Solutions of MATH5360 Assignment 2

1. (a)Set up the tableau and apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & 3 & 2 & 2 & 15 \\ x_2 & 0 & 4^* & 5 & 24 \\ \hline & 3 & 5 & 4 & -12 \end{array} \rightarrow \begin{array}{c|ccc|c} & y_1 & x_2 & y_3 & \\ \hline x_1 & 3^* & -1/2 & -1/2 & 3 \\ y_2 & 0 & 1/4 & 5/4 & 6 \\ \hline & 3 & -5/4 & -9/4 & -42 \end{array} \rightarrow$$

$$\rightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & \\ \hline y_1 & 1/3 & -1/6 & -1/6 & 1 \\ y_2 & 0 & -1/4 & 5/4 & 6 \\ \hline & -1 & -3/4 & -7/4 & -45 \end{array}.$$

Thus an optimal vector for the primal problem is $(y_1, y_2, y_3) = (1, 6, 0)$ and the maximum value f is 45.

The dual problem is

$$\begin{array}{rl} \min & g = 15x_1 + 24x_2 + 12 \\ \text{subject to} & 3x_1 \geq 3 \\ & 2x_1 + 4x_2 \geq 5 \\ & 2x_1 + 5x_2 \geq 4 \end{array}$$

Thus an optimal vector for the primal problem is $(x_1, x_2) = (1, 3/4)$ and the minimum value g is 45. (b)Set up the tableau and apply pivoting operations, we have

$$\begin{array}{c|cccc|c} & y_1 & y_2 & y_3 & y_4 & \\ \hline x_1 & 3 & 1 & 1 & 4 & 12 \\ x_2 & 1 & -3 & 2 & 3 & 7 \\ x_3 & 2 & 1^* & 3 & -1 & 10 \\ \hline & 2 & 4 & 3 & 1 & 0 \end{array} \rightarrow \begin{array}{c|cccc|c} & y_1 & x_3 & y_3 & y_4 & \\ \hline x_1 & 1 & -1 & -2 & 5^* & 2 \\ x_2 & 7 & 3 & 11 & 0 & 37 \\ y_2 & 2 & 1 & 3 & -1 & 10 \\ \hline & -6 & -4 & -9 & 5 & -40 \end{array} \rightarrow$$

		y_1	x_3	y_3	x_1	
	y_4	$1/5$	$-1/5$	$-2/5$	$1/5$	$2/5$
$\to$	x_2	7	3	11	0	37
	y_2	$11/5$	$4/5$	$13/5$	$1/5$	$52/5$
		-7	-3	-7	-1	-42

Thus an optimal vector for the primal problem is $(y_1, y_2, y_3, y_4) = (0, 52/5, 0, 2/5)$ and the maximum value f is 42.

The dual problem is

$$\begin{array}{rl} \min & g = 12x_1 + 7x_2 + 10x_3 \\ \text{subject to} & 3x_1 + x_2 + 2x_3 \geq 2 \\ & x_1 - 3x_2 + x_3 \geq 4 \\ & x_1 + 2x_2 + 3x_3 \geq 3 \\ & 4x_1 + 3x_2 - x_3 \geq 1 \end{array}$$

Thus an optimal vector for the primal problem is $(x_1, x_2, x_3) = (1, 0, 3)$ and the minimum value g is 42.

2.(a) Add $k = 3$ to every entry to get

$$\begin{pmatrix} 5 & 0 & 6 \\ 1 & 6 & 4 \\ 4 & 4 & 8 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	5^*	0	6	1
x_2	1	6	4	1
x_3	4	4	8	1
	1	1	1	0

$\to$

	x_1	y_2	y_3	
y_1	$1/5$	0	$6/5$	$1/5$
x_2	$-1/5$	6	$14/5$	$4/5$
x_3	$-4/5$	4^*	$16/5$	$1/5$
	$-1/5$	1	$-1/5$	$-1/5$

$\to$

	x_1	x_3	y_3	
y_1	$1/5$	0	$6/5$	$1/5$
x_2	1	$-3/2$	-2	$1/2$
y_2	$-1/5$	$1/4$	$4/5$	$1/20$
	0	$-1/4$	-1	$-1/4$

Therefore, $d = 1/4$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = (0, 0, 1),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (4/5, 1/5, 0),$$

the value of the game is $v = \frac{1}{d} - k = 1$.

(b) Add $k = 5$ to every entry to get

$$\begin{pmatrix} 8 & 6 & 0 \\ 4 & 3 & 11 \\ 3 & 4 & 7 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & 8^* & 6 & 0 & 1 \\ x_2 & 4 & 3 & 11 & 1 \\ x_3 & 3 & 4 & 7 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array} \rightarrow \begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & \\ \hline y_1 & 1/8 & 3/4 & 0 & 1/8 \\ x_2 & -1/2 & 0 & 11^* & 1/2 \\ x_3 & -3/8 & 7/4 & 7 & 5/8 \\ \hline & -1/8 & 1/4 & 1 & -1/8 \end{array} \rightarrow$$

$$\rightarrow \begin{array}{c|ccc|c} & x_1 & y_2 & x_2 & \\ \hline y_1 & 1/8 & 3/4^* & 0 & 1/8 \\ y_3 & -1/22 & 0 & 1/11 & 1/22 \\ x_3 & -5/88 & 7/4 & -7/11 & 27/88 \\ \hline & -7/88 & 1/4 & -1/11 & -15/88 \end{array} \rightarrow \begin{array}{c|ccc|c} & x_1 & y_1 & x_2 & \\ \hline y_2 & 1/6 & 4/3 & 0 & 1/6 \\ y_3 & -1/22 & 0 & 1/11 & 1/22 \\ x_3 & -23/66 & -7/3 & -7/11 & 35/132 \\ \hline & -4/33 & -1/3 & -1/11 & -7/33 \end{array}.$$

Therefore, $d = 7/33$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = (4/7, 3/7, 0),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (0, 11/14, 3/14),$$

the value of the game is $v = \frac{1}{d} - k = -2/7$.

(c) Add $k = 2$ to every entry to get

$$\begin{pmatrix} 5 & 2 & 3 \\ 1 & 4 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	5^*	2	3	1
x_2	1	4	0	1
x_3	2	3	1	1
	1	1	1	0

$\to$

	x_1	y_2	y_3	
y_1	$1/5$	$2/5$	$3/5$	$1/5$
x_2	$-1/5$	$18/5^*$	$-3/5$	$4/5$
x_3	$-2/5$	$11/5$	$-1/5$	$3/5$
	$-1/5$	$3/5$	$2/5$	$-1/5$

$\to$

	x_1	x_2	y_3	
y_1	$2/9$	$-1/9$	$2/3^*$	$1/9$
y_2	$-1/18$	$5/18$	$-1/6$	$2/9$
x_3	$-5/18$	$-11/18$	$1/6$	$1/9$
	$-1/6$	$-1/6$	$1/2$	$-1/3$

$\to$

	x_1	x_2	y_1	
y_3	$1/3$	$-1/6$	$3/2$	$1/6$
y_2	0	$1/4$	$1/4$	$1/4$
x_3	$-1/3$	$-7/12$	$-1/4$	$1/12$
	$-1/3$	$-1/12$	$-3/4$	$-5/12$

.

Therefore, $d = 5/12$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = (4/5, 1/5, 0),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (0, 3/5, 2/5),$$

the value of the game is $v = \frac{1}{d} - k = 2/5$.

(d) Add $k = 3$ to every entry to get

$$\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 6 \\ 1 & 5 & 3 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	5^*	3	1	1
x_2	2	0	6	1
x_3	1	5	3	1
	1	1	1	0

$\to$

	x_1	y_2	y_3	
y_1	$1/5$	$3/5$	$1/5$	$1/5$
x_2	$-2/5$	$-6/5$	$28/5^*$	$3/5$
x_3	$-1/5$	$22/5$	$14/5$	$4/5$
	$-1/5$	$3/5$	$2/5$	$-1/5$

$\to$

	x_1	y_2	x_2	
y_1	$3/14$	$9/14$	$-1/28$	$5/28$
y_3	$-1/14$	$-3/14$	$5/28$	$3/28$
x_3	0	5^*	$-1/2$	$1/2$
	$-1/7$	$4/7$	$-1/7$	$-2/7$

$\to$

	x_1	x_3	x_2	
y_1	$3/14$	$-9/70$	$1/35$	$4/35$
y_3	$-1/14$	$-3/70$	$11/70$	$9/70$
y_2	0	$-1/5$	$-1/10$	$1/70$
	$-1/7$	$-4/35$	$-3/35$	$-12/35$

.

Therefore, $d = 12/35$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3) = (5/12, 1/4, 1/3),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (1/3, 7/24, 3/8),$$

the value of the game is $v = \frac{1}{d} - k = -1/12$.

(e) Add $k = 2$ to every entry to get

$$\begin{pmatrix} 3 & 1 & 3 \\ 0 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 3 & 0 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & 3^* & 1 & 3 & 1 \\ x_2 & 0 & 2 & 1 & 1 \\ x_3 & 3 & 0 & 4 & 1 \\ x_4 & 1 & 3 & 0 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array} \to \begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & \\ \hline y_1 & 1/3 & 1/3 & 1 & 1/3 \\ x_2 & 0 & 2 & 1 & 1 \\ x_3 & -1 & -1 & 1 & 0 \\ x_4 & -1/3 & 8/3^* & -1 & 2/3 \\ \hline & -1/3 & 2/3 & 0 & -1/3 \end{array} \to$$

$$\to \begin{array}{c|ccc|c} & x_1 & x_4 & y_3 & \\ \hline y_1 & 9/24 & -1/8 & 9/8^* & 1/4 \\ x_2 & 1/4 & -3/4 & 7/4 & 1/2 \\ x_3 & -9/8 & 3/8 & 5/8 & 1/4 \\ y_2 & -1/8 & 3/8 & -3/8 & 1/4 \\ \hline & -1/4 & -1/4 & 1/4 & -1/2 \end{array} \to \begin{array}{c|ccc|c} & x_1 & x_4 & y_1 & \\ \hline y_3 & 1/3 & -1/9 & 8/9 & 2/9 \\ x_2 & -1/3 & -5/9 & -14/9 & 1/9 \\ x_3 & -4/3 & 4/9 & -5/9 & 1/9 \\ y_2 & 0 & 1/3 & 1/3 & 1/3 \\ \hline & -1/3 & -2/9 & -2/9 & -5/9 \end{array}.$$

Therefore, $d = 5/9$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3, x_4) = (3/5, 0, 0, 2/5),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (0, 3/5, 2/5),$$

the value of the game is $v = \frac{1}{d} - k = -1/5$.

(f) Add $k = 3$ to every entry to get

$$\begin{pmatrix} 0 & 5 & 3 \\ 4 & 1 & 2 \\ 2 & 3 & 5 \\ 4 & 4 & 0 \end{pmatrix}.$$

Set up the tableau and apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	0	5	3	1
x_2	4*	1	2	1
x_3	2	3	5	1
x_4	4	4	0	1
	1	1	1	0

$\rightarrow$

	x_2	y_2	y_3	
x_1	0	5	3	1
y_1	1/4	1/4	1/2	1/4
x_3	−1/2	5/2	4	1/2
x_4	−1	3*	−2	0
	−1/4	−3/4	1/2	−1/4

$\rightarrow$

	x_2	x_4	y_3	
x_1	5/3	−5/3	19/3	1
y_1	1/3	−1/12	2/3	1/4
x_3	1/3	−5/6	17/3*	1/2
y_2	−1/3	1/3	−2/3	0
	0	−1/4	1	−1/4

$\rightarrow$

	x_2	x_4	x_3	
x_1	22/17	−25/34	−19/17	15/34
y_1	5/17	1/68	−2/17	13/68
y_3	1/17	−5/34	3/17	3/34
y_2	−5/17	4/17	2/17	1/17
	−1/17	−7/68	−3/17	−23/68

Therefore, $d = 23/68$ and a maximin strategy for the row player is

$$\mathbf{p} = \frac{1}{d}(x_1, x_2, x_3, x_4) = (0, 4/23, 12/23, 7/23),$$

a minimax strategy for the column player is

$$\mathbf{q} = \frac{1}{d}(y_1, y_2, y_3) = (13/23, 4/23, 6/23),$$

the value of the game is $v = \frac{1}{d} - k = -1/23$.

3.(a) For any $x_1,\ x_2 \in C_1 \cap C_2$, $\lambda \in [0,1]$, by the convexity of C_1, C_2, $\lambda x_1 + (1-\lambda)x_2 \in C_1$ and $\lambda x_1 + (1-\lambda)x_2 \in C_2$. Hence $\lambda x_1 + (1-\lambda)x_2 \in C_1 \cap C_2$, that is $C_1 \cap C_2$ is convex.

(b) For any $x = x_1 + x_2 \in C_1 + C_2$, $y = y_1 + y_2 \in C_1 + C_2$ and $\lambda \in [0,1]$, by the convexity of C_1, C_2, $\lambda x_1 + (1-\lambda)y_1 \in C_1$ and $\lambda x_2 + (1-\lambda)y_2 \in C_2$. Hence $\lambda x + (1-\lambda)y \in C_1 + C_2$, that is $C_1 + C_2$ is convex.

4. Let C be the set of maximin strategy for the row player of A and ν be the value of the game with game matrix A. For any $\mathbf{u} = (u_1, u_2, ..., u_n)$, $\mathbf{v} = (v_1, v_2, ..., v_n) \in C$ and $\lambda \in [0, 1]$, then, by the definition of C, we have $\mathbf{u}A\mathbf{y}^T \geq \nu$ and $\mathbf{v}A\mathbf{y}^T \geq \nu$ for any $\mathbf{y} \in P^m$, and $\sum_{i=1}^n u_i = \sum_{i=1}^n v_i = 1$. Let $\mathbf{w} = (w_1, w_2, ..., w_n) = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$, then $w_i = \lambda u_i + (1-\lambda)v_i$ with $\sum_{i=1}^n w_i = \lambda + (1-\lambda) = 1$, and $\mathbf{w}A\mathbf{y}^T = (\lambda\mathbf{u} + (1-\lambda)\mathbf{v})A\mathbf{y}^T \geq \lambda\nu + (1-\lambda)\nu = \nu$ for any $\mathbf{y} \in P^m$. Hence, $\mathbf{w} \in C$, that is, C is convex.

5.(a) $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ with $\lambda \in \mathbb{R}$. Since $\mathbf{z}$ is orthogonal to $\mathbf{x} - \mathbf{y}$, we have $< \mathbf{x} - \mathbf{y}, \mathbf{z} > = < \mathbf{x} - \mathbf{y}, \lambda\mathbf{x} + (1-\lambda)\mathbf{y} > = 0$. Thus,

$$\lambda = \frac{\|\mathbf{y}\|^2 - < \mathbf{x}, \mathbf{y} >}{\|\mathbf{x}\|^2 - 2 < \mathbf{x}, \mathbf{y} > + \|\mathbf{y}\|^2}.$$

(b)$< \mathbf{x}, \mathbf{y} > < 0$ implies $\lambda \in [0, 1]$. Thus, $\mathbf{z} \in C$ since C is convex.

6. $\nu_c(A) \leq 0$ implies there exists a minimax strategy $\mathbf{q} = (q_1, q_2, ..., q_n)$ for the column player such that $(-\lambda_1, -\lambda_2, ..., -\lambda_m)^T := A\mathbf{q}^T \leq \mathbf{0}^T$, that is, $\lambda_i \geq 0$. Thus, $\mathbf{0}^T = A\mathbf{q}^T + (\lambda_1, \lambda_2, ..., \lambda_m)^T = q_1\mathbf{a_1}^T + q_2\mathbf{a_2}^T + ... + q_n\mathbf{a_n}^T + \lambda_1\mathbf{e_1}^T + ... + \lambda_m\mathbf{e_m}^T$. Therefore, $\mathbf{0}^T = l_1\mathbf{a_1}^T + l_2\mathbf{a_2}^T + ... + l_n\mathbf{a_n}^T + l_{n+1}\mathbf{e_1}^T + ... + l_{n+m}\mathbf{e_m}^T$ with $l_i = \frac{q_i}{\sum q_i + \sum \lambda_j} \in [0, 1]$, $i = 1, 2, ..., n$ and $l_{n+j} = \frac{\lambda_j}{\sum q_i + \sum \lambda_j} \in [0, 1]$, $j = 1, 2, ..., m$ and $\sum_{k=1}^{n+m} l_k = 1$, that is, $\mathbf{0} \in C$.