppose $U = \{(u,v) \in K : u > \mu, v > \nu\} \neq \emptyset$. Then there exists unique $(\alpha,\beta) \in K$ such that

$$g(\alpha,\beta) = \max_{(u,v)\in K} g(u,v)$$



Proof. Since g is continuous and K is closed and bounded, g attains its maximum at some point $(\alpha, \beta) \in K$ and let

$$M = \max_{(u,v)\in K} g(u, v)$$

be the maximum value of g on K. We are going to prove by contradiction that the maximum point of g on K is unique. Suppose on the contrary that there exists $(\alpha', \beta') \in K$ with $(\alpha', \beta') \neq (\alpha, \beta)$ such that

$$g(\alpha', \beta') = g(\alpha, \beta) = M$$

Then either $\alpha' > \alpha$ and $\beta' < \beta$, or $\alpha' < \alpha$ and $\beta' > \beta$. In both case we have $(\alpha - \alpha')(\beta' - \beta) > 0$. Observe that the mid-point $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ of (α, β) and (α', β') lies in K since K is convex. On the other hand, the value of g at $(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2})$ is

$$\begin{aligned}
& g\left(\frac{\alpha+\alpha'}{2}, \frac{\beta+\beta'}{2}\right) \\
= & \left(\frac{\alpha+\alpha'}{2} - \mu, \frac{\beta+\beta'}{2} - \nu\right) \\
= & \frac{1}{4}\left((\alpha-\mu)+(\alpha'-\mu)\right)\left((\beta-\nu)+(\beta'-\nu)\right) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu)+(\alpha-\mu)(\beta'-\nu) \\
& +(\alpha'-\mu)(\beta-\nu)+(\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}((\alpha-\mu)(\beta-\nu)+(\alpha-\mu)((\beta'-\beta)+(\beta-\nu)) \\
& +(\alpha'-\mu)((\beta-\beta')+(\beta'-\nu))+(\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}(2(\alpha-\mu)(\beta-\nu)+(\alpha-\mu)(\beta'-\beta) \\
& +(\alpha'-\mu)(\beta-\beta')+2(\alpha'-\mu)(\beta'-\nu)) \\
= & \frac{1}{4}\left(2g(\alpha,\beta)+(\alpha-\alpha')(\beta'-\beta)+2g(\alpha',\beta')\right) \\
= & \frac{1}{4}\left(2M+(\alpha-\alpha')(\beta'-\beta)+2M\right) \\
> & M
\end{aligned}$$

This contradicts that the maximum value of g on K is M. Therefore g attains its maximum at a unique point. □

Proof of existence of arbitration function. For any closed and bounded convex set $\mathcal{R}$ and $(\mu,\nu)\in\mathcal{R}$, let $K=\{(u,v)\in\mathcal{R}: u\geq\mu, v\geq\nu\}$, $U=\{(u,v)\in\mathcal{R}: u>\mu, v>\nu\}$ and define $(\alpha,\beta)=A(\mathcal{R},(\mu,\nu))$ as follows:

1. If $U\neq\emptyset$, then $(\alpha,\beta)=A(\mathcal{R},(\mu,\nu))\in K$ is the unique maximum point of $g(u,v)=(u-\mu)(v-\nu)$ in K, that is
$$g(\alpha,\beta)=\max_{(u,v)\in K} g(u,v)$$
2. If $U=\emptyset$, then $(\alpha,\beta)=A(\mathcal{R},(\mu,\nu))\in K$ is the unique maximum point of $u+v$ on K, that is
$$\alpha+\beta=\max_{(u,v)\in K}(u+v)$$

We are going to prove that the function $A(\mathcal{R},(\mu,\nu))$ satisfies the Nash bargaining axioms. We prove only for the first case $U\neq\emptyset$ and the second case is obvious.

1. (Individual rationality) It follows by the definition that $(\alpha,\beta)\in K$ and we have $\alpha\geq\mu$ and $\beta\geq\nu$.
2. (Pareto optimality) Suppose there exists $(\alpha',\beta')\in\mathcal{R}$ such that $\alpha'\geq\alpha$ and $\beta'\geq\beta$. Then $g(\alpha',\beta')\geq g(\alpha,\beta)$ which implies that $(\alpha',\beta')=(\alpha,\beta)$ since the maximum point of g on K is unique.
3. (Feasibility) Since $(\alpha,\beta)\in K\subset\mathcal{R}$ by definition, we have $(\alpha,\beta)\in\mathcal{R}$.
4. (Independence of irrelevant alternatives) Suppose $\mathcal{R}'\subset\mathcal{R}$ is a subset of $\mathcal{R}$ which contains both (μ,ν) and (α,β). Since g attains its maximum at (α,β) on K, it also attains its maximum at (α,β) on $K'=K\cap\mathcal{R}'$. Thus
$$A(\mathcal{R}',(\mu,\nu))=(\alpha,\beta)=A(\mathcal{R},(\mu,\nu))$$
5. (Invariant under linear transformation) Let $a,b,c,d\in\mathbb{R}$ with $a,c>0$. Let $\mathcal{R}'=\{(u',v')=(au+b,cv+d):(u,v)\in\mathcal{R}\}$ and $(\mu',\nu')=(a\mu+b,c\nu+d)$. Then
$$\begin{aligned}g'(u',v') &= (u'-\mu')(v'-\nu')\\ &= ((au+b)-(a\mu+b))((cv+d)-(c\nu+d))\\ &= ac(u-\mu)(v-\nu)\\ &= acg(u,v)\end{aligned}$$

Hence g' attains its maximum at $(\alpha', \beta') = (a\alpha + b, c\beta + d)$ on $K' = \{(u', v') = (au + b, cv + d) : (u, v) \in K\}$ since g attains its maximum at (α, β) on K. Therefore $A(\mathcal{R}', (\mu, \nu)) = (\alpha', \beta')$.

6. (Symmetry) Suppose $\mathcal{R}$ is symmetric and $\mu = \nu$. Then

$$g(u, v) = (u - \mu)(v - \mu) = g(v, u)$$

and $(v, u) \in K$ if and only if $(u, v) \in K$. Thus if g attains its maximum at (α, β) on K, then g also attains its maximum at (β, α) on K. By uniqueness of maximum point of g on K, we see that $(\beta, \alpha) = (\alpha, \beta)$ which implies $\alpha = \beta$.

□

Proof of uniqueness of arbitration function. Suppose $A'(\mathcal{R}, (\mu, \nu))$ is another arbitration function satisfying the Nash bargaining axioms. Let $\mathcal{R}$ be a closed and bounded convex set and $(\mu, \nu) \in \mathcal{R}$. By applying a linear transformation, we may assume that $(\mu, \nu) = (0, 0)$ and $(\alpha, \beta) = A(\mathcal{R}, (0, 0)) = (0, 0), (1, 0), (0, 1)$ or $(1, 1)$. We are going to prove that $A'(\mathcal{R}, (0, 0)) = (\alpha, \beta)$.

Case 1. $(\alpha, \beta) = (0, 0)$:

In this case $K = \{(0, 0)\}$ and we have $A'(\mathcal{R}, (0, 0))$ since $(\alpha, \beta) \in K$.

Case 2. $(\alpha, \beta) = (1, 0)$ or $(0, 1)$:

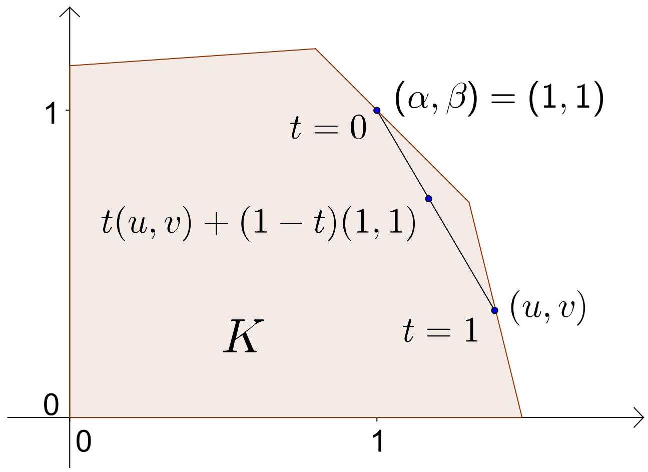
We consider the case for $(\alpha, \beta) = (1, 0)$ and the other case is similar. By definition of (α, β), we must have $K = \{(u, 0) : 0 \leq u \leq 1\}$. By the individual rationality, we have $A'(\mathcal{R}, (0, 0)) \in K$. By Pareto optimality, we have $A'(\mathcal{R}, (0, 0)) = (1, 0)$.

Case 3. $(\alpha, \beta) = (1, 1)$:

First we claim that $u + v \leq 2$ for any $(u, v) \in K$. We prove the claim by contradiction. Suppose there exists $(u, v) \in K$ such that $u + v > 2$. Then for any $0 \leq t \leq 1$, we have

$$t(u, v) + (1 - t)(1, 1) = ((u - 1)t + 1, (v - 1)t + 1) \in K$$

since K is convex. Let $g(t)$ be the value of g at the point $t(u, v) + (1 - t)(1, 1) \in K$ lying on the line segment joining $(1, 1)$ and (u, v).



Then

$$\begin{aligned} g(t) &= g(1 + (u-1)t, 1 + (v-1)t) \\ &= ((u-1)t + 1)((v-1)t + 1) \\ &= (u-1)(v-1)t^2 + (u + v - 2)t + 1 \end{aligned}$$

We have

$$g'(t) = 2(u-1)(v-1)t + u + v - 2$$

which implies

$$g'(0) = u + v - 2 > 0$$

It follows that there exists $0 < t \leq 1$ such that

$$g(t) > g(0) = g(1, 1)$$

which contradicts that g attains its maximum at $(1, 1)$ on K. Hence we proved the claim that $u + v \leq 2$ for any $(u, v) \in K$. Now let $\mathcal{R}'$ be the convex hull of $\{(u, v) : (u, v) \in \mathcal{R}$ or $(v, u) \in \mathcal{R}\}$. Then $u' + v' \leq 2$ for any $(u', v') \in \mathcal{R}'$ since $u + v \leq 2$ for any $(u, v) \in \mathcal{R}$. By symmetry, we have $A'(\mathcal{R}', (0, 0)) = (\alpha', \alpha')$ for some $(\alpha', \alpha') \in \mathcal{R}'$. Now $\alpha' \leq 1$ since $\alpha' + \alpha' \leq 2$. Since $(1, 1) \in K \subset \mathcal{R}'$, we have $A'(\mathcal{R}', (0, 0)) = (1, 1)$ by Pareto optimality. Therefore $A'(\mathcal{R}, (0, 0)) = (1, 1)$ by independence of irrelevant alternative.

This completes the proof that $A'(\mathcal{R},(\mu,\nu)) = A(\mathcal{R},(\mu,\nu))$ for any closed and bounded convex set $\mathcal{R}$ and any point $(\mu,\nu)\in\mathcal{R}$. $\square$

Example 3.3.5 (Dating game)**.** *Consider the dating game given by the bimatrix*

$$(A,B) = \begin{pmatrix} (4,2) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

We use $(\mu,\nu) = (\nu(A),\nu(B^T)) = (\frac{4}{5},\frac{6}{5})$ as the status quo point (see Example 3.1.5). We need to find the payoff pair on

$$K = \left\{(u,v)\in\mathcal{R} : u \geq \frac{4}{5}, v \geq \frac{6}{5}\right\}$$

so that the function

$$g(u,v) = \left(u - \frac{4}{5}\right)\left(v - \frac{4}{5}\right)$$

attains its maximum. Now any payoff pair (u,v) along the line segment joining $(1,3)$ and $(4,2)$ satisfies

$$\begin{aligned} v - 3 &= -\frac{1}{3}(u-1) \\ v &= -\frac{1}{3}u + \frac{10}{3} \end{aligned}$$

Thus

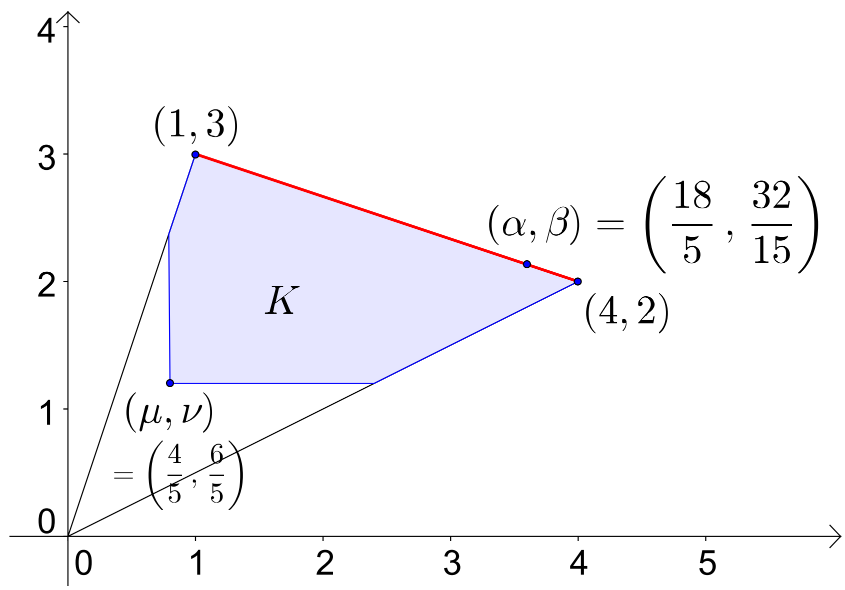
$$\begin{aligned} g(u,v) &= \left(u - \frac{4}{5}\right)\left(v - \frac{6}{5}\right) \\ &= \left(u - \frac{4}{5}\right)\left(-\frac{1}{3}u + \frac{32}{15}\right) \\ &= -\frac{1}{3}u^2 + \frac{12}{5}u - \frac{128}{75} \end{aligned}$$

attains its maximum when

$$u = \frac{18}{5} \text{ and } v = \frac{32}{15}$$

Since this payoff pair lies on the line segment joining $(1,3)$ and $(4,2)$, the arbitration pair of the game with status quo point $(\mu,\nu) = (\frac{4}{5},\frac{6}{5})$ is

$$(\alpha,\beta) = \left(\frac{18}{5},\frac{32}{15}\right)$$



□

To find the arbitration pair, one may use the fact that if $g(u,v) = (u-\mu)(v-\nu)$ attains it maximum at the point (α,β) over the line joining (u_0,v_0) and (u_1,v_1), then the slope of the line joining (α,β) and (μ,ν) would be equal to the negative of the slope of line joining (u_0,v_0) and (u_1,v_1). Using this fact, one may see easily that (α,β) satisfies

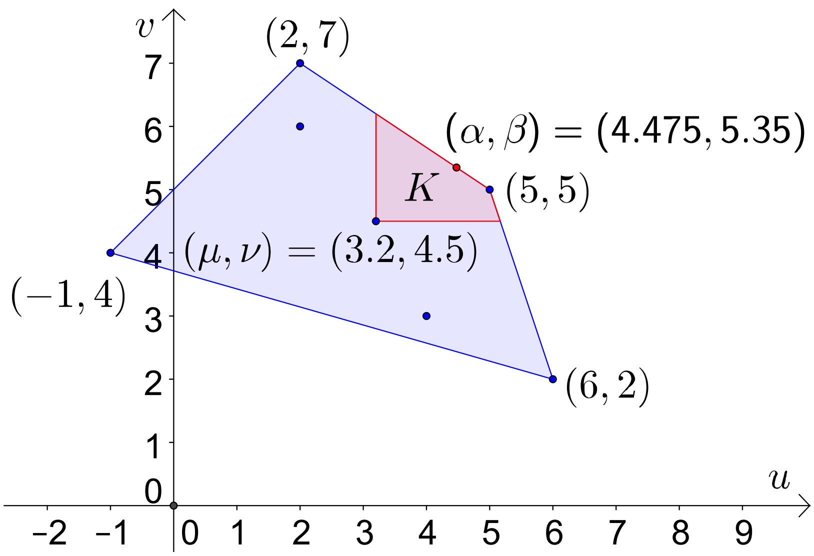
$$\begin{cases} \beta - v_0 = \dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - u_0) \\ \beta - \nu = -\dfrac{v_1 - v_0}{u_1 - u_0}(\alpha - \mu) \end{cases}$$

Hence if the payoff pair (α,β) obtained by solving the above system of equations lies on the line segment joining (u_0,v_0) and (u_1,v_1), which implies that (α,β) lies on the bargaining set, then (α,β) is the arbitrary pair.

Example 3.3.6. *Let*

$$(A,B) = \begin{pmatrix} (2,6) & (6,2) & (-1,4) \\ (4,3) & (2,7) & (5,5) \end{pmatrix}$$

The reader may check that the values of A, B^T are 3.2, 4.5 respectively and we use $(\mu, \nu) = (3.2, 4.5)$ as the status quo point. We need to consider two line segments.



1. The line segment joining $(5, 5)$ and $(6, 2)$:

   The equation of the line segment is given by $v = -3u + 20$. The value of $g(u, v)$ along the line segment is

$$
\begin{aligned}
g(u, v) &= (u - 3.2)(v - 4.5) \\
&= (u - 3.2)(-3u + 15.5) \\
&= -3u^2 + 25.1u + 49.6
\end{aligned}
$$

   which attain its maximum at $(\frac{251}{60}, \frac{149}{20})$. Since this payoff pair lies outside the line segment joining $(5, 5)$ and $(6, 2)$ and thus lies outside K, we know that the arbitration pair does not lie on the line segment joining $(5, 5)$ and $(6, 2)$.

2. The line segment joining $(2, 7)$ and $(5, 5)$:

   The slope of the line joining $(2, 7)$ and $(5, 5)$ is $-\frac{2}{3}$. To find the maximum point of $g(u, v)$ along the line joining $(2, 7)$ and $(5, 5)$, we may solve

$$\begin{cases} v - 7 = -\dfrac{2}{3}(u - 2) \\ v - 4.5 = \dfrac{2}{3}(u - 3.2) \end{cases}$$

which gives $(u, v) = (4.475, 5.35)$. Since this payoff pair lies on the line segment joining $(2, 7)$ and $(5, 5)$, we conclude that the arbitration pair is $(\alpha, \beta) = (4.475, 5.35)$.

□

3.4 Threat solution

In this section, we study two-person **cooperative games with transferable utility**. We assume that the players are 'rational' in the sense that, given a choice between two possible outcomes of differing personal utility, each player will select the one with the higher utility. In the model of the cooperative game with transferable utility, we assume there is a period of preplay negotiation, during which the players meet to discuss the possibility of choosing a joint strategy together with some possible side payment to induce cooperation. They also discuss what will happen if they cannot come to an agreement; each may threaten to use some unilateral strategy that is bad for the opponent. If they do come to an agreement, it may be assumed that the payoff vector is Pareto optimal.

In the discussion, both players may make some threat of what strategy they will take if an agreement is not reached. However, a threat to be believable must not hurt the player who makes it to a greater degree than the opponent. Such a threat would not be credible. For example, consider the following bimatrix game.

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix}$$

If the players come to an agreement, it will be to use the lower right corner because it has the greatest total payoff, namely 9. Player II may argue that she should receive at least half the sum, 4.5. She may even feel generous in 'giving up' as a side payment some of the 6 she would be winning. However, Player I may threaten to use row 1 unless he is given at least 5. That threat is very credible since if Player I uses row 1, Player II cannot make a counter-threat to use column 2 because it would hurt her more than Player I. The counter-threat would not be credible.

In this model of the preplay negotiation, the threats and counter-threats may be made and remade until time to make a decision. Ultimately the players announce what threats they will carry out if agreement is not reached. It is assumed that if agreement is not reached, the players will leave the negotiation table and carry out their threats. However, being rational players, they will certainly reach agreement, since this gives a higher utility. The threats are only a formal method of arriving at a reasonable amount for the side payment, if any, from one player to the other.

The problem then is to choose the threats and the proposed side payment judiciously. The players use threats to influence the choice of the final payoff vector. The problem is how do the threats influence the final payoff vector, and how should the players choose their threat strategies? For two-person games with transferable utility, there is a very convincing answer.

Definition 3.4.1 (Threat solution). *Let (A, B) be a game bimatrix.*

1. *The* **threat matrix** *is the matrix $T = A - B$.*

2. *The* **threat differential** *δ is the value of the threat matrix $T = A - B$. In other words, $\delta = v(T) = v(A - B)$.*

3. *The* **threat strategies** *of Player I and Player II are the maximin strategy $\mathbf{p}_d$ and the minimax strategy $\mathbf{q}_d$ of the threat matrix $T = A - B$ respectively.*

4. *The* **threat point**, *or* **disagreement point**, *is the payoff pair (μ_d, ν_d) when the threat strategies $\mathbf{p}_d$, $\mathbf{q}_d$ are being used. In other words,*

$$(\mu_d, \nu_d) = (\mathbf{p}_d A \mathbf{q}_d^T, \mathbf{p}_d B \mathbf{q}_d^T).$$

Note that $\delta = \mu_d - \nu_d$.

5. *The* **threat solution** *is the payoff pair*

$$(\varphi_1, \varphi_2) = \left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right) = \left(\frac{\sigma + \mu_d - \nu_d}{2}, \frac{\sigma - \mu_d + \nu_d}{2}\right).$$

where

$$\sigma = \max_{i,j}(a_{ij} + b_{ij})$$

is the **maximum total payoff** *which is the maximum entry of the sum matrix $A+B$. Note that (φ_1, φ_2) is the solution to*

$$\begin{cases} \varphi_1 + \varphi_2 = \sigma \\ \varphi_1 - \varphi_2 = \delta \end{cases}$$

If the players come to an agreement, then they will agree to play to achieve the largest possible total payoff $\sigma \max_{i,j}(a_{ij} + b_{ij})$ as the payoff to be divided between them. So it is easy to see that the threat solution (φ_1, φ_2) should satisfy $\varphi_1 + \varphi_2 = \sigma$.

Suppose now that the players have selected their threat strategies, $\mathbf{p}_d$ for Player I and $\mathbf{q}_d$ for Player II. Then if agreement is not reached, Player I receives $\mathbf{p}_d A \mathbf{q}_d^T$ and Player II receives $\mathbf{p}_d B \mathbf{q}_d^T$. The resulting payoff vector, $(\mu_d, \nu_d) = (\mathbf{p}_d A \mathbf{q}_d^T, \mathbf{p}_d B \mathbf{q}_d^T)$ is in the cooperative region and is called the disagreement point or threat point. Once the disagreement point is determined, the players must agree on the point (u, v) on the line $u + v = \sigma$ to be used as the cooperative solution. Player I will accept no less than μ_d and Player II will accept no less than ν_d since these can be achieved if no agreement is reached. But once the disagreement point has been determined, the game becomes symmetric. The players are arguing about which point on the line interval from $(\mu_d, \sigma - \mu_d)$ to $(\sigma - \nu_d, \nu_d)$ to select as the cooperative solution. No other considerations with respect to the matrices A and B play any further role. Therefore, the midpoint of the interval, namely

$$(\varphi_1, \varphi_2) = \left(\frac{\sigma + \mu_d - \nu_d}{2}, \frac{\sigma - \mu_d + \nu_d}{2} \right)$$

is the natural compromise. Both players suffer equally if the agreement is broken. Suppose Player I receives less than φ_1. He may threat Player II by saying that he will use his threat strategy $\mathbf{p}_d$. By doing so, Player I may guarantee that he gets at least $\delta = v(A - B)$ more than Player II. This ensures Player II will suffer more. Similarly, If Player II receives less than φ_2, she may ensure that Player I suffers more by using her threat strategy $\mathbf{q}_d$.

Example 3.4.2. *Find the threat strategies and the threat solution of the game bimatrix*

$$(A, B) = \begin{pmatrix} (0,0) & (6,2) & (-1,2) \\ (4,-1) & (3,6) & (5,5) \end{pmatrix}.$$

Solution. There is a Nash equilibrium in the first row, second column, with payoff vector $(6, 2)$. The maximum total payoff is

$$\sigma = 5 + 5 = 10.$$

If they come to an agreement, Player I will select the second row, Player II will select the third column and both players will receive a payoff of 5. They must still decide on a side payment, if any. They consider the zero-sum game with the threat matrix

$$T = A - B = \begin{pmatrix} 0 & 4 & -3 \\ 5 & -3 & 0 \end{pmatrix}.$$

The first column is strictly dominated by the last. The threat strategies are then easily determined to be

$$\begin{cases} \mathbf{p}_d = (0.3, 0.7) \\ \mathbf{q}_d = (0, 0.3, 0.7) \end{cases}$$

Now the threat differential is $\delta = v(A - B) = -9/10$ and the threat solution is

$$(\varphi_1, \varphi_2) = \left(\frac{10 - \frac{9}{10}}{2}, \frac{10 + \frac{9}{10}}{2} \right) = \left(\frac{91}{20}, \frac{109}{20} \right)$$

Example 3.4.3. *Find the threat solution of the game bimatrix*

$$(A, B) = \begin{pmatrix} (1,5) & (2,2) & (0,1) \\ (4,2) & (1,0) & (2,1) \\ (5,0) & (2,3) & (0,0) \end{pmatrix}.$$

Solution. There are two cooperative strategies giving total payoff $\sigma = 6$. The threat matrix is

$$T = A - B = \begin{pmatrix} -4 & 0 & -1 \\ 2 & 1 & 1 \\ 5 & -1 & 0 \end{pmatrix}.$$

which has a saddle-point at the $2, 3$-entry and the threat differential is $\delta = v(T) = 1$. The threat strategies are

$$\begin{cases} \mathbf{p}_d = (0, 1, 0) \\ \mathbf{q}_d = (0, 0, 1) \end{cases}$$

and the threat solution is

$$(\varphi_1, \varphi_2) = \left(\frac{6+1}{2}, \frac{6-1}{2}\right) = (3.5, 2.5).$$

Exercise 3

1. Find all Nash equilibria of the following bimatrix games. For each of the Nash equilibrium, find the payoff pair.

   (a) $\begin{pmatrix} (1,4) & (5,1) \\ (4,2) & (3,3) \end{pmatrix}$ (c) $\begin{pmatrix} (1,5) & (2,3) \\ (5,2) & (4,2) \end{pmatrix}$

   (b) $\begin{pmatrix} (5,2) & (2,0) \\ (1,1) & (3,4) \end{pmatrix}$ (d) $\begin{pmatrix} (-1,0) & (2,1) \\ (4,3) & (-3,-1) \end{pmatrix}$

2. Find all Nash equilibria of the following bimatirx games

   (a) $\begin{pmatrix} (4,1) & (2,3) & (3,4) \\ (3,2) & (5,5) & (1,2) \end{pmatrix}$ (c) $\begin{pmatrix} (4,6) & (0,3) & (2,-1) \\ (2,4) & (6,5) & (-1,1) \\ (5,0) & (1,2) & (4,3) \end{pmatrix}$

   (b) $\begin{pmatrix} (1,0) & (4,-1) & (5,1) \\ (3,2) & (1,1) & (2,-1) \end{pmatrix}$ (d) $\begin{pmatrix} (3,2) & (4,0) & (7,9) \\ (2,6) & (8,4) & (3,5) \\ (5,4) & (5,3) & (4,1) \end{pmatrix}$

3. The Brouwer's fixed-point theorem states that every continuous map $f : X \to X$ has a fixed-point if X is homeomorphic to a closed unit ball. Find a map $f : X \to X$ which does not have any fixed-point for each of the following topological spaces X. (It follows that the following spaces are not homeomorphic to a closed unit ball.)

   (a) X is the punched closed unit disc $D^2 \setminus \{0\} = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$

   (b) X is the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

   (c) X is the open unit disc $B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

4. For each of the following bimatrices (A, B), find the values ν_A and ν_{B^T} of A and B^T respectively, and the Nash bargaining solution using $(\mu, \nu) = (\nu_A, \nu_{B^T})$ as the status quo point.

(a) $\begin{pmatrix} (4,-4) & (-1,-1) \\ (0,1) & (1,0) \end{pmatrix}$

(b) $\begin{pmatrix} (3,1) & (1,0) \\ (0,-1) & (2,3) \end{pmatrix}$

(c) $\begin{pmatrix} (2,2) & (0,1) & (1,-1) \\ (4,1) & (-2,1) & (1,3) \end{pmatrix}$

(d) $\begin{pmatrix} (6,4) & (0,10) & (4,1) \\ (8,-2) & (4,1) & (0,1) \end{pmatrix}$

5. Two broadcasting companies, NTV and CTV, bid for the exclusive broadcasting rights of an annual sports event. If both companies bid, NTV will win the bidding with a profit of \$20 (million) and CTV will have no profit. If only NTV bids, therell be a profit of \$50 (million). If only CTV bids, therell be a profit of \$40 (million). Find the Nash's solution to the bargaining problem.

6. Let $\mathcal{R} = \{(u,v) : v \geq 0 \text{ and } u^2 + v \leq 4\} \subset \mathbb{R}^2$. Find the arbitration pair $A(\mathcal{R}, (\mu, \nu))$ using the following points as the status quo point (μ, ν).

   (a) $(0,0)$

   (b) $(0,1)$

7. Let $\mathcal{R} \subset \mathbb{R}^2$ be a closed and bounded convex set, $(\mu, \nu) \in \mathcal{R}$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$ be the arbitration pair with $\alpha \neq \mu$. Suppose the boundary of $\mathcal{R}$ is given, locally at (α, β), by the graph of a differentiable function $f(x)$ with $f(\alpha) = \beta$. Prove that $f'(\alpha)$ is equal to the negative of the slope of the line joining (μ, ν) and (α, β).

8. Suppose A is an $n \times n$ matrix such that the sum of entries in any row of A is equal to a constant rn. Let (μ, ν) be the status quo point of the bimatrix (A, A^T).

   (a) Prove that there is a Nash equilibrium of (A, A^T) with (r, r) as payoff pair.

   (b) Prove that the arbitration payoff pair of the bimatrix (A, A^T) is $(\alpha, \beta) = (m, m)$ where m is the maximum entry of $\dfrac{A + A^T}{2}$. (Here in finding the arbitration payoff pair of bimatix (A, B), the status quo point is taken to be $(\mu, \nu) = (v(A), v(B^T))$ where v is the value of a matrix.)

9. Find the threat strategies and the threat solutions of the following game bimatrix.

(a) $\begin{pmatrix} (3,-2) & (2,4) \\ (1,0) & (3,-1) \end{pmatrix}$

(b) $\begin{pmatrix} (5,3) & (1,3) \\ (4,4) & (2,1) \end{pmatrix}$

(c) $\begin{pmatrix} (6,4) & (2,3) & (4,7) \\ (2,6) & (4,2) & (5,4) \end{pmatrix}$

(d) $\begin{pmatrix} (2,8) & (7,5) & (6,3) \\ (0,7) & (4,3) & (5,5) \\ (3,-1) & (-2,6) & (2,7) \end{pmatrix}$

4 Extensive form

In the strategic form of a game, we have assumed that players, when taking their actions, either did so simultaneously, or without knowing the action choice of the other players. Although, this modelling assumption might be appropriate in some settings, there are many situations in the world of business and politics that involve players moving sequentially after observing what the other players have done. For example, a bargaining situation between a seller and a buyer may involve the buyer making an offer and the seller, after observing the buyer's offer, either accepting or rejecting it. Or imagine an incumbent senator deciding whether to run an expensive ad campaign for the upcoming elections and a potential challenger deciding whether to enter the race or not, after observing the campaign decision of the incumbent. Both of these situations involve a player choosing an action after observing the action of the other player. The extensive form of a game, as opposed to the strategic form, provides a more appropriate framework to analyze certain interesting questions that arise in strategic interactions that involve sequential moves.

4.1 Game tree

Games in extensive form are modelled using directed graphs.

Definition 4.1.1 (Directed graph)**.** *A* **directed graph** *is a pair* (T, F) *where* T *is a nonempty set of* **vertices**, *or* **nodes** *and* F *is a function that gives for each* $x \in T$ *a subset* $F(x) \subset T$ *called the* **followers** *of* x. *When a directed graph is used to represent a game, the vertices represent positions of the game. The followers,* $F(x)$*, of a position,* x*, are those positions that can be reached from* x *in one move.*

Definition 4.1.2 (Path)**.** *Let* $G = (T, F)$ *and* $t_0, t \in T$ *be two vertices. A* **path** *from* t_0 *to* t *is a sequence,* $t_0, t_1, \cdots, t_n \in T$*, of vertices such that* $t_n = t$ *and* $t_i \in F(t_{i-1})$ *is a follower of* t_{i-1} *for* $i = 1, 2, \cdots, n$.

For the extensive form of a game, we deal with a particular type of directed graph called a tree.

Definition 4.1.3 (Tree)**.** *A* **tree** *is a directed graph,* (T, F) *in which there is a special vertex,* t_0*, called the root or the initial vertex, such that for every other vertex* $t \in T$*, there is a unique path beginning at* t_0 *and ending at* t.

The existence and uniqueness of the path implies that a tree is connected, has a unique initial vertex, and has no circuits or loops. We will also assume that all trees are finite meaning that there are only finitely many vertices. We say that a vertex of a tree is a terminal vertex if it has no follower and the game ends when a terminal vertex is reached.

Definition 4.1.4 (Terminal vertex)**.** *We say that a vertex $t \in T$ of a tree $G = (T, F)$ is a* **terminal vertex** *if t has no follower. In other words, t is a terminal vertex if $F(t) = \emptyset$.*

In a game tree, play starts at the initial vertex and continues along one of the paths eventually ending in one of the terminal vertices. At terminal vertices, the rules of the game specify the payoff. For n-person games, this would be an n-tuple of payoffs. We will consider two person game and called the two players Player I and Player II. For the nonterminal vertices there are three possibilities. Some nonterminal vertices are assigned to Player I who is to choose the move at that position. Others are assigned to Player II. However, some vertices may be singled out as positions from which a chance move is made.

Now we describe pure strategies of players in a game tree.

Definition 4.1.5 (Pure strategy)**.** *Let $G = (T, F)$ be a game tree and $t_{11}, t_{12}, \cdots, t_{1k_1}$ be the vertices associated with Player I. A* **pure strategy** *of Player I is a choice of vertices $v_{11}, v_{12}, \cdots, v_{1k_1}$ such that $v_{1i} \in F(t_{1i})$ is a follower of t_{1i} for each $i = 1, 2, \cdots, k_1$. Similarly, a pure strategy of Player II is a choice of vertices $v_{21}, v_{22}, \cdots, v_{2k_2}$ such that $v_{2i} \in F(t_{2i})$ is a follower of t_{2i} where $t_{21}, t_{22}, \cdots, t_{2k_2}$ are vertices associated with Player II.*

A game tree can be solved using **backward induction**. Mechanically, backward induction corresponds to the following procedure. Consider any node that comes just before terminal nodes, that is, after each move stemming from this node, the game ends. If the player who moves at this node acts rationally, he chooses the best move for himself at that node. Hence, select one of the moves that give this player the highest payoff. Assigning the payoff vector associated with this move to the node at hand, delete all the moves stemming from this node so that we have a shorter game, where the above node is a terminal node. Repeat this procedure until the origin is the only remaining node. The outcome is the moves that are picked in the way. Since a move is picked at each set, the result is a strategy profile.

Example 4.1.6 (Centipede game)**.** *Consider the* **centipede game** *in Figure 1. This game describes a situation where it is mutually beneficial for all players to stay in a relationship, while a player would like to exit the relationship, if she knows that the other player will exit in the next day.*

In the third day, Player I moves, choosing between going across (α) or down (δ). If he goes across, he would get 2; if he goes down, he would get the higher payoff of 3. Hence, according to the procedure, he goes down. Selecting the move δ for the node at hand, one reduces the game as in Figure 2.

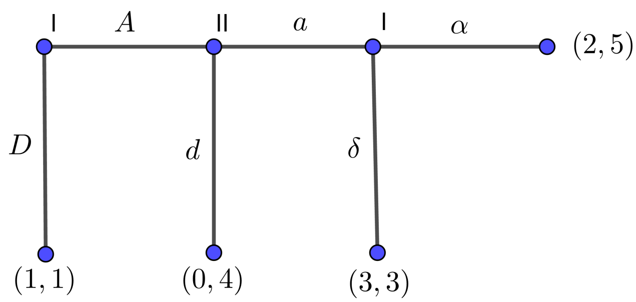


Figure 1: Centipede game

Here, the part of the game that starts at the last decision node is replaced with the payoff vector associated with the selected move, δ, of the player at this decision node. In the second day, Player II moves, choosing between

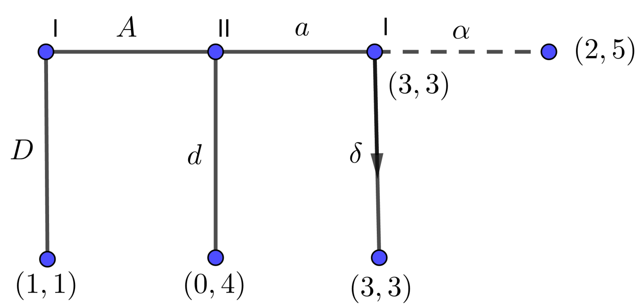


Figure 2: Backward induction

going across (a) or down (d). If she goes across, she get 3; if she goes down, she gets the higher payoff of 4. Hence, according to the procedure, she goes down. Selecting the move d for the node at hand, one reduces the game further as in Figure 3.

Once again, the part of the game that starts with the node at hand is re-

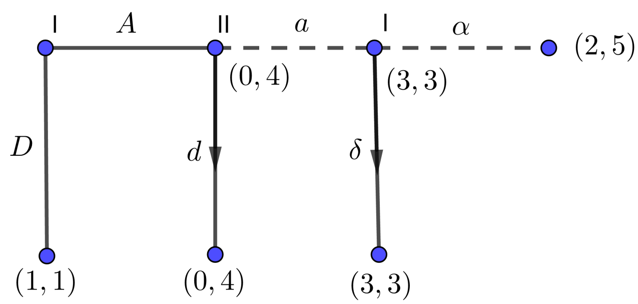


Figure 3: Backward induction

placed with the payoff vector associated with the selected move d. Now, Player I gets 0 if he goes across (A), and gets 1 if he goes down (D). Therefore, he goes down. The procedure results in the following pure strategies: At each node, the player who is to move goes down, exiting the relationship. The pure strategies obtained by using backward induction can be described as Player I uses $D\delta$ *and Player* II *uses d.*

Let's go over the assumptions that we have made in constructing this pure

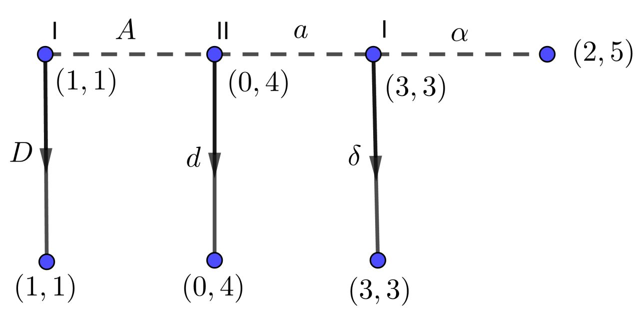


Figure 4: Backward induction

strategies. We assumed that Player I will act rationally at the last date, when we reckoned that he goes down. When we reckoned that Player II goes down in the second day, we assumed that Player II assumes that Player I will act rationally on the third day, and also assumed that she is rational, too. On the first day, Player I anticipates all these. That is, he is assumed to know that Player II is rational, and that she will keep believing that Player I will act rationally on the third day. This example also illustrates another notion associated with backward induction - commitment (or the lack of commitment). Note that the outcomes on the third day (i.e., $(3, 3)$ *and* $(2, 5)$*) are both strictly better than the equilibrium outcome* $(1, 0)$*. But they cannot reach these outcomes, because Player II cannot commit to go across, and anticipating that Player II will go down, Player I exits the relationship in the first day. There is also a further commitment problem in this example. If Player I were able to commit to go across on the third day, then Player II would definitely go across on the second day. In that case, Player I would go across on the first. Of course, Player I cannot commit to go across on the third day, and the game ends in the first day, yielding the low payoffs* $(1, 1)$.

It is not difficult to see that the pure strategies resulting from backward induction is always a Nash equilibrium. But not all Nash equilibria can be obtained by backward induction.

Example 4.1.7 (Battle of sexes)**.** *Consider the game of the* **battle of the sexes**. *There are two players Alice and Bob. The game tree is shown in Figure 5.*

In this game, backward induction leads to pure strategies Alice uses O *and Bob uses* OF *as shown in Figure 6.*

There is another Nash equilibrium: Alice uses F*, and Bob uses* FF*. Let's see why this is a Nash equilibrium. Alice plays a best response to the strategy of Bob: if she goes to Football she gets 1, and if she goes to Opera she gets 0 (as they do not meet). Bob's strategy (*FF*) is also a best response to Alice's strategy: under this strategy he gets 2, which is the highest he can get in this game.*

One can, however, discredit the latter Nash equilibrium because it relies on an sequentially irrational move at the node after Alice goes to Opera. This node does not happen according to Alice's strategy, and it is therefore ignored in Nash equilibrium. Nevertheless, if Alice goes to Opera, going to football game would be irrational for Bob, and he would rationally go to Opera as well.

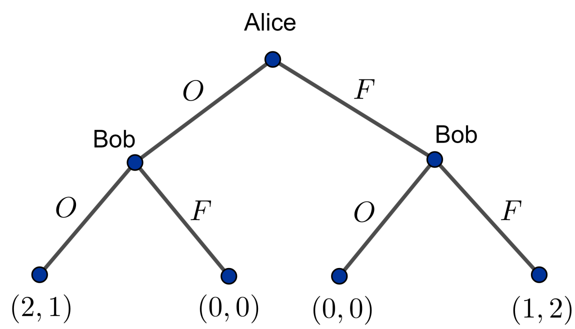


Figure 5: Battle of the sexes

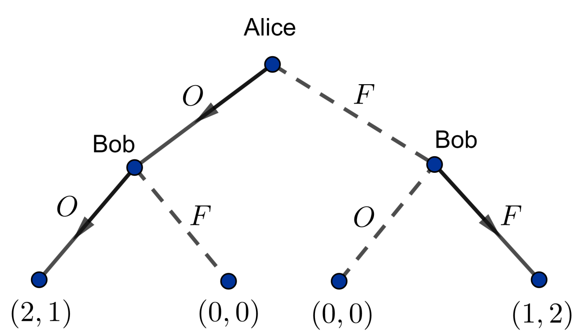


Figure 6: Battle of the sexes

And Alice should foresee this and go to Opera. Sometimes, we say that this equilibrium is based on "an incredible threat", with the obvious interpretation.

The battle of sexes game (Example 4.1.7) illustrates a shortcoming of the usual rationality condition, which requires that one must play a best response (as a complete contingent plan) at the beginning of the game. While this requires that the player plays a best response at the nodes that he assigns positive probability, it leaves the player free to choose any move at the nodes that he puts zero probability because all the payoffs after those nodes are multiplied by zero in the expected utility calculation. Since the likelihoods of the nodes are determined as part of the solution, this may lead to somewhat erroneous solutions in which a node is not reached because a player plays irrationally at the node, anticipating that the node will not be reached, as in (F, FF) equilibrium. Of course, this is erroneous in that when that node is reached the player cannot pretend that the node will not be reached as he will know that the node is reached by the definition of information set. Then, he must play a best response taking it given that the node is reached.

Many games involve **chance moves**. Examples include the rolling of dice in board games like monopoly or backgammon or gambling games, the dealing of cards as in bridge or poker. In these games, chance moves play an important role. Even in chess, there is generally a chance move to determine which player gets the white pieces (and therefore the first move which is presumed to be an advantage). It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move.

Another important aspect we must consider in studying the extensive form of games is the amount of information available to the players about past moves of the game. In poker for example, the first move is the chance move of shuffling and dealing the cards, each player is aware of certain aspects of the outcome of this move (the cards he received) but he is not informed of the complete outcome (the cards received by the other players). This leads to the possibility of "bluffing". This type of games are called games with **imperfect information**. Here is an example.

Example 4.1.8 (Bluffing game)**.** *The* **bluffing game** *is played as follows. Both players put 1 dollar, called the ante, in the center of the table. The money in the center of the table, so far two dollars, is called the pot. Then Player I is dealt a card from a deck. It is a winning card with probability* $1/4$ *and a losing card with probability* $3/4$. *Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.) Player I then* *checks or bets. If he checks, then his card is inspected; if he has a winning card he wins the pot and hence wins the 1 dollar ante from Player II, and otherwise he loses the 1 dollar ante to Player II. If Player I bets, he puts 2 dollars more into the pot. Then Player II, not knowing what card Player I has, must fold or call. If she folds, she loses the 1 dollar ante to Player I no matter what card Player I has. If Player II calls, she adds 2 dollars to the pot. Then Player I's card is exposed and Player I wins 3 dollars (the ante plus the bet) from Player II if he has a winning card, and loses 3 dollars to Player II otherwise.*

The tree for the bluffing game is shown in Figure 7. There are at most three moves in this game: (1) the chance move that chooses a card for Player I, (2) Player I's move in which he checks or bets, and (3) Player II's move in which she folds or calls. To each vertex of the game tree, we attach a label indicating which player is to move from that position. Chance moves refer to as moves by nature and use the label N. Each edge is labelled to identify the move. The moves leading from a vertex at which nature moves are labelled with the probabilities with which they occur. At each terminal vertex, we write the numerical value of Player I's winnings (Player II's losses).

There is only one feature lacking from the above figure. From the tree we

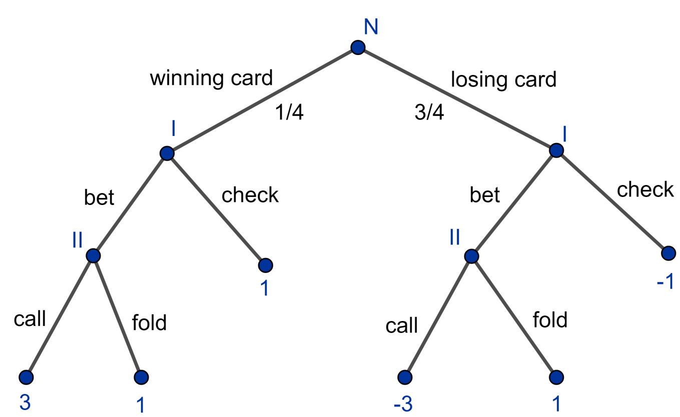


Figure 7: Bluffing game

should be able to reconstruct all the essential rules of the game. That is not the case with the tree of Figure 7 since we have not indicated that at the time Player II makes her decision she does not know which card Player I has *received. That is, when it is Player II's turn to move, she does not know at which of her two possible positions she is. We indicate this on the diagram by joining the two positions with a dotted line, and we say that these two vertices constitute an* **information set***. The two vertices at which Player I is to move constitute two separate information sets since he is told the outcome of the chance move. The completed game tree is shown is Figure 8.*

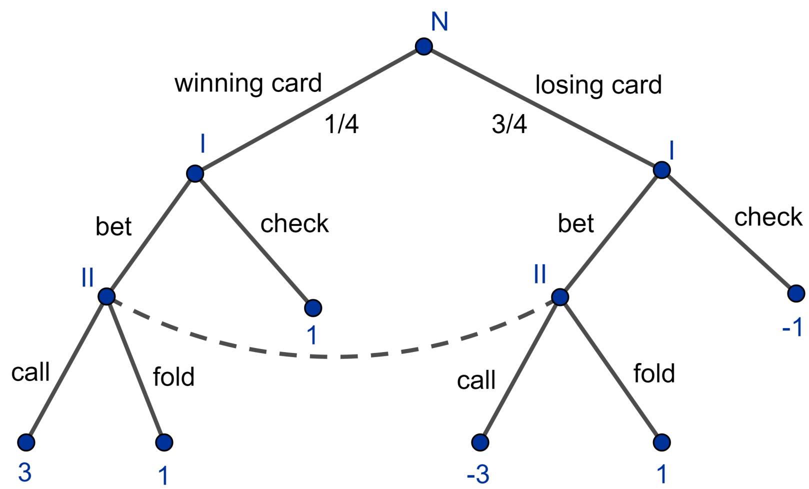


Figure 8: Bluffing game

The game tree with all the payoffs, information sets, and labels for the edges and vertices included is known as the **Kuhn tree**. We now give the formal definition of a Kuhn tree. Not every set of vertices can form an information set. In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it. Furthermore, it is important that the edges from each vertex of an information set have the same set of labels. The player moving from such an information set really chooses a label. It is presumed that a player makes just one choice from each information set.

Definition 4.1.9 (Extensive form)**.** *A finite two-person game in* **extensive form** *is given by*

1. *a finite tree with vertices T,*

2. *a payoff function that assigns a payoff pair to each terminal vertex,*

3. *a set T_0 of non-terminal vertices (representing positions at which chance moves occur) and for each $t \in T_0$, a probability distribution on the edges leading from t,*

4. *a partition of the rest of the vertices (not terminal and not in T_0) into two groups $T_{11}, T_{12}, \cdots, T_{1k_1}$ (for Player I) and $T_{21}, T_{22}, \cdots, T_{2k_2}$ (for Player II), and*

5. *for each information set T_{jk} for player j, a set of labels L_{jk}, and for each vertex $t \in T_{jk}$, a one-to-one mapping of T_{jk} onto the set of edges leading from t,*

Here is an example of games in extensive form.

Example 4.1.10 (Matching pennies)**.** *Consider the game in Figure 9. The*

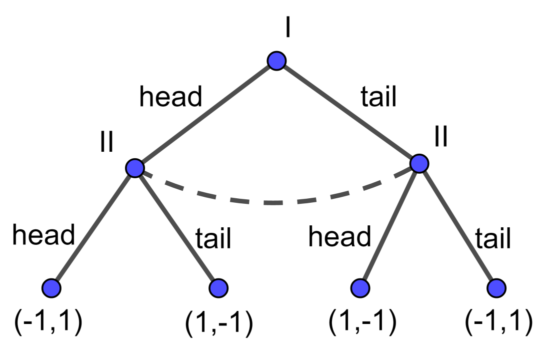


Figure 9: Matching pennies

tree consists of 7 nodes. The first one is allocated to Player I, and the next two to Player II. The four terminal vertices have payoffs attached to them. Since there are two players, payoff vectors have two elements. The first number is the payoff of Player I and the second is the payoff of Player II.

The information structure in a game in extensive form can be quite complex. It may involve lack of knowledge of the other player's moves or of some of the chance moves. It may indicate a lack of knowledge of how many moves have already been made in the game. It may describe situations in which one player has forgotten a move he has made earlier. In fact, one way to try to model the game of bridge as a two-person zero-sum game involves the use of this idea. In bridge, there are four individuals forming two teams or partnerships of two players each. The interests of the members of a partnership are identical, so it makes sense to describe this as a two-person game. But the members of one partnership make bids alternately based on cards that one member knows and the other does not. This may be described as a single player who alternately remembers and forgets the outcomes of some of the previous random moves. A kind of degenerate situation exists when an information set contains two vertices which are joined by a path, as is the case with Player I's information set in Figure 10.

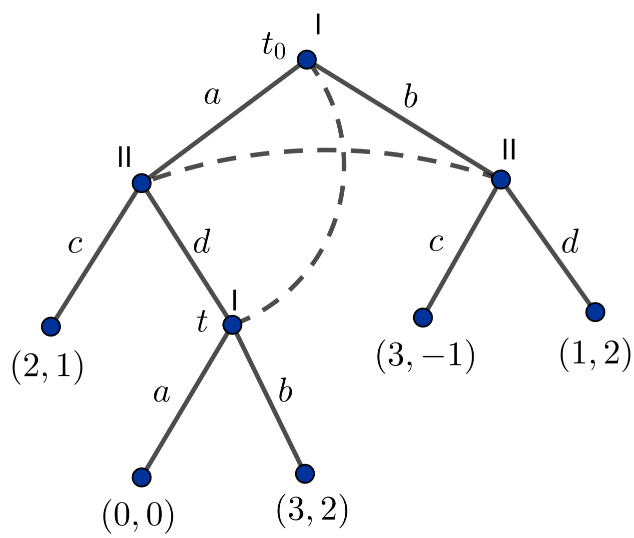


Figure 10: When t is reached, Player I has forgotten he had chosen a at t_0.

Definition 4.1.11 (Perfect recall)**.** *We say that a game in extensive form is of* **perfect recall** *if all players remember all past information they once knew and all past moves they made.*

Consider the game tree in Figure 11. Suppose Player I plays a and then Player II plays d. Player I does not know whether he arrives t_1 or t_2 because t_1, t_2 are in the same information set even though he has played a in the first move. Player I has forgotten what he has played in the first move. So this game is not of perfect recall. If then Player I plays e, the payoff pair would be $(0, 0)$.

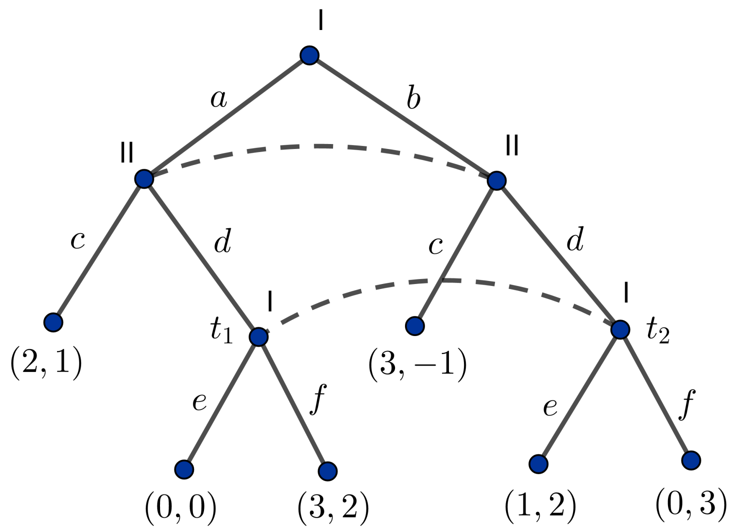


Figure 11: Player I does not know whether he is at t_1 and t_2 because he has forgotten whether he has chosen a or b previously.

Games in which both players know the rules of the game, that is, in which both players know the Kuhn tree, are called games of **complete information**. Games in which one or both of the players do not know some of the payoffs, or some of the probabilities of chance moves, or some of the information sets, or even whole branches of the tree, are called games with incomplete information, or pseudogames. We assume in the following that we are dealing with games of complete information.

Extensive form games describe strategic interactions in which moves may occur sequentially. The two main classes of extensive form games are games of perfect information and games of imperfect information. Now we can make a formal definition of games of perfect information.

Definition 4.1.12 (Game of perfect information)**.** *A* **game of perfect in-**

formation *is a game in extensive form in which each information set of every player contains a single vertex.*

The centipede (Example 4.1.6) and battle of the sexes (Example 4.1.7) are examples of games of perfect information. In such games a player's choices are always immediately observed by his opponents. Simultaneous moves are modeled by incorporating unobserved moves, and so lead to imperfect information games. Extensive form games may also include chance events, modeled as moves by Nature. For our categorization, it is most convenient to understand 'game of perfect information' to refer only to games without moves by Nature. The bluffing game (Example 4.1.8) and matching pennies (Example 4.1.10) are examples of games of imperfect information.

4.2 Reduction of extensive form to strategic form

In this section, we study how to transform a game in extensive form to strategic form.

Given a game in extensive form, we first find X and Y, the sets of pure strategies of the players to be used in the strategic form. A pure strategy for Player I is a rule that tells him exactly what move to make in each of his information sets. Let $T_{11}, T_{12}, \cdots, T_{1k_1}$ be the information sets for Player I and let $L_{11}, L_{12}, \cdots, L_{1k_1}$ be the corresponding sets of labels. A pure strategy for Player I is a k_1-tuple $x = (x_1, x_2, \cdots, x_{k_1})$ where for each i, x_i is one of the elements of L_{1i}. If there are m_i elements in L_{1i}, the number of such k_1-tuples and hence the number of Player I's pure strategies is the product $m_1 m_2 \cdots m_{k_1}$. The set of all such strategies is X. Similarly, if $T_{21}, T_{22}, \cdots, T_{2k_2}$ represent Player II's information sets and $L_{21}, L_{22}, \cdots, L_{2k_2}$ the corresponding sets of labels, a pure strategy for Player II is a k_2-tuple, $y = (y_1, y_2, \cdots, y_{k_2})$ where $y_j \in L_{2j}$ for each j. Player II has $n_1 n_2 \cdots n_{k_2}$ pure strategies if there are n_j elements in L_{2j}. Y denotes the set of these strategies.

Definition 4.2.1 (Pure strategy)**.** *A* **pure strategy** *for Player I of a game in extensive form is a k_1-tuple $x = (x_1, x_2, \cdots, x_{k_1})$ where for each i, x_i an element in the set of labels L_{1i} associated with the information set T_{1i}. Similarly a pure strategy for Player II is a k_2-tuple $y = (y_1, y_2, \cdots, y_{k_2})$ where y_j is an element in L_{2j}. Denoted by X and Y the set of pure strategies of Player I and II respectively.*

Note that since the set of vertices T is finite, both X and Y are finite sets and we have

$$|X| = |L_{11}| \times |L_{12}| \times \cdots \times |L_{1k_1}| \text{ and } |Y| = |L_{21}| \times |L_{22}| \times \cdots \times |L_{2k_2}|.$$

A referee, given $(x, y) \in X \times Y$, could play the game, playing the appropriate move from x whenever the game enters one of Player I's information sets, playing the appropriate move from y whenever the game enters one of Player II's information sets.

Example 4.2.2 (Matching pennies)**.** *Consider the matching pennies game (Example 4.1.10). Player I has one information set (consisting of one vertex) with two labels "head" and "tail". So Player I has two strategies "head" and "tail". On the other hand, Player II has one information set (consisting of two vertices) with two labels "head" and "tail". So Player II also has two pure strategies "head" and "tail".*

Example 4.2.3 (Battle of the sexes)**.** *Consider the battle of the sexes game (Example 4.1.7). Alice has two pure strategies O and F. Bob has two information sets and each of them has two labels O and F. So Bob has 4 strategies OO, OF, FO, FF. For example, OO mean Bob uses O no matter what Alice uses, and FO means Bob uses F if Alice uses O while Bob would use O if Alice uses F. If Alice uses F and Bob uses FO, then Alice plays F, Bob plays O and the payoff pair is* $(0, 0)$*. If Alice uses O and Bob uses OF, then Alice plays O, Bob plays O and the payoff pair is* $(2, 1)$*.*

If there are chance moves, the move is played at random with the indicated probabilities. The actual outcome of the game for given $(x, y) \in X \times Y$ depends on the chance moves selected, and is therefore a random quantity. Strictly speaking, random payoffs were not provided for in our definition of games in normal form. However, we are quite used to replacing random payoffs by their average values (expected values) when the randomness is due to the use of mixed strategies by the players. We adopt the same convention in dealing with random payoffs when the randomness is due to the chance moves. If for fixed pure strategies of the players, $(x, y) \in X \times Y$, the payoff is a random variable, we replace the payoff by the expected value, and denote this expected payoff by $A(x, y)$.

Example 4.2.4 (Silver dollar)**.** *Player II chooses one of two rooms, room A and room B, in which to hide a silver dollar. Then, Player I, not knowing* *which room contains the dollar, selects one of the rooms to search. However, the search is not always successful. In fact, if the dollar is in room A and Player I searches there, then (by a chance move) he has only probability* 1/2 *of finding it, and if the dollar is in room B and Player I searches there, then he has only probability* 1/4 *of finding it. Of course, if he searches the wrong room, he certainly won't find it. If he does find the coin, Player II pays* \$8 *to Player I; otherwise Player I pays* \$2 *to Player II.*

The game tree of the silver dollar game is shown in Figure 12.

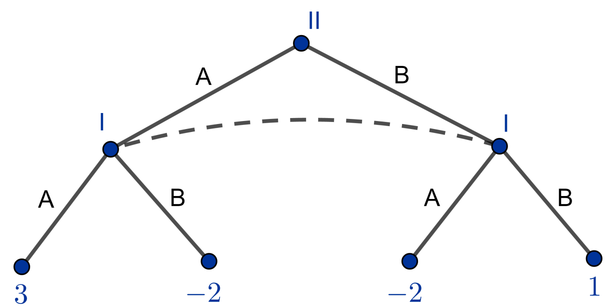


Figure 12: Silver dollar

There are 3 non-terminal vertices in the game tree. One vertex, namely the initial vertex, is associated with Player II. The other two vertices, which are followers of the initial vertex, are associated with Player I. Theses two vertices are connected by a dotted line which shows that they belong to the same information set. Both Player I and Player II have one information set and each of them has two strategies labeled A and B.

To find the expected payoff of Player I, if the two players use different strategies, then Player can never find the silver dollar and the expected payoff is -2*. If both players play A, then the probability that Player I finds the silver dollar is* 1/2 *and the expected payoff to player I is*

$$8 \times (1/2) + (-2) \times (1/2) = 3.$$

If both players play B, then the probability that Player I finds the silver dollar is 1/4 *and the expected payoff to player I is*

$$8 \times (1/4) + (-2) \times (3/4) = 1.$$

Thus the strategic form of the game is given by the 2×2 *matrix*

$$\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}.$$

Now we can calculate that the maximin strategy for Player I is $(0.375, 0.625)$ *and the minimax strategy for Player II is* $(0.375, 0.625)$. *The value of the game is* $v = -1/8$.

Example 4.2.5 (Bluffing game)**.** *Let us find the equivalent strategic form to the bluffing game (Example 4.1.8), whose tree is shown in Figure 8. Player I has two information sets. In each set he must make a choice from among two options. He therefore has* $2 \times 2 = 4$ *pure strategies. We may denote them by*

- (b, b)*: bet with a winning card or a losing card.*
- (b, c)*: bet with a winning card, check with a losing card.*
- (c, b)*: check with a winning card, bet with a losing card.*
- (c, c)*: check with a winning card or a losing card.*

Therefore, $X = \{(b, b), (b, c), (c, b), (c, c)\}$. *We include in* X *all pure strategies whether good or bad (in particular,* (c, b) *seems a rather perverse sort of strategy.) Player II has only one information set. Therefore,* $Y = \{c, f\}$, *where*

- c*: if Player I bets, call.*
- f*: if Player I bets, fold.*

Now we find the payoff matrix. Suppose Player I uses (b, b) *and Player II uses* c. *Then if Player I gets a winning card (which happens with probability* $1/4$*), he bets, Player II calls, and Player I wins 3 dollars. But if Player I gets a losing card (which happens with probability* $3/4$*), he bets, Player II calls, and Player I loses 3 dollars. Player I's average or expected winnings is*

$$A((b, b), c) = \frac{1}{4}(3) + \frac{3}{4}(-3) = -\frac{3}{2}.$$

This gives the upper left entry in the following matrix. The other entries may be computed similarly and are left as exercises.

$$\begin{array}{c} \\ (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} \begin{array}{c} \begin{array}{cc} \quad c & \quad\; f \end{array} \\ \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

Let us solve this 4 by 2 game. The third row is dominated by the first row, and the fourth row is dominated by the second row. In terms of the original form of the game, this says something you may already have suspected: that if Player I gets a winning card, it cannot be good for him to check. By betting he will win at least as much, and maybe more. With the bottom two rows eliminated the matrix becomes

$$\begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \end{pmatrix},$$

whose solution is easily found. The value is $v = -1/4$*. The maximin strategy for Player I is* $(1/6, 5/6, 0, 0)$ *and the minimax strategy for Player II is* $(1/2, 1/2)$*. That means Player I's optimal strategy is to mix* (b, b) *and* (b, c) *with probabilities* $1/6$ *and* $5/6$ *respectively, while Player II's optimal strategy is to mix c and f with equal probabilities* $1/2$ *each. The strategy* (b, b) *is Player I's "bluffing" strategy. Its use entails betting with a losing hand. The strategy* (b, c) *is Player I's "honest" strategy, bet with a winning hand and check with a losing hand. Player I's optimal strategy requires some bluffing and some honesty.*

In a game of perfect information, each player when called upon to make a move knows the exact position in the tree. In particular, each player knows all the past moves of the game including the chance ones. Examples include tic-tac-toe, chess, backgammon, craps, etc. Games of perfect information have a particularly simple mathematical structure. The main result is that every game of perfect information when reduced to strategic form has a saddle point; both players have optimal pure strategies. Moreover, the saddle point can be found by removing dominated rows and columns. This has an interesting implication for the game of chess for example. Since there are no chance moves, every entry of the game matrix for chess must be either $+1$ (a win for Player I), or -1 (a win for Player II), or 0 (a draw). A saddle point must be one of these numbers. Thus, either Player I can guarantee himself a win, or Player II can guarantee himself a win, or both players can assure themselves at least a draw. From the game-theoretic viewpoint, chess is a very simple game. One needs only to write down the matrix of the game. If there is a row of all $+1$'s, Player I can win. If there is a column of all -1's, then Player II can win. Otherwise, there is a row with all $+1$'s and 0's and a column with all -1's and 0's, and so the game is drawn with best play. Of course, the real game of chess is so complicated, there is virtually no hope of ever finding an optimal strategy.

For games in extensive form, it is useful to consider a different method of randomization for choosing among pure strategies. All a player really needs to do is to make one choice of an edge for each of his information sets in the game. A behavioral strategy is a strategy that assigns to each information set a probability distributions over the choices of that set. For example, suppose the first move of a game is the deal of one card from a deck of 52 cards to Player I. After seeing his card, Player I either bets or passes, and then Player II takes some action. Player I has 52 information sets each with 2 choices of action, and so he has 2^{52} pure strategies. Thus, a mixed strategy for Player I is a vector of 2^{52} components adding to 1 (a probability vector of dimension 2^{52}) which is an element in $\mathcal{P}^{2^{52}}$. On the other hand, a behavioral strategy for Player I simply given by the probability of betting for each card he may receive, and so is specified by only 52 numbers (52 probability vectors of dimension 2) which is an element in $(\mathcal{P}^2)^{52}$.

Definition 4.2.6 (Behavioral strategy)**.** *A* **behavioral strategy** *of a player of a game in extensive form is a strategy that assigns to each information set of the player a probability distribution over the choices of that set. In other words, a behavioral strategy of Player I is an element of the form*

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k_1})$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{im_i}) \in \mathcal{P}^{m_i}$, $m_i = |L_{1i}|$, *is a probability vector and* $0 \leq x_{ik} \leq 1$ *represents the probability that Player I would choose the edge labeled with* k *in the set* L_{1i} *when player I arrives at the information set* T_{1i}*. Similarly a behavioral strategy of Player II is an element of the form*

$$\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{k_2})$$

where $\mathbf{y}_j = (y_{j1}, \cdots, y_{jn_j}) \in \mathcal{P}^{n_j}$, $n_j = |L_{2j}|$, *is a probability vector.*

In general, the space of behavioral strategies for the two players is

$$(\mathcal{P}^{m_1} \times \cdots \times \mathcal{P}^{m_{k_1}}) \times (\mathcal{P}^{n_1} \times \cdots \times \mathcal{P}^{n_{k_2}})$$

which is much smaller than the space of mixed strategies

$$\mathcal{P}^{|X|} \times \mathcal{P}^{|Y|} = \mathcal{P}^{m_1 m_2 \cdots m_{k_1}} \times \mathcal{P}^{n_1 n_2 \cdots n_{k_2}}.$$

Here is an example of a mixed strategy that is not a behavioral strategy.

Example 4.2.7 (Battle of sexes)**.** *Consider the battle of the sexes (Example 4.1.7) with perfect information, where Alice moves first. The game tree is shown in Figure 5. Bob has four pure strategies OO, OF, FO, FF. A mixed strategy of Bob is using OO with a probability* 0.3 *and using FF with a probability* 0.7*. The strategies OF and FO will never be used. We can use the vector* $\mathbf{y}_1 = (0.3, 0, 0, 0.7)$ *to represent this mixed strategy where the numbers in the vector represents the probability that Bob uses OO, OF, FO, FF respectively. This mixed strategy is not induced by any behavioral strategy because the actions Bob takes at its two nodes are correlated. A mixed strategy induced by behavioral strategy would be* $\mathbf{y}_2 = (0.09, 0.21, 0.21, 0.49)$ *where Bob is to play O with probability* 0.3 *and to play F with a probability* 0.7 *at each node independently. Note that for any strategy of Alice, the expected payoff for the two players are the same with the strategies* $\mathbf{y}_1$ *and* $\mathbf{y}_2$*. Suppose Alice uses O. If Bob uses* $\mathbf{y}_1 = (0.3, 0, 0, 0, 7)$*, the expected payoff pair is*

$$A(O, \mathbf{y}_1) = 0.3(2, 1) + 0.7(0, 0) = (0.6, 0.3).$$

and if Bob uses $\mathbf{y}_2 = (0.09, 0.21, 0.21, 0.49)$*, the expected payoff pair is*

$$A(O, \mathbf{y}_2) = 0.09(2, 1) + 0.21(2, 1) + 0.21(0, 0) + 0.49(0, 0) = (0.6, 0.3).$$

Similarly, the expected payoff pair are the same if Alice uses F and we have

$$A(F, \mathbf{y}_1) = A(F, \mathbf{y}_2) = (0.7, 1.4).$$

The question arises: Can we do as well with behavioral strategies as we can with mixed strategies? The answer is we can if both players in the game have perfect recall. The basic theorem, due to Kuhn says that in finite games with perfect recall, any distribution over the payoffs achievable by mixed strategies is achievable by behavioral strategies as well.

Theorem 4.2.8 (Kuhn's theorem)**.** *Consider a game in extensive form of perfect recall. For any mixed strategies* $\mathbf{x}$ *and* $\mathbf{y}$ *of Player I and Player II respectively, there exists strategies* $\mathbf{x}_b$ *and* $\mathbf{y}_b$ *induced by behavioral strategies such that* $(\mathbf{x}, \mathbf{y})$*,* $(\mathbf{x}_b, \mathbf{y})$*,* $(\mathbf{x}, \mathbf{y}_b)$ *have the same expected payoff pairs, that is,*

$$A(\mathbf{x}, \mathbf{y}) = (\mathbf{x}_b, \mathbf{y}) = (\mathbf{x}, \mathbf{y}_b).$$

While a Nash equilibrium in mixed strategies always exists via reduction to the normal form case, it is not obvious that a Nash equilibrium in behavioral strategies exists. This is true thanks to Kuhn's theorem (Theorem 4.2.8).

Theorem 4.2.9. *In a finite game in extensive form of perfect recall, there is a Nash equilibrium in behavioral strategies.*

4.3 Subgame perfect Nash equilibrium

Backward induction is a powerful solution concept with some intuitive appeal. Unfortunately, it can be applied only to perfect information games. Its intuition, however, can be extended beyond these games through subgame perfection. This section defines the concept of subgame-perfect equilibrium and illustrates how one can check whether pure strategy pair is a subgame perfect equilibrium.

Definition 4.3.1 (Subgame)**.** *A* **subgame** *of a game G in extensive form is a game G' such that*

1. *all vertices of G' are vertices of G, and*
2. *if x is a vertex in G', then all followers of x are vertices of G'.*

A main property of backward induction is that, when restricted to a subgame of the game, the equilibrium computed using backward induction remains an equilibrium of the subgame.

Definition 4.3.2 (Subgame perfect Nash equilibrium)**.** *A Nash equilibrium of a game in extensive form is said to be* **subgame perfect** *if it is Nash equilibrium in every subgame of the game.*

Any subgame other than the entire game itself is called **proper subgame**. Observe that the pure strategy pair of a subgame obtained by backward induction coincides with the strategy pair obtained by applying backward induction to the entire game. We see that the strategies obtained by backward induction is a subgame perfect Nash equilibrium.

Theorem 4.3.3. *Every finite game in extensive form with perfect information has a subgame perfect pure Nash equilibrium which can be computed by backward induction.*

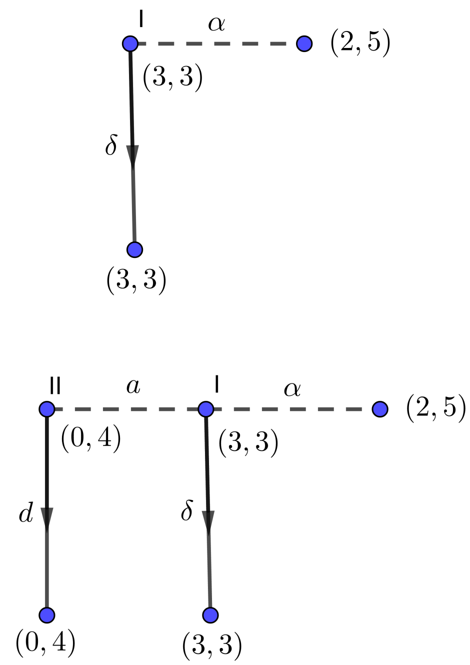


Figure 13: Proper subgames of centipede

Example 4.3.4. *Then centipede game (Example 4.1.6) has two proper subgames as shown in Figure 13. The equilibrium obtained by backward induction, that is player I uses* $D\delta$ *and Player II uses* d*, remains to be an equilibrium of each subgame.*

Example 4.3.5 (Matching pennies)**.** *Consider the matching penny game with perfect information (Example 4.1.10). This game has three subgames: one after Player I chooses Head, one after Player I chooses Tail, and the game itself. Again, the equilibrium computed through backward induction is a Nash equilibrium at each subgame.*

We consider a game with imperfect information.

Example 4.3.6 (Imperfect information game)**.** *Consider the game in Figure 14.*

One cannot apply backward induction in this game because it is not a perfect

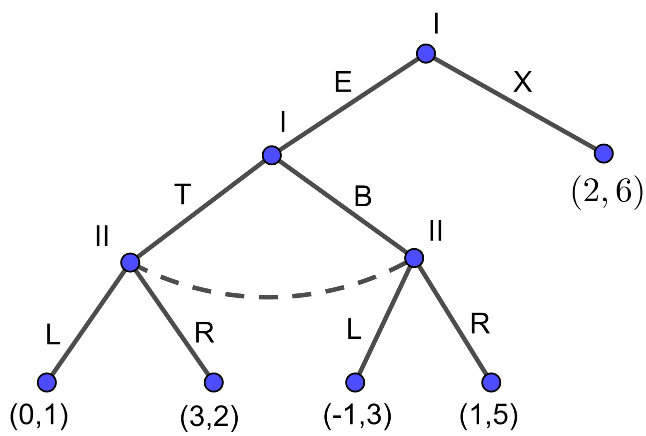


Figure 14: Imperfect information game

information game. One can compute the subgame-perfect equilibrium, however. This game has two subgames: one starts after Player I plays E; the second one is the game itself. The subgame perfect equilibria are computed as follows. First compute a Nash equilibrium of the subgame, then fixing the equilibrium actions as they are (in this subgame), and taking the equilibrium *payoffs in this subgame as the payoffs for entering the subgame, compute a Nash equilibrium in the remaining game.*

The subgame has only one Nash equilibrium, as T dominates B, and R dominates L. In the unique Nash equilibrium, Player I plays T and Player II plays R, yielding the payoff vector $(3, 2)$, as illustrated in Figure 15.

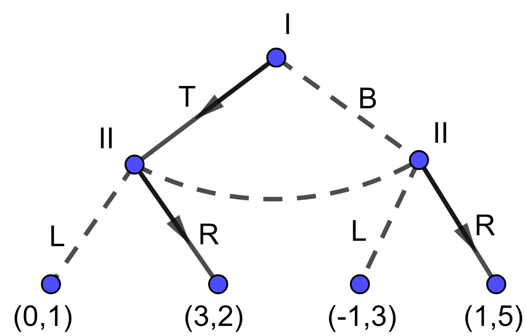


Figure 15: Equilibrium in a subgame

Given this, the game reduces to a game shown in Figure 16.

Player I chooses E in this reduced game. Therefore, the subgame-perfect

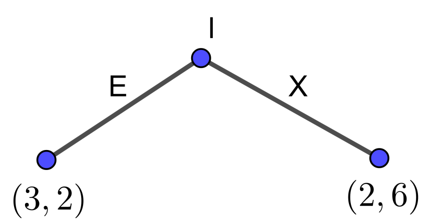


Figure 16: Reduced game

equilibrium is as in Figure 17. First, Player I uses ET and Player II uses R.

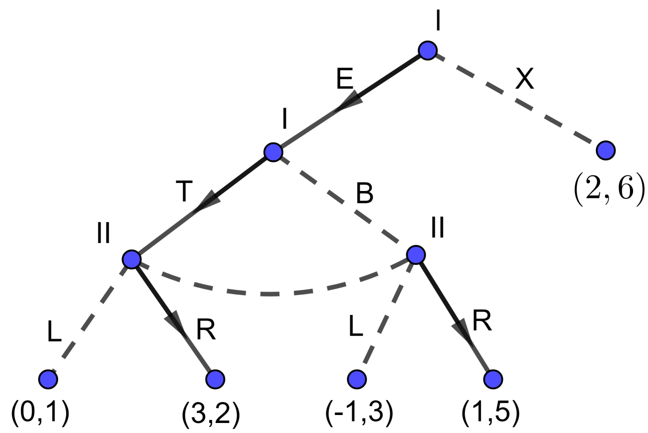


Figure 17: Subgame perfect equilibrium

The above example illustrates a technique to compute the subgame-perfect equilibria in finite games.

- Step 1. Pick a subgame that does not contain any other subgame.
- Step 2. Compute a Nash equilibrium of this game.
- Step 3. Assign the payoff vector associated with this equilibrium to the starting node, and eliminate the subgame.
- Step 4. Iterate this procedure until a move is assigned at every contingency, when there remains no subgame to eliminate.

As in backward induction, when there are multiple equilibria in the picked subgame, one can choose any of the Nash equilibrium, including one in a mixed strategy. Every choice of equilibrium leads to a different subgame-perfect Nash equilibrium in the original game. By varying the Nash equilibrium for the subgames at hand, one can compute all subgame perfect Nash equilibria. A subgame-perfect Nash equilibrium is a Nash equilibrium because the entire game is also a subgame. The converse is not true. There can be a Nash equilibrium that is not subgame-perfect.

Example 4.3.7. *The game in Example 4.3.6 has the following equilibrium: Player I uses XB and Player II uses L. You should be able to check that* *this is a Nash equilibrium. (Player II cannot improve his payoff by changing his strategy L alone.) But it is not subgame perfect: Player II plays a strictly dominated strategy in the proper subgame.*

Sometimes subgame-perfect equilibrium can be highly sensitive to the way we model the situation. For example, consider the game in Figure 18.
This is essentially the same game as Example 4.3.6. The only difference is

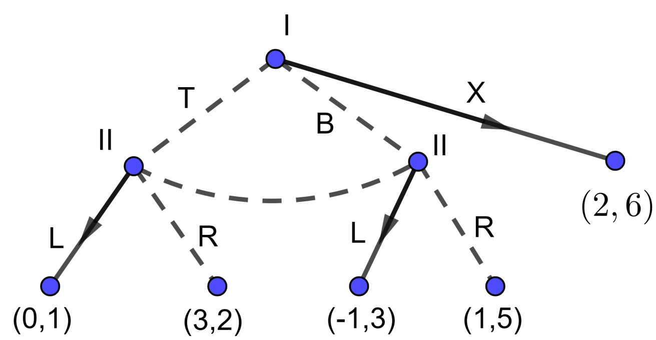


Figure 18: Game equivalent to Figure 14

that Player I makes his choices here at once. (There is no strategy E in the first move.) One would have thought that such a modeling choice should not make a difference in the solution of the game. It does make a huge difference for subgame-perfect Nash equilibrium nonetheless. In the new game, the only subgame of this game is itself, hence any Nash equilibrium is subgame perfect. In particular, the non-subgame-perfect Nash equilibrium of the game above is subgame perfect. In the new game, it is formally written as Player I uses X and Player II uses L as shown in Figure 18. Clearly, one could have used the idea of sequential rationality to solve this game. That is, by sequential rationality of Player II at her information set, she must choose R. Knowing this, Player I must choose T. Therefore, subgame perfect equilibrium does not fully formalize the idea of sequential rationality. It does yield reasonable solutions in many games, and it is widely used in game theory. It will also be used in this course frequently.

Example 4.3.8 (Market entry)**.** *Suppose Pluto is considering whether to enter a market and Venus is a provider of the market. If Pluto enters, both* *firms simultaneously decide whether to act tough (T) or accommodate (A). This leads to an extensive form game with imperfect information whose game tree representation is given in Figure 19, where the first number in a payoff vector belongs to Pluto and the second to Venus. In this game, Pluto has*

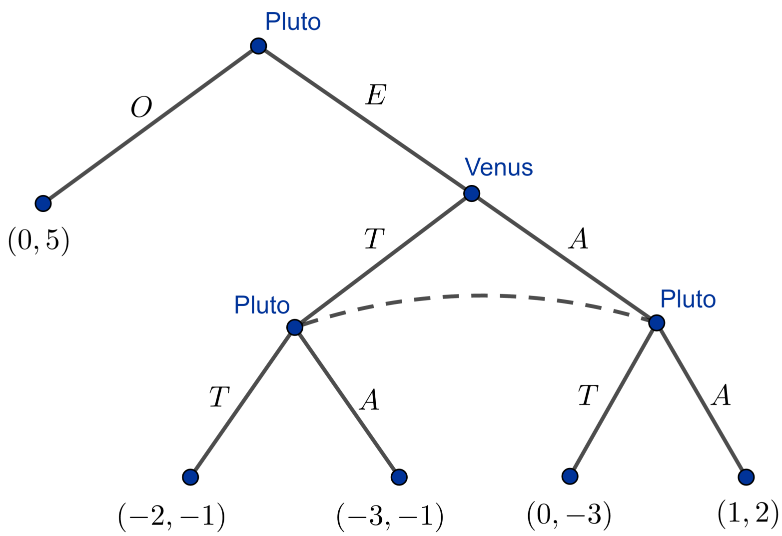


Figure 19: Market entry game

two information sets and each information set has two labels. Thus Pluto has 4 strategies OT, OA, ET, EA. *Here* O *stands for "out" and* E *stands for "enter". For example,* OA *means Pluto stays out of the market and chooses to accommodate if he enters the market.* ET *means Pluto enters the market and chooses to act tough. Venus has two strategies* T *and* A. *The strategic form of the market entry game is given below.*

	T	A
OT	$(0, 5)$	$(0, 5)$
OA	$(0, 5)$	$(0, 5)$
ET	$(-2, -1)$	$(0, -3)$
EA	$(-3, 1)$	$(1, 2)$

There are three Nash equilibria of this game: (OT, T), (OA, T), (EA, A). *In the second Nash equilibrium Pluto is supposed to accommodate and Venues is supposed to act tough, following Pluto entering the market. Is that reasonable? In other words, suppose, the game actually reached that stage, that* *is Pluto actually entered. Now, is* (A, T) *a reasonable outcome? One way of asking the same question is to check if both players are acting rationally, i.e., best responding to each other's strategies, conditional upon Pluto entering the market. Notice that conditional upon Pluto entering the market we have the subgame in Figure 20. The strategic form of the subgame is given below.*

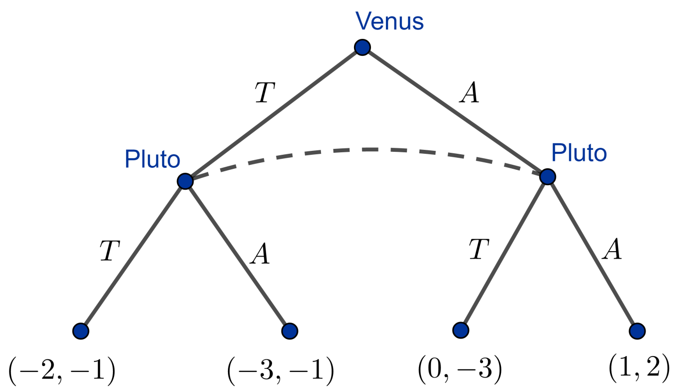


Figure 20: Subgame of market entry

	T	A
T	$(-2,-1)$	$(0,-3)$
A	$(-3,1)$	$(1,2)$

If Pluto anticipates Venus to play T*, then its best response is* T *as well, not* A*. (Neither is* T *a best response for Pluto to* A*.) Therefore, to the extent that we regard only Nash equilibrium outcomes as reasonable, we conclude that* (A, T) *is not reasonable. It follows that the Nash equilibrium* (OA, T) *is not subgame-perfect. In contrast,* (OT, T) *and* (EA, A) *are subgame-perfect equilibria.*

4.4 Recursive games

We consider now matrix games in which the outcome of a particular choice of pure strategies of the players may be that the players have to play another game. Let us take a simple example.

Example 4.4.1. *Let G_1 and G_2 denote 2×2 games with matrices*

$$G_1 = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$$

and let G denote the 2×2 game whose matrix is represented by

$$G = \begin{pmatrix} G_1 & 4 \\ 5 & G_2 \end{pmatrix}.$$

The game G is played in the usual manner with Player I choosing a row and Player II choosing a column. If the entry in the chosen row and column is a number, Player II pays Player I that amount and the game is over. If Player I chooses row 1 and Player II chooses column 1, then the game G_1 is played. If Player I chooses row 2 and Player II chooses column 2, then G_2 is played. We may analyze the game G by first analyzing G_1 and G_2.

G_1: *Maximin strategy for Player I is* $(1/2, 1/2)$
Minimax strategy for Player II is $(2/3, 1/3)$
value of G_1 is $v(G_1) = 1$
G_2: *Maximin strategy for Player I is* $(0, 1)$
Minimax strategy for Player II is $(0, 1)$
value of G_2 is $v(G_2) = 3$

If after playing the game G the players end up playing G_1, then they can expect a payoff of the value of G_1, namely 1, on the average. If the players end up playing G_2, they can expect an average payoff of the value of G_2, namely 3. Therefore, the game G can be considered equivalent to the game with matrix

$$\begin{pmatrix} 1 & 4 \\ 5 & 3 \end{pmatrix}$$

G: *Maximin strategy for Player I is* $(2/5, 3/5)$
Minimax strategy for Player II is $(1/5, 4/5)$
value of G is $v(G) = 17/5$

This method of solving the game G may be summarized as follows. If the matrix of a game G has other games as components, the solution of G is the solution of the game whose matrix is obtained by replacing each game in the matrix of G by its value.

This example may be written as a 4×4 matrix game. The four pure strategies of Player I may be denoted $\{(1,1),(1,2),(2,1),(1,2)\}$, where (i,j) represents: use row i in G, and if this results in G_i being played use row j. A similar notation may be used for Player II. The 4×4 game matrix becomes

$$G = \left(\begin{array}{cc|cc} 0 & 3 & 4 & 4 \\ 2 & -1 & 4 & 4 \\ \hline 5 & 5 & 0 & 1 \\ 5 & 5 & 4 & 3 \end{array} \right).$$

Conversely, suppose we are given a game G and suppose after some rearrangement of the rows and of the columns the matrix may be decomposed into the form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

where G_{11} and G_{22} are arbitrary matrices and G_{12} and G_{21} are constant matrices. (A constant matrix has the same numerical value for all of its entries.) Then we can solve G by the above method, pretending that as the first move the players choose a row and column from the 2×2 decomposed matrix.

Of course, a game that is the component of some matrix game may itself have other games as components, in which case one has to iterate the above method to obtain the solution. This works if there are a finite number of stages.

Example 4.4.2 (Inspection game)**.** *Player II must try to perform some forbidden action in one of the next n time periods. Player I is allowed to inspect Player II secretly just once in the next n time periods. If Player II acts while Player I is inspecting, Player II loses 1 unit to Player I. If Player I is not inspecting when Player II acts, the payoff is zero. Let G_n denotes this game. Player I has two strategies: inspect and wait; and Player II has two strategies: act and wait. The game tree of the game is shown is Figure 21 We obtain the iterative strategic form*

$$G_n = \begin{pmatrix} 1 & 0 \\ 0 & G_{n-1} \end{pmatrix}$$

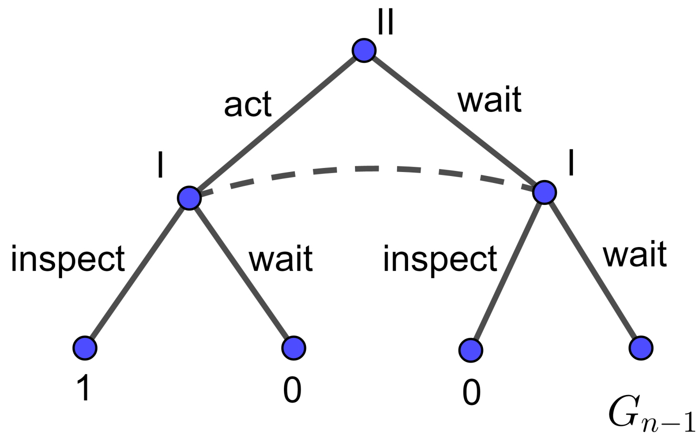


Figure 21: Game tree of G_n

for $n = 2, 3, 4, \cdots$, with boundary condition $G_1 = (1)$. We may solve for value of G_n iteratively as

$$
\begin{aligned}
v(G_1) &= 1 \\
v(G_2) &= v\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1/2 \\
v(G_3) &= v\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} = 1/3 \\
&\cdots \\
v(G_n) &= v\begin{pmatrix} 1 & 0 \\ 0 & 1/(n-1) \end{pmatrix} = 1/n
\end{aligned}
$$

since inductively, the value of G_n is $v(G_n) = 1/n$. The optimal strategy in the game G_n for both players is $(1/n, (n-1)/n)$.

The following multistage game is loosely related to the game of Cluedo.

Example 4.4.3 (Cluedo)**.** *From a deck with $m+n+1$ distinct cards, m cards are dealt to Player I, n cards are dealt to Player II, and the remaining card, called the "target card", is placed face down on the table. Players knows their own cards but not those of their opponent. The objective is to guess correctly the target card. Players alternate moves, with Player I starting. At each move, a player may either*

- *guess at the target card, in which case the game ends, with the winner being the player who guessed if the guess is correct, and his opponent if the guess is incorrect, or*
- *ask if the other player holds a certain card. If the other player has the card, that card must be shown and is removed from play.*

With a deck of say 11 *cards and each player receiving 5 cards, this is a nice playable game that illustrates need for bluffing in a clear way. If a player asks about a card that is in his own hand, he knows what the answer will be. We call such a play a bluff. If a player asks about a card not in his hand, we say he is honest. If a player is always honest and the card he asks about is the target card, the other player will know that the requested card is the target card and so will win. Thus a player must bluff occasionally. Bluffing may also lure the opponent into a wrong guess at the target card.*

Let us denote this game with Player I to move by $G_{m,n}$*. The game* $G_{m,0}$ *is easy to play. Player I can win immediately. Since his opponent has no cards, he can tell what the target card is. Similarly, the game* $G_{0,n}$ *is easy to solve. If Player I does not make a guess immediately, his opponent will win on the next move. However, his probability of guessing correctly is only* $1/(n+1)$*. Valuing 1 for a win and 0 for a loss from Player I's viewpoint, the value of the game is just the probability Player I wins under optimal play. We have*

$$\begin{cases} v(G_{m,0}) = 1, & \textit{for any } m \geq 0, \\ v(G_{0,n}) = \dfrac{1}{n+1}, & \textit{for any } n \geq 0. \end{cases}$$

If Player I asks for a card that Player II has, that card is removed from play and it is Player II's turn to move, holding $n-1$ *cards to her opponent's* m *cards. This is exactly the game* $G_{n-1,m}$ *but with Player II to move. We denote this game by* $\bar{G}_{n-1,m}$*. Since the probability that Player I wins is one minus the probability that Player II wins, we have*

$$v(\bar{G}_{n-1,m}) = 1 - v(G_{n,m}), \ \textit{for any } m, n.$$

Suppose Player I asks for a card that Player II does not have. Player II must immediately decide whether or not Player I was bluffing. If she decides Player I was honest, she will announce the card Player I asked for as her guess at the target card, and win if she was right and lose if she was wrong. If she decides Player I was bluffing and she is wrong, Player I will win on *his turn. If she is correct, the card Player I asked for is removed from his hand, and the game played next is* $\bar{G}_{n,m-1}$*. Using such considerations, we may write the game as a multistage game in which a stage consists of three pure strategies for Player I (honest, bluff, guess) and two pure strategies for Player II (ignore the asked card, call the bluff by guessing the asked card). In summary, Player I has three strategies: honest, bluff, guess; and Player II has two strategies: ignore, call. The game tree is shown in Figure 22.*

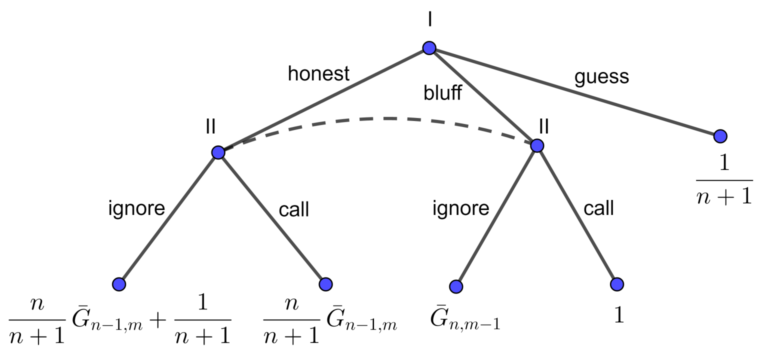


Figure 22: Game tree of $G_{m,n}$

The game matrix becomes, for $m \geq 1$ *and* $n \geq 1$*,*

$$G_{m,n} = \begin{pmatrix} \frac{n}{n+1}\bar{G}_{n-1,m} + \frac{1}{n+1} & \frac{n}{n+1}\bar{G}_{n-1,m} \\ \bar{G}_{n,m-1} & 1 \\ \frac{1}{n+1} & \frac{1}{n+1} \end{pmatrix}.$$

This assumes that if Player I asks honestly, he chooses among the $n+1$ *unknown cards with probability* $1/(n+1)$ *each; also if he bluffs, he chooses among his* m *cards with probability* $1/m$ *each.*

As an example, the upper left entry of the matrix is found as follows. With probability $n/(n+1)$*, Player I asks a card that is in Player II's hand and the game becomes* $\bar{G}_{n-1,m}$*; with probability* $1/(n+1)$*, Player I asks the target card, Player II ignores it and Player I wins on his next turn, i.e. gets 1. The upper right entry is similar, except this time if the asked card is the target card, Player II guesses it and Player I gets 0. It is reasonable to assume that if* $m \geq 1$ *and* $n \geq 1$*, Player I should not guess, because the* *probability of winning is too small. In fact if* $m \geq 1$ *and* $n \geq 1$*, there is a strategy for Player I that dominates guessing, so that the last row of the matrix may be deleted. This strategy is: On the first move, ask any of the* $m+n+1$ *cards with equal probability* $1/(m+n+1)$ *(i.e. use row 1 with probability* $(n+1)/(m+n+1)$ *and row 2 with probability* $m/(m+n+1)$*, and if Player II doesn't guess at her turn, then guess at the next turn. We must show that Player I wins with probability at least* $1/(n+1)$ *whether or not Player II guesses at her next turn. If Player II guesses, her probability of win is exactly* $1/(m+1)$ *whether or not the asked card is one of hers. So Player I's win probability is* $m/(m+1) \geq 1/2 \geq 1/(n+1)$*. If Player II does not guess, then at Player I's next turn, Player II has at most* n *cards (she may have* $n-1$*) so again Player I's win probability is at least* $1/(n+1)$*. So the third row of* $G_{m,n}$ *may be removed and the games reduce to*

$$G_{m,n} = \begin{pmatrix} \frac{n}{n+1}\bar{G}_{n-1,m} + \frac{1}{n+1} & \frac{n}{n+1}\bar{G}_{n-1,m} \\ \bar{G}_{n,m-1} & 1 \end{pmatrix}.$$

for $m \geq 1$ *and* $n \geq 1$*. These* 2×2 *games are easily solved recursively, using the boundary conditions* $v(G_{m,0}) = 1$ *and* $v(G_{0,n}) = \dfrac{1}{n+1}$ *for* $m, n \geq 0$*. One can find the value and optimal strategies of* $G_{m,n}$ *after one finds the values of* $G_{n,m-1}$ *and* $G_{n-1,m}$*. For example, the game* $G_{1,1}$ *reduces to the game with matrix*

$$G_{1,1} = \begin{pmatrix} \frac{1}{2}\bar{G}_{0,1} + \frac{1}{2} & \frac{1}{2}\bar{G}_{0,1} \\ \bar{G}_{1,0} & 1 \end{pmatrix}.$$

Thus the value of the game is

$$\begin{aligned} v(G_{1,1}) &= v\begin{pmatrix} \frac{1}{2}(1-v(G_{0,1})) + \frac{1}{2} & \frac{1}{2}(1-v(G_{0,1})) \\ 1-v(G_{1,0}) & 1 \end{pmatrix} \\ &= v\begin{pmatrix} \frac{1}{2}(1-\frac{1}{2}) + \frac{1}{2} & \frac{1}{2}(1-\frac{1}{2}) \\ 1-1 & 1 \end{pmatrix} \\ &= v\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2}. \end{aligned}$$

The maximin strategy for Player I is $(2/3, 1/3)$ *(i.e. bluff with probability* $1/3$*), and the minimax strategy of player II is* $(1/2, 1/2)$*. One can also show* *that* $G_{m,n}$ *has no saddle point for* $m, n \geq 1$. *If we let* $v_{m,n} = v(G_{m,n})$, *the game* $G_{m,n}$ *is equivalent to the game with* 2×2 *game matrix*

$$\begin{pmatrix} \frac{n+1-nv_{n-1,m}}{n+1} & \frac{n-nv_{n-1,m}}{n+1} \\ 1 - v_{n,m-1} & 1 \end{pmatrix}.$$

Recall that for 2×2 *game matrix with no saddle points*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we have

$$\begin{array}{ll} \text{value of } A: & v = \dfrac{ad - bc}{a - b - c + d} \\ \text{maximin strategy for Player } I: & \mathbf{p} = \left(\dfrac{d - c}{a - b - c + d}, \dfrac{a - b}{a - b - c + d} \right) \\ \text{minmax strategy for Player } II: & \mathbf{q} = \left(\dfrac{d - b}{a - b - c + d}, \dfrac{a - c}{a - b - c + d} \right) \end{array}$$

We obtain the recursive formula for the value $v_{m,n}$, *the maximin strategy* $\mathbf{p}_{m,n}$ *and the minimax strategy* $\mathbf{q}_{m,n}$ *of* $G_{m,n}$ *given by*

$$\begin{aligned} v_{m,n} &= v \begin{pmatrix} \frac{n+1-nv_{n-1,m}}{n+1} & \frac{n-nv_{n-1,m}}{n+1} \\ 1 - v_{n,m-1} & 1 \end{pmatrix} \\ &= \frac{1 + n(1 - v_{n-1,m})v_{n,m-1}}{1 + (n+1)v_{n,m-1}} \\ \mathbf{p}_{m,n} &= \left(\frac{(n+1)v_{n,m-1}}{1 + (n+1)v_{n,m-1}}, \frac{1}{1 + (n+1)v_{n,m-1}} \right) \\ \mathbf{q}_{m,n} &= \left(\frac{1 + nv_{n-1,m}}{1 + (n+1)v_{n,m-1}}, \frac{(n+1)v_{n,m-1} - v_{n-1,m}}{1 + (n+1)v_{n,m-1}} \right) \end{aligned}$$

for $m, n \geq 1$ *and*

$$\begin{cases} v_{m,0} = 1, & \text{for } m \geq 0 \\ v_{0,n} = \frac{1}{n+1}, & \text{for } n \geq 0 \end{cases}$$

This provides a simple direct way to compute the values recursively. For *example*

$$\begin{aligned} v_{2,1} &= \frac{1+(1-v_{0,2})v_{1,1}}{1+2v_{1,1}} \\ &= \frac{1+(1-\frac{1}{3})\frac{1}{2}}{1+2(\frac{1}{2})} \\ &= \frac{2}{3} \\ v_{1,2} &= \frac{1+2(1-v_{1,1})v_{2,0}}{1+3v_{2,0}} \\ &= \frac{1+2(1-\frac{1}{2})}{1+3} \\ &= \frac{1}{2} \\ v_{2,2} &= \frac{1+2(1-v_{1,2})v_{2,1}}{1+3v_{2,1}} \\ &= \frac{1+2(1-\frac{1}{2})\frac{2}{3}}{1+3(\frac{2}{3})} \\ &= \frac{5}{9} \end{aligned}$$

The following table shows the values of $v_{m,n}$, $\mathbf{p}_{m,n}$ *and* $\mathbf{q}_{m,n}$ *for* $2 \leq m+n \leq 4$.

$G_{m,n}$	$G_{1,1}$	$G_{1,2}$	$G_{2,1}$	$G_{1,3}$	$G_{2,2}$	$G_{3,1}$
value $v_{m,n}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{5}{9}$	$\frac{11}{16}$
maximin strategy $\mathbf{p}_{m,n}$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{1}{4},\frac{3}{4})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{4}{5},\frac{1}{5})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{1}{2},\frac{1}{2})$
minimax strategy $\mathbf{q}_{m,n}$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{3}{5},\frac{2}{5})$	$(\frac{1}{3},\frac{2}{3})$	$(\frac{5}{8},\frac{3}{8})$

Exercise 4

1. In a bargaining game, the buyer moves first by offering either \$500 or \$100 for a product that she values \$600. The seller, for whom the value of the object is \$50, responds by either accepting (A) or rejecting (R) the offer.

   (a) Draw the game tree of the bargaining game.

(b) Use backward induction to solve the game.

2. Albert and Benson start with \$16 in each of their piles. They take turns choosing one of two actions, continue or stop with Albert chooses first. Each time a player says continue, half of the amount in his pile will move to the other player's pile, and then extra \$16 will be added to his pile. The game automatically stop when the total amount in their piles reaches \$96.

   (a) Draw the game tree of the game.

   (b) Use backward induction to solve the game and write down the payoffs of the players in the solution.

3. Armies I and II are fighting over an island initially held by a battalion of army II. Army I has 3 battalions and army II has 4, including the battalion occupying the island. Whenever the island is occupied by one army the opposing army can launch an attack with all its battalions. The outcome of the attack is that the army with more battalions will win and occupy the island with the surviving battalions which is equal to the difference of the number of battalions while the battalions of the loser will all be destroyed. The commander of each army is interested in maximizing the number of surviving battalions but also regards the occupation of the island as worth one and a half battalions.

   (a) Draw the game tree of the game.

   (b) Solve the game.

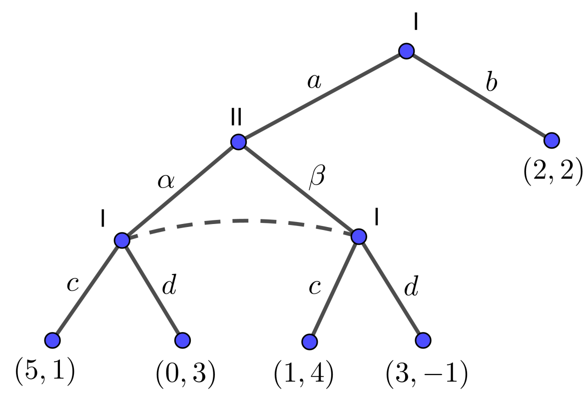
4. In a senate race game, a senate seat is currently occupied by Gray (the incumbent). A potential challenger for Gray's seat is Green. Gray moves first and decide whether to launch a preemptive advertising campaign and Green has to decide whether to enter the race. Green will win the senate seat only if Gray does not advertise and Green enters the race. Otherwise Gray will win the Senate Seat. Both Gray and Green value the senate seat as 5 units. However, 2 units of advertising cost will be deducted from the payoff of Gray if he launches the advertising and 1 unit of running cost will be deducted from the payoff of Green if he enters the race.

   (a) Draw the game tree of the senate race game.

(b) Use backward induction to solve the game.

(c) Suppose Green does not know whether Gray has launched an advertising before he decides whether to enter the race. Draw the game tree and write down the strategic form of the game.

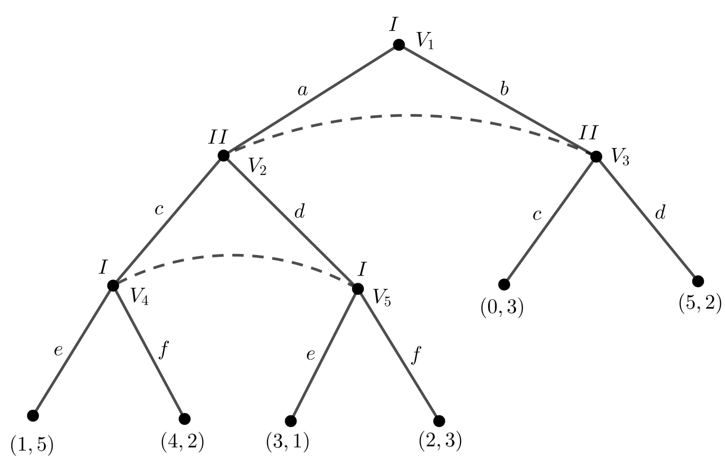
5. Consider the game tree



(a) Write down all pure strategies for Player I and Player II.

(b) Write down the strategic form (game bimatrix) of the game.

6. Consider the game tree



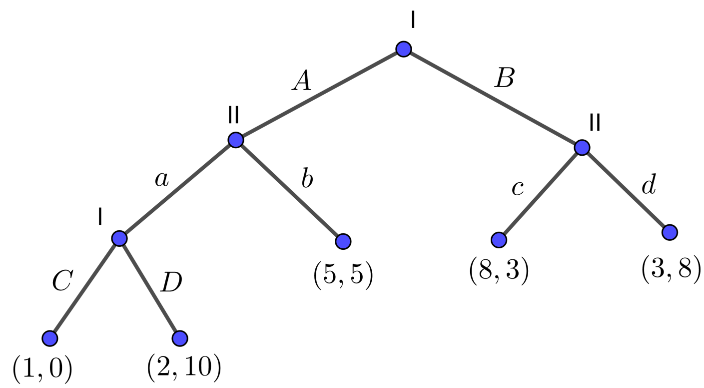
(a) Write down all pure strategies for Player I and Player II.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Solve the subgame after Player I chooses a.

(d) Find all Nash equilibria of the game.

7. Consider the game tree



(a) Write down all pure strategies for Player I and Player II.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Find all pure Nash equilibrium of the game.

(d) State whether each of the Nash equilibria is subgame perfect.

8. In a game show, there is \$5 in a green envelope and \$7 in a yellow envelope. A player Alan chooses an envelope and the amount inside the envelope is increased by \$4. Another player Bonnie, not knowing which envelope Alan has chosen, chooses an envelope and the amount inside the envelope is doubled. Then Alan, not knowing which envelope Bonnie has chosen, chooses envelope and gets the money inside. Bonnie will get the money inside the other envelope.

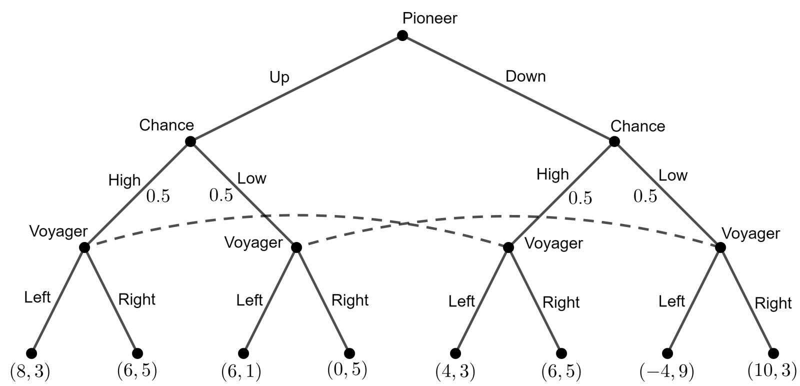
(a) Draw the game tree of the game.

(b) Write down all strategies of Alan and Bonnie.

(c) Write down the strategic form (game bimatrix) of the game.

(d) Find the Nash equilibrium of the game.

9. Consider the game tree



After Pioneer has made the first move, the chance of High and Low are equal.

(a) Write down the strategies of Pioneer and Voyager.

(b) Write down the strategic form (game bimatrix) of the game.

(c) Find the Nash equilibrium of the game.

10. Both Steve and Tony put \$1 to the pot. Steve draws a card from a winning card and a losing card randomly. Steve sees his card but keeps it hidden from Tony. Steve then bets or Checks. If Steve bets, he puts \$6 more into the pot and Tony, not knowing what card Steve has, must fold or call. If Tony folds, he loses \$1 to Steve no matter what card Steve has. If Tony calls, Steve wins \$7 from Tony if Steve has the winning card and Steve loses \$7 to Tony if Steve has the losing card. If Steve checks, his card is inspected. Steve wins \$1 from Tony if he has the winning card, and otherwise he loses \$1 to Tony.

(a) Draw the game tree of the game.

(b) Write down the strategies of Steve and Tony.

(c) Write down the strategic form (game matrix) of the game.

(d) Solve the game.

11. Players I and II play the following bluffing game. Each player bet \$1. Player I is given a card which is high or low; each is equally likely.

Player I sees the card, player II doesn't. Player I can raise the bet to \$2 or fold. If player I raises, player II can call or fold. If player II folds, he loses \$1 to player I no matter what the card is. If player II calls, then player I wins \$2 from player II if his card is high and loses \$2 to player II if the card is low.

(a) Draw the game tree of the game.

(b) Write down all pure strategies of the players.

(c) Write down the strategic form (game matrix) of the game.

(d) Solve the game.

12. Two firms, an entrant (I) and an incumbent (II) play an market entry game. The entrant moves first, deciding to stay Out or to Enter the market. If the entrant stays Out, he gets a payoff of 0, while the incumbent gets the monopoly profit of 3. If the entrant Enters, the incumbent must choose between Fighting (so that both players obtain -1) or Accommodating (so that both players obtain the duopoly profit of 1).

    (a) Draw the game tree of the game.

    (b) Write down all pure strategies of the players.

    (c) Write down the strategic form of the game.

    (d) Solve the game.

13. Player I has two coins. One is fair (probability 1/2 of heads and 1/2 of tails) and the other is biased with probability 1/3 of heads and 2/3 of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to Player II. Then II must guess whether I chose the fair or biased coin. If II is correct there is no payoff. If II is incorrect, she loses 1 dollar.

    (a) Draw the game tree.

    (b) Solve the game.

14. A fair coin (probability 1/2 of heads and 1/2 of tails) is tossed and the outcome is shown to Player I. On the basis of the outcome of this toss, Player I decides whether to bet 1 or 2. Then Player II hearing

the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Player II wins if her guess is correct and loses if her guess is incorrect. The absolute value of the amount won is the amount bet if the coin comes up tail and the amount bet plus 1 if the coin comes up heads.

(a) Draw the game tree.

(b) Write down the strategic form of the game.

(c) Solve the game.

15. Coin A has probability 1/2 of heads and 1/2 of tails. Coin B has probability 1/3 of heads and 2/3 of tails. Player I must predict "heads" or "tails". If he predicts heads, coin A is tossed. If he predicts tails, coin B is tossed. Player II is informed as to whether I's prediction was right or wrong (but she is not informed of the prediction or the coin that was used), and then must guess whether coin A or coin B was used. If Player II guesses correctly she wins 1 dollar from Player I. If Player II guesses incorrectly and Player I's prediction was right, Player I wins 2 dollars from Player II. If both are wrong there is no payoff.

(a) Draw the game tree of the game.

(b) Write down the strategic form of the game.

(c) Solve the game.

16. Consider the two games

$$G_1 = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

One of these games is chosen to be played at random with probability 1/3 for G_1 and probability 2/3 for G_2. The game chosen is revealed to Player I but not to Player II. Then Player I selects a row, 1 or 2, and simultaneously Player II chooses a column, 1 or 2, with payoff determined by the selected game.

(a) Draw the game tree.

(b) Solve the game.

17. Player I draws a card at random from a full deck of 52 cards. After looking at the card, he bets either 1 or 5 that the card he drew is a face card (king, queen or jack). Then Player II either concedes or doubles. If she concedes, she pays Player I the amount bet (no matter what the card was). If she doubles, the card is shown to her, and Player I wins twice his bet if the card is a face card, and loses twice his bet otherwise.

    (a) Draw the game tree.

    (b) Write down the strategic form of the game.

    (c) Solve the game.

18. Player II must count from n down to zero by subtracting either one or two at each stage. Player I must guess at each stage whether Player II is going to subtract one or two. If Player I ever guesses incorrectly at any stage, the game is over and there is no payoff. Otherwise, if Player I guesses correctly at each stage, he wins 1 from Player II. Let G_n denote this game, and use the initial conditions $G_0 = (1)$ and $G_1 = (1)$. Let v_n be the value of G_n.

    (a) Find v_3, v_4 and v_5.

    (b) Find v_n. (You may use F_n to denote the Fibonacci sequence, $0, 1, 1, 2, 3, 5, 8, 13, \cdots$, with definition $F_0 = 0, F_1 = 1$, and for $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.)

19. There is one point to go in the match. The player that wins the last point while serving wins the match. The server has two strategies, high and low. The receiver has two strategies, near and far. The probability the server wins the point is given in the accompanying table.

	near	far
high	0.8	0.5
low	0.6	0.7

If the server misses the point, the roles of the players are interchanged and the win probabilities for given pure strategies are the same for the new server. Find optimal strategies for server and receiver, and find the probability the server wins the match.

20. Player I tosses a coin with probability p of heads. For each $k = 1, 2, \cdots,$ if Player I tosses k heads in a row he may stop and challenge Player II to toss the same number of heads; then Player II tosses the coin and wins if and only if he tosses k heads in a row. If Player I tosses tails before challenging Player II, then the game is repeated with the roles of the players reversed. If neither player ever challenges, the game is a draw.

    (a) Solve the game when $p = 1/2$.

    (b) For arbitrary p, find the optimal strategies of the players and the probability that Player I wins.

5 Cooperative games

In a **cooperative game**, players can make binding agreements about which strategies to play. In the last chapter, we studied Nash bargaining solution for 2-person cooperative games with non-transferable utility. In this chapter, we study n-person cooperative games with **transferable utility**. In such a game, players may share their payoffs according to the agreements made by the players in advance. However there is no universally accepted rules to determine how the payoffs should be shared among the players. Different solution concepts may be used in different situations. In this chapter, we are going to study two solution concepts namely core and Shapley value.

5.1 Characteristic form and imputations

First we define the strategic form of a cooperative game.

Definition 5.1.1. *Let $A = \{A_1, A_2, \cdots, A_n\}$ be the set of players. Let X_i, $i = 1, 2, \cdots, n$, be the set of strategies of player $A_i \in A$.*

1. *The* **strategic form** *of a game is a function*

$$\pi = (\pi_1, \pi_2, \cdots, \pi_n) : X_1 \times X_2 \times \cdots \times X_n \to \mathbb{R}^n$$

2. *A* **coalition** *is a subset $S \subset A$ of the set of players. For each $i = 1, 2, \cdots, n$, the set $\{A_i\}$, consists of one player, is a coalition. The whole set A of all players is also a coalition which is called the* **grand coalition**.

3. *Let $S \subset A$ be a coalition. The* **counter coalition** *of S is the complement $S^c = A \setminus S \subset A$ of S in A.*

4. *The* **characteristic function** *is the function $\nu : \mathcal{P}(A) \to \mathbb{R}$, where $\mathcal{P}(A)$ is the power set of A, defined as follows. For any coalition $S \subset A$, define $\nu(S)$ as the maximin total payoff to the players in S when the game is considered as a 2-person non-cooperative game between S and S^c. For a coalition with one single player $S = \{A_i\}$, $A_i \in A$, we will use an abuse of notation and write $\nu(A_i)$ for $\nu(\{A_i\})$.*

Example 5.1.2 (3-person constant sum game)**.** *Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.*

Strategy	*Payoff vector*
$(0,0,0)$	$(-2,1,2)$
$(0,0,1)$	$(1,1,-1)$
$(0,1,0)$	$(0,-1,2)$
$(0,1,1)$	$(-1,2,0)$
$(1,0,0)$	$(1,-1,1)$
$(1,0,1)$	$(0,0,1)$
$(1,1,0)$	$(1,0,0)$
$(1,1,1)$	$(1,2,-2)$

For coalition $S = \{A_1, A_2\}$, we compute $\nu(S)$ and $\nu(S^c)$ as follows. First the game bimatrix for the 2-person game between S and S^c is

		Strategy of A_3	
		0	1
Strategy of $\{A_1, A_2\}$	$(0,0)$	$(-1,2)$	$(2,-1)$
	$(0,1)$	$(-1,2)$	$(1,0)$
	$(1,0)$	$(0,1)$	$(0,1)$
	$(1,1)$	$(1,0)$	$(3,-2)$

The game has a saddle point with payoff pair $(1,0)$. Thus $\nu(\{A_1, A_2\}) = 1$ and $\nu(\{A_3\}) = 0$. For $S = \{A_1, A_3\}$, the game bimatrix is

		Strategy of A_2	
		0	1
Strategy of $\{A_1, A_3\}$	$(0,0)$	$(0,1)$	$(2,-1)$
	$(0,1)$	$(0,1)$	$(-1,2)$
	$(1,0)$	$(2,-1)$	$(1,0)$
	$(1,1)$	$(1,0)$	$(-1,2)$

Now the payoff matrix for the coalition $S = \{A_1, A_3\}$ is

$$\begin{pmatrix} 0 & 2 \\ 0 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Observe that the sum of the payoffs to S and S^c is always equal to 1. The non-cooperative game between S and S^c can be considered as a zero sum game. The value of $\nu(\{A_2, A_3\})$ is equal to the value of the above game matrix which is equal to $\frac{4}{3}$. Moreover, $\nu(\{A_1\}) = -\frac{1}{3}$ since the sum of the payoffs to S and S^c is always equal to 1. The values of $\nu(S)$ for varies coalitions S are given in the following table.

S	$\nu(S)$
$\emptyset$	0
$\{A_1\}$	$\frac{1}{4}$
$\{A_2\}$	$-\frac{1}{3}$
$\{A_3\}$	0
$\{A_1, A_2\}$	1
$\{A_2, A_3\}$	$\frac{3}{4}$
$\{A_1, A_3\}$	$\frac{4}{3}$
$\{A_1, A_2, A_3\}$	1

□

Suppose S and T are two disjoint coalitions. The two coalitions can combine and form a larger coalition $S \cup T$ which is called the **union coalition**. We always have $\nu(S \cup T) \geq \nu(S) + \nu(T)$. This property is called superadditivity.

Theorem 5.1.3 (Superadditivity)**.** *Let ν be the characteristic function of a game in strategic form. Then ν is* **superadditive**. *That is to say, if $S, T \subset A$ are two coalitions with $S \cap T = \emptyset$, then*

$$\nu(S \cup T) \geq \nu(S) + \nu(T)$$

In particular

$$\nu(A) \geq \sum_{i=1}^{n} \nu(A_i)$$

Proof. Let S and T be two coalitions with $S \cap T = \emptyset$. Let $\mathbf{p}$ and $\mathbf{q}$ be the maximin strategies for the coalitions S and T respectively. By combining $\mathbf{p}$ and $\mathbf{q}$ which is a strategy of $S \cup T$, the coalition $S \cup T$ may guarantee a payoff of at least $\nu(S) + \nu(T)$. Therefore we have $\nu(S \cup T) \geq \nu(S) + \nu(T)$. The second statement is a direct consequence of the first. □

Definition 5.1.4 (Characteristic form)**.** *The* **characteristic form** *of a game is an ordered pair (A, ν) where A is the set of player and $\nu : \mathcal{P}(A) \to \mathbb{R}$, where $\mathcal{P}(A)$ is the power set of A, is a function, such that*

1. $\nu(\emptyset) = 0$
2. *(Superadditivity) If $S, T \subset A$ are subset of A with $S \cap T = \emptyset$, then*

$$\nu(S \cup T) \geq \nu(S) + \nu(T)$$

The function ν is called the **characteristic function** *of the game.*

The players have a tendency to cooperate only when the game is essential.

Definition 5.1.5. *We say that a game (A, ν) in characteristic form is* **essential** *if*

$$\nu(A) > \sum_{i=1}^{n} \nu(A_i)$$

Otherwise, it is said to be **inessential**.

If a game is essential, then the total payoff to all players when they cooperate is larger than the sum of the payoffs to the players when they play the game individually. This gives an incentive for the players to cooperate. If a game is inessential, then no player can gain more by cooperation.

Theorem 5.1.6. *If (A, ν) is inessential, then for any coalition $S \subset A$, we have*

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

Proof. For any coalition $S \subset A$, by superadditivity, we have

$$\nu(S) \geq \sum_{A_i \in S} \nu(A_i) \text{ and } \nu(S^c) \geq \sum_{A_j \in S^c} \nu(A_j)$$

Now if (A, ν) is inessential, then

$$\nu(A) \leq \sum_{i=1}^{n} \nu(A_i)$$

which implies, by superadditivity again,

$$\nu(A) = \sum_{i=1}^{n} \nu(A_i)$$

Hence

$$\begin{aligned}\nu(A) &= \sum_{i=1}^{n} \nu(A_i) \\ &= \sum_{A_i \in S} \nu(A_i) + \sum_{A_j \in S^c} \nu(A_j) \\ &\leq \nu(S) + \nu(S^c) \\ &\leq \nu(A)\end{aligned}$$

Thus all inequalities above become equality and therefore

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

□

In a cooperative game with transferable utility, the players may benefit by forming the grand coalition A. The total amount received by the players is $\nu(A)$. The problem is to agree on how this amount should be split among the players. The first criterion is that each player should receive no less than the amount before cooperation. We call a splitting of total payoffs to the players an imputation if it satisfies this criterion.

Definition 5.1.7 (Imputation)**.** *Let $\nu : \mathcal{P}(A) \to \mathbb{R}$ be a characteristic function. A vector $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is called an* **imputation** *for ν if*

1. *(Individual rationality) For any $i = 1, 2, \cdots, n$, we have $x_i \geq \nu(A_i)$.*
2. *(Efficiency[4]) $\sum_{i=1}^{n} x_i = \nu(A)$*

The set of imputations for ν is denoted by $I(\nu)$.

In an inessential game, no player may receive more by cooperation and there is only one imputation for the game. For essential games, there are infinitely many ways to split the payoffs which satisfy individual rationality.

Theorem 5.1.8. *Let ν be a characteristic function and $I(\nu)$ be the set of imputations.*

[4] It is also called collective rationality.

1. *If ν is inessential, then $I(\nu) = \{(\nu(A_1), \nu(A_2), \cdots, \nu(A_n))\}$.*

2. *If ν is essential, then $I(\nu)$ is an infinite set.*

Proof. 1. If ν is inessential, then for any imputation $(x_1, x_2, \cdots, x_n) \in I(\nu)$, we have

$$\nu(A) = \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} \nu(A_i) = \nu(A)$$

Thus $x_i = \nu(A_i)$ for $i = 1, 2, \cdots, n$ and $I(\nu) = \{(\nu(A_1), \nu(A_2), \cdots, \nu(A_n))\}$.

2. Suppose ν is essential. Let

$$\beta = \nu(A) - \sum_{i=1}^{n} \nu(A_i) > 0$$

Then there are infinitely many solutions to the equation $\sum_{i=1}^{n} \alpha_i = \beta$ for variables $\alpha_1, \alpha_2, \cdots, \alpha_n > 0$ and each of the solutions gives an imputation by putting $x_i = \nu(A_i) + \alpha_i$ for $i = 1, 2, \cdots, n$.

□

5.2 Core

The core of a cooperative game is the set of imputations that are not dominated by other imputations through any coalition.

Definition 5.2.1. *Let $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in I(\nu)$ be two imputations. We say that $\mathbf{x}$ is dominated by $\mathbf{y}$ through a coalition $S \subset A$ and write $\mathbf{x} \prec_S \mathbf{y}$ if*

1. *If $A_i \in S$, then $x_i < y_i$.*

2. $\sum_{A_i \in S} y_i \leq \nu(S)$

We write $\mathbf{x} \not\prec_S \mathbf{y}$ if $\mathbf{x}$ is not dominated by $\mathbf{y}$ through S.

Example 5.2.2. *Consider the 3-person constant sum game (Example 5.1.2) with characteristic function*

S	$\nu(S)$
$\emptyset$	0
$\{A_1\}$	$\frac{1}{4}$
$\{A_2\}$	$-\frac{1}{3}$
$\{A_3\}$	0
$\{A_1, A_2\}$	1
$\{A_2, A_3\}$	$\frac{3}{4}$
$\{A_1, A_3\}$	$\frac{4}{3}$
$\{A_1, A_2, A_3\}$	1

We have

$$\begin{aligned} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) &\prec_{\{A_1,A_2\}} \left(\frac{1}{2}, \frac{1}{2}, 0\right) \\ (1, 0, 0) &\prec_{\{A_2,A_3\}} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \\ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) &\prec_{\{A_2,A_3\}} \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right) \end{aligned}$$

□

For imputation $\mathbf{x} \in I(\nu)$, if there exists imputation $\mathbf{y} \in I(\nu)$ and coalition $S \subset A$ such that $\mathbf{x} \not\succ_S \mathbf{y}$, then there will be a tendency for coalition S to form and upset the proposal $\mathbf{x}$ because such a coalition could guarantee each of its members more than they could receive from $\mathbf{x}$. Thus it reasonable to require the splitting of payoff to the players to be an imputation which is not dominated by any other imputation through any coalition.

Definition 5.2.3 (Core)**.** *The* **core** $C(\nu)$ *of a characteristic function is the set of all imputations that are not dominated by any other imputation through any coalition, that is*

$$C(\nu) = \{\mathbf{x} \in I(\nu) : \mathbf{x} \not\succ_S \mathbf{y} \text{ for any } \mathbf{y} \in I(\nu) \text{ and } S \subset A\}$$

There is an easy way to check whether an imputation lies in the core.

Theorem 5.2.4. *Let* $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ *be an imputation. Then* $\mathbf{x} \in C(\nu)$ *if and only if*

$$\sum_{A_i \in S} x_i \geq \nu(S)$$

for any coalition $S \subset A$*.*

Proof. Suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ does not lie in the core $C(\nu)$. Then there exists imputation $\mathbf{y} \in I(\nu)$ and coalition $S \subset A$ such that $x_i < y_i$ for any $A_i \in S$ and $\sum_{A_i \in S} y_i \leq \nu(S)$. Thus we have

$$\sum_{A_i \in S} x_i < \sum_{A_i \in S} y_i \leq \nu(S)$$

On the other hand, suppose $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in I(\nu)$ is an imputation such that

$$\sum_{A_i \in S} x_i < \nu(S)$$

for some coalition S. Then $S \neq A$ and since

$$\sum_{A_i \in S} x_i + \sum_{A_j \in S^c} x_j = \sum_{i=1}^{n} x_i = \nu(A) \geq \nu(S) + \nu(S^c) > \sum_{A_i \in S} x_i + \sum_{A_j \in S^c} \nu(A_j)$$

by superadditivity, there exists $A_k \in S^c$ such that $x_k > \nu(A_k)$. Define

$$y_i = \begin{cases} x_i + \frac{\alpha}{|S|} & \text{for } A_i \in S \\ x_k - \alpha & \text{for } i = k \\ x_i & \text{for } A_i \in S^c \text{ and } i \neq k \end{cases}$$

where

$$\alpha = \min \left\{ x_k - \nu(A_k), \nu(S) - \sum_{A_i \in S} x_i \right\} > 0$$

By taking $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, we have $\mathbf{x} \prec_S \mathbf{y}$. Therefore $\mathbf{x}$ does not lie in the core $C(\nu)$ and the proof of the theorem is complete. □

Theorem 5.2.5. *The core $C(\nu)$ is a convex set if it is not empty.*

Proof. Let $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in C(\nu)$ be two imputations in the core. Then for any coalition S, we have

$$\sum_{A_i \in S} x_i, \sum_{A_i \in S} y_i \geq \nu(S)$$

by Theorem 5.2.4. Now for any real number $0 \leq \lambda \leq 1$, we have

$$\sum_{A_i \in S} (\lambda x_i + (1 - \lambda) y_i) \geq \nu(S)$$

which implies $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in C(\nu)$. Therefore $C(\nu)$ is convex. □

Example 5.2.6 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.2.2). Find the core $C(\nu)$ of ν.*

Solution. For any imputation $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$, we have $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1 \geq \dfrac{1}{4}, x_2 \geq -\dfrac{1}{3}, x_3 \geq 0 \\ x_1 + x_2 \geq 1, x_2 + x_3 \geq \dfrac{3}{4}, x_1 + x_3 \geq \dfrac{4}{3} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

First of all, we have

$$x_3 = (x_1 + x_2 + x_3) - (x_1 + x_2) \leq 1 - 1 = 0$$

which implies $x_3 = 0$. Then

$$x_1 + x_2 = (x_1 + x_3) + (x_2 + x_3) \geq \frac{4}{3} + \frac{3}{4} > 1$$

which leads to a contradiction. Therefore $C(\nu) = \emptyset$. □

Example 5.2.7. *Suppose $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$ and*

S	$\nu(S)$
$\{A_1, A_2\}$	$\frac{1}{3}$
$\{A_1, A_3\}$	$\frac{1}{2}$
$\{A_2, A_3\}$	$\frac{1}{4}$
$\{A_1, A_2, A_3\}$	1

Find the core $C(\nu)$ of ν.

Solution. Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation. Then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq \dfrac{1}{3}, x_1 + x_3 \geq \dfrac{1}{2}, x_2 + x_3 \geq \dfrac{1}{4} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

Now

$$\begin{array}{c} 0 \leq x_1 = 1 - x_2 - x_3 \leq 1 - \frac{1}{4} = \frac{3}{4} \\ 0 \leq x_2 = 1 - x_1 - x_3 \leq 1 - \frac{1}{2} = \frac{1}{2} \\ 0 \leq x_3 = 1 - x_1 - x_2 \leq 1 - \frac{1}{3} = \frac{2}{3} \end{array}$$

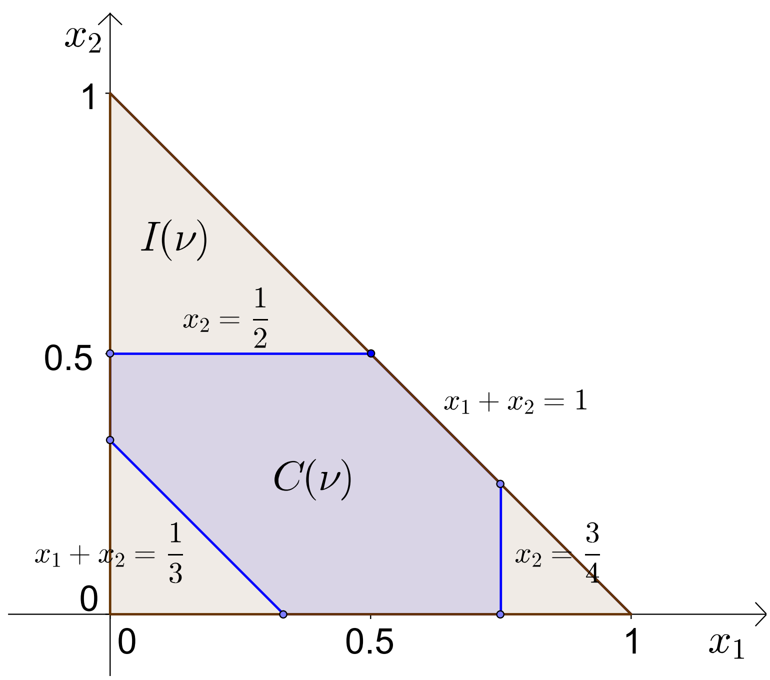
The above system of inequalities is equivalent to

$$\begin{cases} 0 \le x_1 \le \frac{3}{4} \\ 0 \le x_2 \le \frac{1}{2} \\ 0 \le x_3 \le \frac{2}{3} \\ x_1 + x_2 + x_3 = \nu(A) = 1 \end{cases}$$

We may consider x_1 and x_2 as independent variables and $x_3 = 1 - x_1 - x_2$ depends on x_1, x_2. Then x_1 and x_2 satisfy the constraints

$$\begin{cases} 0 \le x_1 \le \frac{3}{4} \\ 0 \le x_2 \le \frac{1}{2} \\ \dfrac{1}{3} \le x_1 + x_2 \le 1 \end{cases}$$

We may represent the core on the $x_1 - x_2$ plane



□

Example 5.2.8 (Used car game)**.** *A man named Andy has an old car he wishes to sell. He no longer drives it, and it is worth nothing to him unless he can sell it. Two people are interested in buying it, Ben and Carl. Bill values the car at \$500 and Carl thinks it is worth \$700. The game consists of each of the prospective buyers bidding on the car, and Andy either accepting one of the bids (presumably the higher one), or rejecting both of them. Find the core of the game and represent it on the* $x_1 - x_2$ *plane.*

Solution. If there is no deal, no player gets anything and we have $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$. The characteristic values of other coalitions are listed below.

S	$\nu(S)$
$\{A_1, A_2\}$	500
$\{A_1, A_3\}$	700
$\{A_2, A_3\}$	0
$\{A_1, A_2, A_3\}$	700

Let $\mathbf{x} = (x_1, x_2, x_3) \in I(\nu)$ be an imputation. Then $\mathbf{x} \in C(\nu)$ if and only if

$$\begin{cases} x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 \geq 500, x_1 + x_3 \geq 700, x_2 + x_3 \geq 0 \\ x_1 + x_2 + x_3 = 700 \end{cases}$$

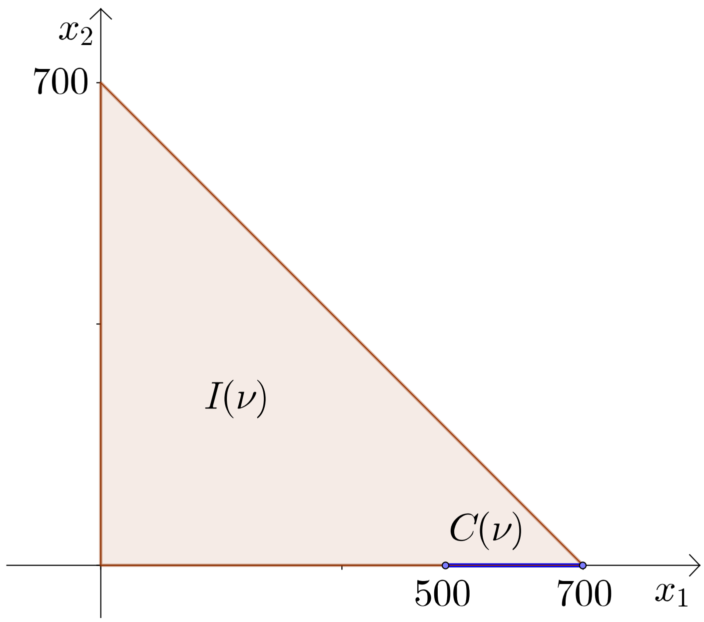
Observe that

$$\begin{array}{l} 0 \leq x_2 = (x_1 + x_2 + x_3) - (x_1 + x_3) \leq 700 - 700 = 0 \\ 0 \leq x_3 = (x_1 + x_2 + x_3) - (x_1 + x_2) \leq 700 - 500 = 200 \\ x_1 = x_1 + x_2 \geq 500 \\ x_1 \leq x_1 + x_2 + x_3 \leq 700 \end{array}$$

The above system of inequalities is equivalent to

$$\begin{cases} 500 \leq x_1 \leq 700 \\ x_2 = 0 \\ x_3 = 700 - x_1 \end{cases}$$

The core of the used car game is shown in the following figure.



□

Example 5.2.9 (Mayor and council)**.** *In a city, there is a Mayor and a city council with 7 members. A bill can be passed to a law if either*

1. *the majority of the council members passes it and the Mayor signs it, or*

2. *the Mayor vetoes it but at least 6 council members vote to override the veto.*

Find the core of the game.

Solution. Let $A = \{M, 1, 2, 3, 4, 5, 6, 7\}$ be the set of players. Then

- $\nu(S) = 1$ if
  1. S contains the mayor and at least 4 council members, or
  2. S contains at least 6 council members.
- $\nu(S) = 0$ otherwise.

Then $\mathbf{x} = (x_M, x_1, \cdots, x_7) \in I(\nu)$ if and only if

$$\begin{cases} x_M, x_1, x_2, \cdots, x_7 \geq 0 \\ x_M + x_1 + x_2 + \cdots + x_7 = 1 \end{cases}$$

Suppose $\mathbf{x} \in C(\nu)$. Then for any $k = 1, 2, \cdots, 7$,

$$\sum_{i \neq k} x_i \geq 1$$

which implies $x_1 + x_2 + \cdots + x_7 \geq 1$ and

$$x_M = x_M + x_1 + x_2 \cdots + x_7 - (x_1 + x_2 + \cdots + x_7) \leq 1 - 1 = 0$$

Moreover for any $k = 1, 2, \cdots, 7$,

$$x_k = (x_1 + x_2 + \cdots + x_7) - \sum_{i \neq k} x_i \leq 1 - 1 = 0$$

which contradicts $x_M + x_1 + x_2 + \cdots + x_7 = 1$. Therefore $C(\nu) = \emptyset$. □

Definition 5.2.10. *A characteristic function* ν *is* **constant sum** *if*

$$\nu(S) + \nu(S^c) = \nu(A)$$

for any coalition $S \subset A$.

Theorem 5.2.11. *If* ν *is both essential and constant sum, then* $C(\nu) = \emptyset$.

Proof. Suppose ν is constant sum and its core $C(\nu)$ is nonempty. It suffices to show that ν is inessential. To this end, let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in C(\nu)$ be an imputation lying in the core. Then for any $k = 1, 2, \cdots, n$, we have

$$x_k \geq \nu(A_k) \text{ and } \sum_{i \neq k} x_i \geq \nu(\{A_k\}^c)$$

Thus by Theorem 5.2.4, we have

$$\nu(A) = \nu(A_k) + \nu(\{A_k\}^c) \leq x_k + \sum_{i \neq k} x_i = \nu(A)$$

It follows that $x_k = \nu(A_k)$ and we have

$$\sum_{k=1}^{n} \nu(A_k) = \nu(A)$$

which means ν is inessential and the proof of the theorem is complete. □

Definition 5.2.12. *Two characteristic functions μ and ν are* **strategically equivalent** *if there exists real numbers $k > 0$ and $c_1, c_2, \cdots, c_n$ such that for any coalition $S \subset A$,*

$$\mu(S) = k\nu(S) + \sum_{A_i \in S} c_i$$

It is obvious that being strategically equivalent is an equivalence relation. Two games share very similar properties when their characteristic functions are strategically equivalent.

Theorem 5.2.13. *Suppose μ and ν are strategically equivalent characteristic functions. Let $k > 0$, $c_1, c_2, \cdots, c_n$ be real numbers such that*

$$\mu(S) = k\nu(S) + \sum_{A_i \in S} c_i$$

Write $\mathbf{c} = (c_1, c_2, \cdots, c_n)$. We have

1. *μ is essential if and only if ν is essential.*
2. *$I(\mu) = \{\mathbf{y} : \mathbf{y} = k\mathbf{x} + \mathbf{c}$ for some $\mathbf{x} \in I(\nu)\}$*
3. *$C(\mu) = \{\mathbf{y} : \mathbf{y} = k\mathbf{x} + \mathbf{c}$ for some $\mathbf{x} \in C(\nu)\}$*

Definition 5.2.14 ($(0, 1)$ reduced form)**.** *We say that a characteristic function μ is a $(0, 1)$* **reduced form** *if*

1. *$\mu(A_i) = 0$ for any $i = 1, 2, \cdots, n$*
2. *$\mu(A) = 1$*

Every inessential game is strategically equivalent to a trivial game. Every essential game is strategically equivalent to a unique $(0, 1)$ reduced form.

Theorem 5.2.15. *Let ν be a characteristic function.*

1. *If ν is inessential, then ν is strategically equivalent to the zero game, that is, a game with characteristic function identically equal to zero.*
2. *If ν is essential, then ν is strategically equivalent to a unique game in $(0, 1)$ reduced form.*

Proof. Let ν be a characteristic function.

1. Suppose ν is inessential. By Theorem 5.1.6, for any coalition $S \subset A$,

$$\nu(S) = \sum_{A_i \in S} \nu(A_i)$$

Taking $k = 1$ and $c_i = -\nu(A_i)$ for $i = 1, 2, \cdots, n$, we have ν is strategically equivalent to the characteristic function

$$\mu(S) = \nu(S) - \sum_{A_i \in S} \nu(A_i)$$

and $\mu(S) = 0$ for any coalition S which means μ is the trivial game.

2. Suppose ν is essential. Taking

$$k = \frac{1}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)} \text{ and } c_i = \frac{-\nu(A_i)}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)} \text{ for } i = 1, 2, \cdots, n$$

ν is strategically equivalent to the characteristic function

$$\mu(S) = \frac{\nu(S) - \sum_{A_i \in S} \nu(A_i)}{\nu(A) - \sum_{j=1}^{n} \nu(A_j)}$$

for $S \subset A$. Now $\mu(A) = 1$ and $\mu(A_i) = 0$ for any $i = 1, 2, \cdots, n$. Therefore ν is strategically equivalent to the $(0, 1)$ reduced form μ. Suppose μ' is another $(0, 1)$ reduced form strategically equivalent to ν. Then μ' is strategically equivalent to μ. Thus there exists constants $k > 0$ and $c_1, c_2, \cdots, c_n$ such that

$$\mu'(S) = k\mu(S) + \sum_{A_i \in S} c_i$$

for any coalition S. Taking $S = \{A_i\}$, $i = 1, 2, \cdots, n$, we have $c_1 = c_2 = \cdots = c_n = 0$ since $\mu'(\{A_i\}) = \mu(\{A_i\}) = 0$. Moreover taking $S = A$, we have $\mu'(A) = k\mu(A)$ which implies $k = 1$ since $\mu'(A) = \mu(A) = 1$. Therefore $\mu' = \mu$ and the $(0, 1)$ reduced form of ν is unique.

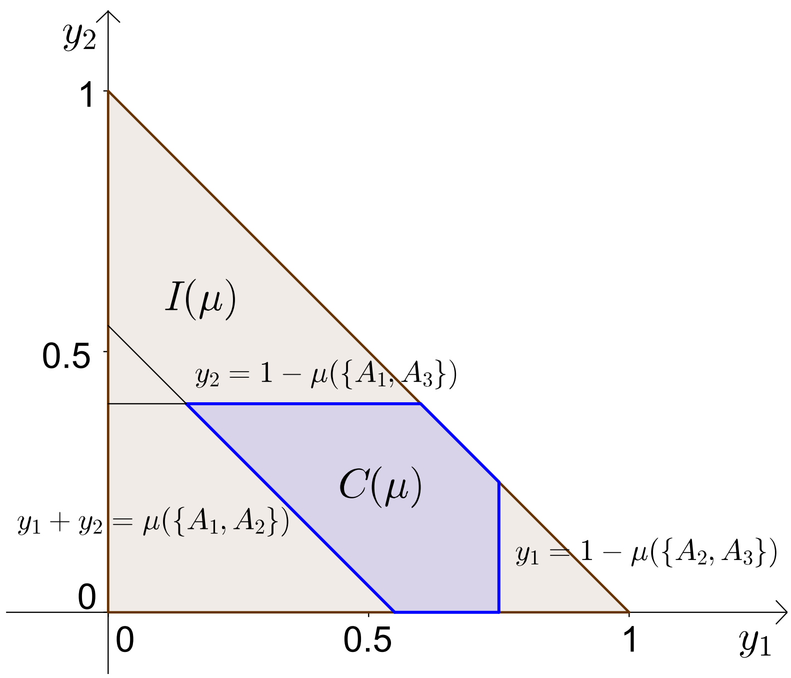
□

Suppose ν is the characteristic function of a 3-person game and μ is the $(0,1)$ reduced form of ν. Then an imputation $(y_1, y_2, y_3) \in I(\mu)$ of μ lies in the core of μ if and only if

$$\begin{cases} 0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) \\ 0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) \\ 0 \le y_3 \le 1 - \mu(\{A_1, A_2\}) \\ y_1 + y_2 + y_3 = 1 \end{cases}$$

and on the $x_1 - x_2$ plane, it can be represented by the region

$$\begin{cases} 0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) \\ 0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) \\ \mu(\{A_1, A_2\}) \le y_1 + y_2 \le 1 \end{cases}$$



Example 5.2.16 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.1.2 and Example 5.2.2). Let μ be the $(0,1)$ reduced form of ν. Find μ and its core $C(\mu)$.*

Solution. First we have

$$\mu(A_1) = \mu(A_2) = \mu(A_3) = 0$$

and

$$\mu(A) = 1$$

Next we calculate

$$k = \frac{1}{\nu(A) - (\nu(A_1) + \nu(A_2) + \nu(A_3))} = \frac{1}{1 - (\frac{1}{4} + (-\frac{1}{3}) + 0)} = \frac{12}{13}$$

and we have

$$\begin{aligned}
\mu(\{A_1, A_2\}) &= k(\nu(\{A_1, A_2\}) - (\nu(A_1) + \nu(A_2))) \\
&= \frac{12}{13}\left(1 - \left(\frac{1}{4} - \frac{1}{3}\right)\right) \\
&= 1 \\
\mu(\{A_1, A_3\}) &= k(\nu(\{A_1, A_3\}) - (\nu(A_1) + \nu(A_3))) \\
&= \frac{12}{13}\left(\frac{4}{3} - \left(\frac{1}{4} + 0\right)\right) \\
&= 1 \\
\mu(\{A_2, A_3\}) &= k(\nu(\{A_2, A_3\}) - (\nu(A_2) + \nu(A_3))) \\
&= \frac{12}{13}\left(\frac{3}{4} - \left(-\frac{1}{3} + 0\right)\right) \\
&= 1
\end{aligned}$$

Now an imputation $(y_1, y_2, y_3) \in I(\mu)$ lies in the core $C(\mu)$ of μ if and only if

$$\begin{cases} 0 \leq y_1 \leq 1 - \mu(\{A_2, A_3\}) = 0 \\ 0 \leq y_2 \leq 1 - \mu(\{A_1, A_3\}) = 0 \\ 0 \leq y_3 \leq 1 - \mu(\{A_1, A_2\}) = 0 \\ y_1 + y_2 + y_3 = 1 \end{cases}$$

which has no solution. Thus $C(\mu) = \emptyset$. (Note that $C(\nu)$ is also empty.) □

Example 5.2.17 (Used car game)**.** *Let ν be the characteristic function of the used car game (Example 5.2.8). Let μ be the* $(0, 1)$ *reduced form of ν. Find μ and the core $C(\nu)$ of ν.*

Solution. First we have

$$\mu(A_1) = \mu(A_2) = \mu(A_3) = 0 \text{ and } \mu(A) = 1$$

Now

$$k = \frac{1}{\nu(A) - (\nu(A_1) + \nu(A_2) + \nu(A_3))} = \frac{1}{700}$$

and we have

$$\begin{aligned}
\mu(\{A_1, A_2\}) &= k(\nu(\{A_1, A_2\}) - (\nu(A_1) + \nu(A_2))) \\
&= \frac{500 - 0}{700} \\
&= \frac{5}{7} \\
\mu(\{A_1, A_3\}) &= k(\nu(\{A_1, A_3\}) - (\nu(A_1) + \nu(A_3))) \\
&= \frac{700 - 0}{700} \\
&= 1 \\
\mu(\{A_2, A_3\}) &= k(\nu(\{A_2, A_3\}) - (\nu(A_2) + \nu(A_3))) \\
&= \frac{0 - 0}{700} \\
&= 0
\end{aligned}$$

Now an imputation $(y_1, y_2, y_3) \in I(\mu)$ lies in the core $C(\mu)$ of μ if and only if

$$\begin{cases}
0 \le y_1 \le 1 - \mu(\{A_2, A_3\}) = 1 - 0 = 1 \\
0 \le y_2 \le 1 - \mu(\{A_1, A_3\}) = 1 - 1 = 0 \\
0 \le y_3 \le 1 - \mu(\{A_1, A_2\}) = 1 - \frac{5}{7} = \frac{2}{7} \\
y_1 + y_2 + y_3 = 1
\end{cases}$$

which is equivalent to

$$\begin{cases}
\frac{5}{7} \le y_1 \le 1 \\
y_2 = 0 \\
y_3 = 1 - y_1
\end{cases}$$

□

5.3 Shapley value

In the last section, we studied cores of characteristic functions. The core has a disadvantage that it may be empty and usually contains an infinite number of elements when it is nonempty. In this section, we study another solution concept called Shapley value which always exists and is unique.

Definition 5.3.1 (Shapley value)**.** *Let ν be a characteristic function. The* **Shapley value** *of the player A_k, $k = 1, 2, \cdots, n$, is defined as*

$$\phi_k = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} \frac{(n - |S|)!(|S| - 1)!}{n!} (\nu(S) - \nu(S \setminus \{A_k\}))$$

The vector $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ is called the **Shapley vector** *of ν.*

The Shapley value of a player can be interpreted in the following way. Suppose we form the grand coalition A by entering the players one after another. As player A_k enters the coalition, he receives the amount by which his entry inceases the value of the coalition he enters. This amount is equal to $\delta_k(S) = \nu(S) - \nu(S \setminus \{A_k\})$ where S is the coalition after A_k has entered. The amount a player receives depends on the order in which the players are entered. The Shapley value ϕ_k is the average amount that A_k receives over all orders of entering of players in forming the grand coalition.

Let S be a coalition which contains player A_k. There are $(|S|-1)!$ number of ways for other players in S to enter the coalition before A_k. Then player A_k enters the coalition to form the coalition S and there are $(n-|S|)!$ number of ways for the remaining players to enter to form the grand coalition. Thus among all $n!$ permutations of players in forming the grand coalition, there are $(n - |S|)!(|S| - 1)!$ of which the coalition S would form at the moment that player A_k enters into the coalition and A_k would receive $\nu(S) - \nu(S \setminus \{A_k\})$. Therefore the average amount that A_k receives is given by the formula in Definition 5.3.1. This also shows the following alternative formula for the Shapley values.

Theorem 5.3.2. *The Shapley value of the player A_k is given by*

$$\phi_k = \frac{1}{n!} \sum_{\sigma \in S_n} (\nu(S_k^\sigma) - \nu(S_k^\sigma \setminus \{A_k\}))$$

where S_n is the set of all permutations of $1, 2, \cdots, n$, and $S_k^\sigma = \{A_{\sigma(1)}, A_{\sigma(2)}, \cdots, A_{\sigma(i)}\}$ where i is determined by $\sigma(i) = k$. In other words, S_k^σ is the set of players in A which precede A_k in permutation σ, including A_k.

Remarks:

1. The quantity $\delta_k(S) = \nu(S) - \nu(S \setminus \{A_k\})$ is the amount the player A_k contributes to the coalition S. In particular $\delta_k(S) = 0$ if $A_k \notin S$. Therefore to find ϕ_k, we only need to sum over S with $A_k \in S$.

2. The formula for ϕ_k can also be written as

$$\phi_k = \sum_{S \in \mathcal{P}(A) \setminus \{A\}} \frac{(n - |S| - 1)!|S|!}{n!} (\nu(S \cup \{A_k\}) - \nu(S))$$

3. Suppose $n = 3$ and $\nu(A_k) = 0$ for $k = 1, 2, 3$. To find the Shapley value ϕ_1 of A_1, we need to calculate, for each permutation of players, the value of $\delta_1(S)$ where S is the coalition right after the joining of A_1. The values of $\delta_1(S)$ for the permutations of players are shown in the following table.

Permutation	S	$S \setminus \{A_1\}$	$\delta_1(S)$
123	$\{A_1\}$	$\emptyset$	0
132	$\{A_1\}$	$\emptyset$	0
213	$\{A_1, A_2\}$	$\{A_2\}$	$\nu(\{A_1, A_2\})$
231	$\{A_1, A_2, A_3\}$	$\{A_2, A_3\}$	$\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})$
312	$\{A_1, A_3\}$	$\{A_3\}$	$\nu(\{A_1, A_3\})$
321	$\{A_1, A_2, A_3\}$	$\{A_2, A_3\}$	$\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})$

The Shapley value ϕ_1 of A_1 is the average value in the last column. Thus we have

$$\phi_1 = \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_1, A_3\}) - 2\nu(\{A_2, A_3\})}{6}$$

We have similar formula for ϕ_2 and ϕ_3.

Now we prove that the Shapley vector is always an imputation.

Theorem 5.3.3. *Let ν be a characteristic function and $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$ be the Shapley vector of ν. Then $\phi \in I(\nu)$. In other words, we always have*

1. $\phi_i \geq \nu(A_i)$ *for any* $i = 1, 2, \cdots, n$

2. $\sum_{i=1}^{n} \phi_i = \nu(A)$

Proof. 1. For any A_i and any coalition $S \subset A$ with $A_i \in S$, we have

$$\nu(A_i) + \nu(S \setminus \{A_i\}) \leq \nu(S)$$

by supperadditivity. Therefore

$$\begin{aligned} \phi_i &= \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\ &\geq \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\nu(A_i) \\ &= \nu(A_i) \end{aligned}$$

2.

$$\begin{aligned} \sum_{i=1}^{n} \phi_i &= \sum_{i=1}^{n} \sum_{\substack{S \subset A \\ A_i \in S}} \frac{(n-|S|)!(|S|-1)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\ &= \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}(\nu(S) - \nu(S \setminus \{A_i\})) \\ &= \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\ &\quad - \sum_{S \subset A} \sum_{A_i \in S} \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S \setminus \{A_i\}) \\ &= \sum_{S \subset A} |S| \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\ &\quad - \sum_{T \subset A} \sum_{A_j \notin T} \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\ &= \sum_{S \subset A} |S| \frac{(n-|S|)!(|S|-1)!}{n!}\nu(S) \\ &\quad - \sum_{T \subset A} (n-|T|) \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\ &= \sum_{S \subset A} \frac{(n-|S|)!|S|!}{n!}\nu(S) \\ &\quad - \sum_{\substack{T \subset A \\ T \neq A}} \frac{(n-|T|)!|T|!}{n!}\nu(T) \\ &= \nu(A) \end{aligned}$$

□

Example 5.3.4 (3-person constant sum game)**.** *Let ν be the characteristic function of the 3-person constant sum game (Example 5.2.2). Find the Shapley value of each player.*

Solution. To find the Shapley value ϕ_1 of A_1, observe that the coalitions containing A_1 are $\{A_1\}$, $\{A_1, A_2\}$, $\{A_1, A_3\}$ and $\{A_1, A_2, A_3\}$. Thus

$$
\begin{aligned}
& \phi_1 \\
= & \frac{(3-1)!(1-1)!}{3!}(\nu(A_1) - \nu(\emptyset)) + \frac{(3-2)!(2-1)!}{3!}(\nu(\{A_1, A_2\}) - \nu(A_2)) \\
& + \frac{(3-2)!(2-1)!}{3!}(\nu(\{A_1, A_3\}) - \nu(A_3)) \\
& + \frac{(3-3)!(3-1)!}{3!}(\nu(\{A_1, A_2, A_3\}) - \nu(\{A_2, A_3\})) \\
= & \frac{2}{6}\left(\frac{1}{4}\right) + \frac{1}{6}\left(1 - \left(-\frac{1}{3}\right)\right) + \frac{1}{6}\left(\frac{4}{3} - 0\right) + \frac{2}{6}\left(1 - \frac{3}{4}\right) \\
= & \frac{11}{18}
\end{aligned}
$$

Similarly, we have

$$\phi_2 = \frac{1}{36} \text{ and } \phi_3 = \frac{13}{36}$$

□

Example 5.3.5 (Used car game)**.** *Let ν be the characteristic function of the used car game (Example 5.2.8). Find the Shapley values of the players.*

Solution. Since $\nu(A_1) = \nu(A_2) = \nu(A_3) = 0$, we may use the formula

$$
\begin{aligned}
\phi_1 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_1, A_3\}) - 2\nu(\{A_2, A_3\})}{6} \\
&= \frac{2(700) + 500 + 700 - 2(0)}{6} \\
&= \frac{1300}{3}
\end{aligned}
$$

$$
\begin{aligned}
\phi_2 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_2\}) + \nu(\{A_2, A_3\}) - 2\nu(\{A_1, A_3\})}{6} \\
&= \frac{2(700) + 500 + 0 - 2(700)}{6} \\
&= \frac{250}{3}
\end{aligned}
$$

$$\begin{aligned}
\phi_3 &= \frac{2\nu(\{A_1, A_2, A_3\}) + \nu(\{A_1, A_3\}) + \nu(\{A_2, A_3\}) - 2\nu(\{A_1, A_2\})}{6} \\
&= \frac{2(700) + 700 + 0 - 2(500)}{6} \\
&= \frac{550}{3}
\end{aligned}$$

Hence the Shapley vector is $\phi = (\frac{1300}{3}, \frac{250}{3}, \frac{550}{3})$. □

Example 5.3.6 (Mayor and council)**.** *Let ν be the characteristic function of the Mayor and council game (Example 5.2.9). Find the Shapley values of the players.*

Solution. Recall that

1. $\nu(S) = 1$ if $M \in S$ and $|S \setminus \{M\}| \geq 4$, or $|S| \geq 6$.

2. $\nu(S) = 0$ otherwise.

Thus we have

$$\phi_M = \binom{7}{4}\frac{(8-5)!(5-1)!}{8!}(1) + \binom{7}{5}\frac{(8-6)!(6-1)!}{8!}(1) = \frac{1}{4}$$

By symmetry, for each $i = 1, 2, \cdots, 7$, we have

$$\phi_i = \frac{1}{7}\left(1 - \frac{1}{4}\right) = \frac{3}{28}$$

□

Example 5.3.7 (Voting game)**.** *In a council there are 100 members. The red, blue, green, white parties has 40, 30, 25, 5 members in the council. For a resolution to pass, it is necessary to have more than 50 affirmative votes. The set of players is $A = \{R, B, G, W\}$. For any coalition $S \subset A$, define*

1. $\nu(S) = 1$ if the total votes of S is larger than 50.

2. $\nu(S) = 0$ otherwise.

Find the Shapley values of the players.

Solution. We have $\nu(R) = \nu(B) = \nu(G) = \nu(W) = 0$ and

$$\begin{cases} \nu(\{R, B\} = \nu(\{R, G\} = \nu(\{B, G\} = 1 \\ \nu(\{R, W\} = \nu(\{B, W\} = \nu(\{G, W\} = 0 \\ \nu(S) = 1 \text{ for any } S \text{ with } |S| \geq 3 \end{cases}$$

Thus

$$\begin{aligned} \phi_R &= 2\left(\frac{(4-1)!(2-1)!}{4!}\right)(1-0) + 2\left(\frac{(4-3)!(3-1)!}{4!}\right)(1-0) \\ &= \frac{1}{3} \end{aligned}$$

Similarly, we have

$$\phi_B = \phi_G = \frac{1}{3}$$
$$\phi_W = 1 - (\phi_R + \phi_B + \phi_G) = 0$$

□

The Shapley value can be defined using the axiomatic approach as follows. The Shapley vector is the unique allocation of payoffs which satisfies the 4 properties listed in the following theorem. The efficiency property requires that ϕ allocates the total worth of the grand coalition $\nu(A)$. The symmetry property asks ϕ to allocate same payoff to players with identical contributions to coalitions. The null player properties says that players who contribute nothing to every coalition should receive nothing. The linearity properties looks very natural mathematically but there is no good reason to impose such condition in the sense of fairness.

Theorem 5.3.8 (Axioms for Shapley values)**.** *The Shapley vector* $\phi(\nu) = (\phi_1, \cdots, \phi_n)$ *is the unique payoff allocation which satisfies the following* **axioms for Shapley values**.

1. *(Efficiency)* $\sum_{i=1}^{n} \phi_i = \nu(A)$
2. *(Symmetry) If* $A_i, A_j \in A$ *satisfy* $\nu(S \cup \{A_i\}) = \nu(S \cup \{A_j\})$ *for any coalition* S *not containing* A_i *and* A_j*, then* $\phi_i = \phi_j$.
3. *(Null player) If* $\nu(S \cup \{A_i\}) = \nu(S)$ *for any coalition* S*, then* $\phi_i = 0$.

4. *(Linearity) Let μ and ν be two characteristic functions and a, b be two real numbers. Then*

$$\phi(a\mu + b\nu) = a\phi(\mu) + b\phi(\nu)$$

Proof. First we prove that $\phi(\nu)$ satisfies the 4 axioms for Shapley values.

1. It has been proved in Theorem 5.3.3.

2. Suppose $\nu(S \cup \{A_i\}) = \nu(S \cup \{A_j\})$ for any coalition S not containing A_i and A_j. For any coalition $S \subset A$, denote by S' the coalition obtained by replacing A_i by A_j if $A_i \in S$ and replacing A_j by A_i if $A_j \in S$. Note that $|S'| = |S|$. We are going to prove that for any coalition S, we have

$$\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$$

First if $A_j \in S$, then $S \cup \{A_i\} = S' \cup \{A_j\}$ and thus $\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$. On the other hand, if $A_j \notin S$, then $S \setminus \{A_i\} = S' \setminus \{A_j\}$ and we also have

$$\begin{aligned}
\nu(S \cup \{A_i\}) &= \nu((S \setminus \{A_i\}) \cup \{A_i\}) \\
&= \nu((S' \setminus \{A_j\}) \cup \{A_i\}) \\
&= \nu((S' \setminus \{A_j\}) \cup \{A_j\}) \\
&= \nu(S' \cup \{A_j\})
\end{aligned}$$

Thus we proved that $\nu(S \cup \{A_i\}) = \nu(S' \cup \{A_j\})$ for any coalition S.

Therefore

$$\begin{aligned}
&\phi_i \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(\nu(S\cup\{A_i\})-\nu(S)) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S\cup\{A_i\}) - \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S'|-1)!|S'|!}{n!}\nu(S'\cup\{A_j\}) - \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}\nu(S) \\
= &\sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}\nu(T\cup\{A_j\}) - \sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}\nu(T) \\
= &\sum_{T\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|T|-1)!|T|!}{n!}(\nu(T\cup\{A_j\})-\nu(T)) \\
= &\phi_j
\end{aligned}$$

3. Suppose $\nu(S\cup\{A_i\})=\nu(S)$ for any coalition S. Then

$$\begin{aligned}
\phi_i &= \sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(\nu(S\cup\{A_i\})-\nu(S)) \\
&= 0
\end{aligned}$$

4. Let μ and ν be two characteristic functions and a, b be two real numbers. Then

$$\begin{aligned}
&\phi_i(a\mu+b\nu) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}((a\mu+b\nu)(S\cup\{A_i\})-(a\mu+b\nu)(S)) \\
= &\sum_{S\in\mathcal{P}(A)\backslash\{A\}} \frac{(n-|S|-1)!|S|!}{n!}(a(\mu(S\cup\{A_i\})-\mu(S))+b(\nu(S\cup\{A_i\})-\nu(S))) \\
= &a\phi_i(\mu)+b\phi_i(\nu)
\end{aligned}$$

Next we prove the uniqueness. Suppose ϕ satisfies the four axioms for Shapley values. For each non-empty coalition $S\subset A$, $S\neq\emptyset$, define a characteristic function ν_S by

$$\nu_S(T) = \begin{cases} 1 & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases}$$

Observe that if $A_i \notin S$, then $\nu_S(T \cup \{A_i\}) = \nu_S(T)$ for any $T \subset A$. Thus A_i is a null player of ν_S and we have

$$\phi_i(\nu_S) = 0 \text{ if } A_i \notin S$$

by the axiom for null player. By symmetry, we have $\phi_i(\nu_S) = \phi_j(\nu_S)$ whenever $A_i, A_j \in S$ which implies, by efficiency, that

$$\phi_i(\nu_S) = \frac{1}{|S|} \text{ if } A_i \in S$$

In conclusion we have

$$\phi_i(\nu_S) = \begin{cases} \frac{1}{|S|} & \text{if } A_i \in S \\ 0 & \text{if } A_i \notin S \end{cases}$$

To prove uniqueness, it suffices to prove that any characteristic function ν can be written uniquely as

$$\nu = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} c_S \nu_S$$

for some constants c_S, $S \in \mathcal{P}(A) \setminus \{\emptyset\}$. Then

$$\phi(\nu) = \sum_{S \in \mathcal{P}(A) \setminus \{\emptyset\}} c_S \phi(\nu_S)$$

is uniquely determined. We are going to determined c_S by induction on $|S|$. Suppose $|S| = 1$, that is $S = \{A_i\}$ for some $i = 1, 2, \cdots, n$. Now for any coalition $T \subset A$, if $T = \{A_i\}$, then $T \subset S$ and $\nu_T(S) = \nu_T(A_i) = 1$. On the other hand, if $T \neq \{A_i\}$, then $\nu_T(S) = 0$. Thus for $S = \{A_i\}$, we have

$$\nu_T(S) = \nu_T(A_i) = \begin{cases} 1 & \text{if } T = \{A_i\} \\ 0 & \text{if } T \neq \{A_i\} \end{cases}$$

Hence we must have

$$\nu(A_i) = \sum_{T \in \mathcal{P}(A)\setminus\{\emptyset\}} c_T \nu_T(A_i) = c_{\{A_i\}}$$

and thus

$$c_{\{A_i\}} = \nu(A_i)$$

for $i = 1, 2, \cdots, n$. Suppose c_S is determined for each $\emptyset \neq S \subset A$ with $0 < |S| < k$. Now fix $S \subset A$ with $|S| = k$. Recall that for any coalition $T \subset A$, we have $\nu_T(S) = 1$ if $T \subset S$ and $\nu_T(S) = 0$ if T is not a subset of S. Thus we have

$$\nu(S) = \sum_{T \in \mathcal{P}(A)\setminus\{\emptyset\}} c_T \nu_T(S) = \sum_{\emptyset \neq T \subset S} c_T = c_S + \sum_{\emptyset \neq T \subsetneq S} c_T$$

and hence

$$c_S = \nu(S) - \sum_{\emptyset \neq T \subsetneq S} c_T$$

is determined because all c_T had been determined for any $\emptyset \neq T \subsetneq S$. Hence we proved that any characteristic function ν can be written uniquely as

$$\nu = \sum_{S \in \mathcal{P}(A)\setminus\{\emptyset\}} c_S \nu_S$$

and the proof of the theorem is complete. □

In Section 5.2, we introduced the core of a cooperative game. One may ask whether the Shapley vector always lies in the core whenever the core is not empty. The answer is negative. We need an extra condition for it to be true.

Definition 5.3.9 (Convex game)**.** *We say that a characteristic function ν is* **convex** *if for any $S, T \subset A$, we have*

$$\nu(S \cup T) \geq \nu(S) + \nu(T) - \nu(S \cap T)$$

Suppose S and T are two coalitions with $T \subset S$. The contribution of $S \setminus T$ to the coalition S is $\nu(S) - \nu(T)$. In a convex game, this contribution of $S \setminus T$ cannot be larger if the coalition T gets smaller. More precisely, we have

Theorem 5.3.10. *Suppose ν is a convex game. For any coalitions R, S, T with $R \subset T \subset S$, we have*

$$\nu(S) - \nu(T) \geq \nu(S \setminus R) - \nu(T \setminus R)$$

Proof. Consider $S = (S \setminus R) \cup T$. By convexity of ν, we have

$$\begin{aligned} \nu(S) &= \nu((S \setminus R) \cup T) \\ &\geq \nu(S \setminus R) + \nu(T) - \nu((S \setminus R) \cap T) \\ &= \nu(S \setminus R) + \nu(T) - \nu(T \setminus R) \end{aligned}$$

□

Now we can prove

Theorem 5.3.11. *The Shapley vector of a convex game always lies in the core. In particular, the core of a convex game is not empty.*

Proof. Let ν be a convex game with player set $A = \{1, 2, \cdots, n\}$ and ϕ be the Shapley vector of ν. For any permutation $\sigma \in S_n$, define $\phi^\sigma = (\phi_1^\sigma, \phi_2^\sigma, \cdots, \phi_n^\sigma) \in \mathbf{R}^n$ with

$$\phi_k^\sigma = \nu(S_k^\sigma) - \nu(S_k^\sigma \setminus \{k\})$$

where $S_k^\sigma = \{\sigma(1), \sigma(2), \cdots, \sigma(i)\}$ and i is the integer determined by $\sigma(i) = k$. We have seen (Theorem 5.3.2) that

$$\phi = \frac{1}{n!} \sum_{\sigma \in S_n} \phi^\sigma$$

Since the core $C(\nu)$ is convex, it suffices to prove that $\phi^\sigma \in C(\nu)$ for any $\sigma \in S_n$. Without loss of generality, we may assume that σ is the identity, that is, $\sigma(k) = k$ for any $k =, 1, 2, \cdots, n$. In this case, for any coalition $S = \{s_1 < s_2 < \cdots < s_m\} \subset A$ and $s_i \in S$, $i = 1, 2, \cdots, m$, we have $S_{s_i}^\sigma = \{1, 2, \cdots, s_i\}$ and

$$\begin{aligned} \phi_{s_i}^\sigma &= \nu(S_{s_i}^\sigma) - \nu(S_{s_i}^\sigma \setminus \{s_i\}) \\ &= \nu(\{1, 2, \cdots, s_i\}) - \nu(\{1, 2, \cdots, s_i - 1\}) \\ &\geq \nu(\{s_1, s_2, \cdots, s_i\}) - \nu(\{s_1, s_2, \cdots, s_{i-1}\}) \end{aligned}$$

where in the last line, we removed those elements not in S in both sets and the inequality follows from Theorem 5.3.10. Thus

$$\begin{aligned}
\sum_{s_i \in S} \phi^\sigma_{s_i} &= \sum_{s_i \in S} (\nu(S^\sigma_k) - \nu(S^\sigma_k \setminus \{k\})) \\
&\geq \sum_{s_i \in S} (\nu(\{s_1, s_2, \cdots, s_i\}) - \nu(\{s_1, s_2, \cdots, s_{i-1}\})) \\
&= \nu(\{s_1, s_2, \cdots, s_m\}) - \nu(\emptyset) \\
&= \nu(S)
\end{aligned}$$

Hence we have $\phi^\sigma \in C(\nu)$ by Theorem 5.2.4. Therefore $\phi = \frac{1}{n!} \sum_{\sigma \in S_n} \phi^\sigma \in C(\nu)$ since $C(\nu)$ is a convex set. □

As a matter of fact, Shapley proved that if ν is convex, then $C(\nu)$ is a convex polyhedron of dimension $n-1$ with $2^n - 2$ faces and $n!$ vertices located exactly at ϕ^σ's, $\sigma \in S_n$. Therefore the Shapley vector ϕ is precisely the center of mass of the vertices of the core when ν is convex.

Exercise 5

1. Let $A = \{A_1, A_2, A_3\}$ be the player set and $X_i = \{0, 1\}$, for $i = 1, 2, 3$, be the strategy set for A_i. Suppose the payoffs to the players are given by the following table.

Strategy	Payoff vector
$(0,0,0)$	$(-2,3,5)$
$(0,0,1)$	$(1,-2,7)$
$(0,1,0)$	$(1,5,0)$
$(0,1,1)$	$(10,-3,-1)$
$(1,0,0)$	$(-1,0,7)$
$(1,0,1)$	$(-4,4,6)$
$(1,1,0)$	$(12,-4,-2)$
$(1,1,1)$	$(-1,5,2)$

   (a) Find the characteristic function of the game.

   (b) Show that the core of the game is empty.

2. Consider a three-person game with characteristic function

$$\begin{aligned} \nu(\{1\}) &= 27 \\ \nu(\{2\}) &= 8 \\ \nu(\{3\}) &= 18 \\ \nu(\{1,2\}) &= 36 \\ \nu(\{1,3\}) &= 50 \\ \nu(\{2,3\}) &= 27 \\ \nu(\{1,2,3\}) &= 60 \end{aligned}$$

   Find the core of the game and draw the region representing the core on the $x_1 - x_2$ plane.

3. Let ν be the characteristic function defined by $\nu(\{1\}) = 3, \nu(\{2\}) = 4, \nu(\{3\}) = 6, \nu(\{1,2\}) = 9, \nu(\{1,3\}) = 12, \nu(\{2,3\}) = 15, \nu(\{1,2,3\}) = 20$.

   (a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{1,2\}), \mu(\{1,3\}), \mu(\{2,3\})$.

   (b) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

   (c) Find the Shapley values of the players.

4. Three towns A, B, C are considering whether to built a joint water distribution system. The costs of the construction works are listed in the following table

Coalition	Cost(in million dollars)
$\{A\}$	11
$\{B\}$	7
$\{C\}$	8
$\{A,B\}$	15
$\{A,C\}$	14
$\{B,C\}$	13
$\{A,B,C\}$	20

   For any coalition $S \subset \{A, B, C\}$, define $\nu(S)$ to be the amount saved if they build the system together. Find the Shapley values of A, B, C and the amount that each of them should pay if they cooperate.

5. Players 1, 2, 3 and 4 have 45, 25, 15, and 15 votes respectively. In order to pass a certain resolution, 51 votes are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain resolution. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

6. Players 1, 2, 3 and 4 have 40, 30, 20, and 10 shares if stocks respectively. In order to pass a certain decision, 50 shares are required. For any coalition S, define $\nu(S) = 1$ if S can pass a certain decision. Otherwise $\nu(S) = 0$. Find the Shapley values of the players.

7. Consider the following market game. Each of the 5 players starts with one glove. Two of them have a right-handed glove and three of them have a left-handed glove. At the end of the game, an assembled pair is worth \$1 to whoever holds it. Find the Shapley value of the players.

8. Let $\mathcal{A} = \{1, 2, 3\}$ be the set of players and ν be a game in characteristic form with

$$\begin{aligned} \nu(\{1\}) &= -a \\ \nu(\{2\}) &= -b \\ \nu(\{3\}) &= -c \\ \nu(\{2,3\}) &= a \\ \nu(\{1,3\}) &= b \\ \nu(\{1,2\}) &= c \\ \nu(\{1,2,3\}) &= 1 \end{aligned}$$

   where $0 \leq a, b, c \leq 1$.

   (a) Let μ be the $(0,1)$ reduced form of ν. Find $\mu(\{1,2\})$, $\mu(\{1,3\})$, $\mu(\{2,3\})$ in terms of a, b, c.

   (b) Suppose $a + b + c = 2$. Find an imputation $\mathbf{x}$ of ν which lies in the core $C(\nu)$ in terms of a, b, c and prove that $C(\nu) = \{\mathbf{x}\}$.

9. Aaron (A), Benny (B) and Carol (C) each has to buy a book on Game Theory. The list price of the book is \$200. Alan has a discount card which allow him to buy two books for \$360, and three books for \$480. Benny has a coupon which allows him to have 20% off for the whole bill. The discount card and coupon can be used at the same time. Let

$\nu(S)$ be the amount that a coalition $S \subset \{A, B, C\}$ may save by buying the books together comparing with buying them separately.

(a) Find $\nu(\{A, B\})$, $\nu(\{B, C\})$, $\nu(\{A, C\})$ and $\nu(\{A, B, C\})$

(b) Find $\mu(\{A, B\})$ where μ is the $(0, 1)$ reduced form of ν.

(c) Find the core of ν and draw the region representing the core on the $x_1 - x_2$ plane.

10. Let $a > 0$ be a positive real number. Let $f : [0, a] \to \mathbb{R}$ be a differentiable function such that $f(u) \geq 0$ for any $u \in [0, a]$ and $f(a) = 0$. It is given that the set $\mathcal{R} = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq a, 0 \leq v \leq f(u)\}$ is convex. Suppose $(\mu, \nu) \in \mathcal{R}$ and $(\alpha, \beta) = A(\mathcal{R}, (\mu, \nu))$, where A is the arbitration function.

    (a) Show that $f'(\alpha) = -\dfrac{\beta - \nu}{\alpha - \mu}$.

    (b) Let $\mathcal{R} = \{(u, v) \in \mathbb{R}^2 : 0 \leq v \leq 14 + 5u - u^2\}$. Find $(\alpha, \beta) = A(\mathcal{R}, (0, 6))$.

11. Let $\mathcal{A} = \{1, 2, \cdots, N\}$. Prove that for any $i \in \mathcal{A}$

$$\sum_{\{i\} \subset S \subset \mathcal{A}} (N - |S|)!(|S| - 1)! = N!$$

12. Consider an airport game which is a cost allocation problem. Let $N = \{1, 2, \cdots, n\}$ be the set of players. For each $i = 1, 2, \cdots, n$, player i requires an airfield that costs c_i to build. To accommodate all the players, the field will be built at a cost of $\max_{1 \leq i \leq n} c_i$. Suppose all the costs are distinct and $c_1 < c_2 < \cdots < c_n$. Take the characteristic function of the game to be

$$\nu(S) = -\max_{i \in S} c_i$$

For each $k = 1, 2, \cdots, n$, let $R_k = \{k, k + 1, \cdots, n\}$ and define

$$\nu_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset \\ 0 & \text{if } S \cap R_k = \emptyset \end{cases}$$

    (a) Show that $\nu = \sum_{k=1}^{n} \nu_k$

(b) Show that for each $k = 1, 2, \cdots, n$, if $i \notin R_k$, then player i is a null player of ν_k.

(c) Show that for each $k = 1, 2, \cdots, n$, if $i, j \in R_k$, then player i and player j are symmetric players of ν_k.

(d) Find the Shapley value $\phi_k(\nu)$ of player k, $k = 1, 2, \cdots, n$, of the airport game ν.

13. Let $A = \{1, 2, \ldots, n\}$ and $\nu : \mathcal{P}(A) \to \mathbb{R}$ be the characteristic function defined by

$$\nu(S) = |S| \sum_{i \in S} i$$

where $|S|$ denotes the number of elements in S. Let $\phi_k(\nu)$ be the Shapley value of $k \in A$ in the game (A, ν).

(a) Show that ν is superadditive.

(b) For each $i = 1, 2, \ldots, n$, let $\nu_i : \mathcal{P}(A) \to \mathbb{R}$ be the characteristic function defined by

$$\nu_i(S) = \begin{cases} 0, & \text{if } i \notin S \\ i|S|, & \text{if } i \in S \end{cases}.$$

Let $\phi_k(\nu_i)$ be the Shapley value of $k \in A$ in the game (A, ν_i).

(i) Show that $\phi_k(\nu_k) = \dfrac{k(n+1)}{2}$.

(ii) Find $\phi_k(\nu_i)$ for $i \neq k$.

(c) Using the results in (b), or otherwise, find $\phi_k(\nu)$ in terms of k and n.

14. In a game there are three boxes, Bronze Box, Silver Box and Gold Box. Ada puts \$1,001 into the boxes in any way she likes. The money in Bronze Box will be doubled, the money in Silver Box will be tripled and the money in Gold Box will become 4 times the original amount. Then Bella, without knowing how Ada puts the money, chooses one of the boxes and gets the money inside. Ada will get the money inside the other two boxes.

(a) How should Ada split the money so that the payoff of Bella are the same no matter what strategy Bella uses.

(b) Find the strategy of Bella in the Nash's equilibrium.

(c) Find the expected payoffs of Ada and Bella in the Nash's equilibrium.

(d) Suppose Ada and Bella decided to cooperate. Using Nash's solution to the bargaining problem and the answer in (c) as the status quo point, determine how much Ada and Bella should get from the boxes.

15. In a money sharing game, three players Alex, Beatrice and Christine put money into a Magic Box. Alex may put from \$0 to \$8, Beatrice may put from \$0 to \$20 and Christine may put from \$0 to \$50. After they put the money, the amount in the Magic Box will be doubled. Then the money in the Magic Box will be evenly distributed to the three players.

    (a) Find the amount that Alex, Beatrice and Christine should put in the Nash equilibrium.

    (b) Find the maximum total profit that Alex and Beatrice may guarantee themselves if they choose to cooperate.

    (c) The three players decide to cooperate. Use Shapley value to find a suitable way to split the money in the Magic Box at the end of the game.

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