1.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$g(x)$	0	0	1	1	2	0	3	1	2	0	0	1	1	2.

$$g(x) = \begin{cases} 0, & \text{if } x \equiv 0,1,5 \pmod 9 \\ 1, & \text{if } x \equiv 2,3,7 \pmod 9 \\ 2, & \text{if } x \equiv 4,8 \pmod 9 \\ 3, & \text{if } x \equiv 6 \pmod 9. \end{cases}$$

$g(6) = 3$, $g(13) = 2$, $g(34) = 1$.

b) $13-2 \equiv 2 \pmod 9$, $13-3 \equiv 1 \pmod 9$, $13-6 \equiv 7 \pmod 9$

$\Rightarrow$ The winning moves of $g(13)$ is to remove 3 chips, as $g(10)=0$.

$34-2 \equiv 5 \pmod 9$, $34-3 \equiv 4 \pmod 9$, $34-6 \equiv 1 \pmod 9$

$\Rightarrow$ The winning moves of $g(34)$ is to remove 2 chips of 6 chips

c) $P = \{x : x \equiv 0,1,5 \pmod 9\}$.

i) 0, 1 are the terminal positions and $0,1 \equiv 0,1 \pmod 9$

ii) Let $p \in P$, then $p \equiv 0$ or $p \equiv 1$ or $p \equiv 5 \pmod 9$. Then

$p-2 \equiv 7,8,3$, $p-3 \equiv 6,7,2$, $p-5 \equiv 3,2,8 \pmod 9$,

which all are not in P.

iii) Suppose $q \notin P$. Then $q \equiv 2,3,4,6,7,8 \pmod 9$. Then

when $q \equiv 2$, $q-2 \equiv 0 \pmod 9$

when $q \equiv 3$, $q-3 \equiv 0 \pmod 9$

when $q \equiv 4$, $q-3 \equiv 1 \pmod 9$

when $q \equiv 6$, $q-6 \equiv 0 \pmod 9$

when $q \equiv 7$, $q-6 \equiv 1 \pmod 9$

when $q \equiv 8$, $q-3 \equiv 5 \pmod 9$.

2a) $x \oplus 10 = 25 \oplus 29$

$x = 25 \oplus 29 \oplus 10$

$= 1110_2$

$= 13.$

$$\begin{array}{r} 11001 \\ 11101 \\ \oplus\ 01010 \\ \hline 1110_2 \end{array}$$

b) From (a), we know that $(10, 25, 29)$ is not a P-position.

$25 \oplus 29 = 10111_2 = 23 < 25$

$25 \oplus 10 = 10011_2 = 19 < 29$

$25 \oplus 29 = 00100_2 = 4 < 10$

$\Rightarrow$ $(10, 23, 29)$, $(10, 25, 19)$, $(4, 23, 29)$ are winning move!

3a)

x	0	1	2	3	4	5	6	7	8	9	10	11	12
$g_1(x)$	0	0	0	1	0	2	1	3	0	4	2	5	1

13	14	15	16	17	18
6	3	7	0	8	4.

$g_1(x) = 0$ if $x = 0$ or $x = 2^k$, $k \in \mathbb{Z}^+$

$\Rightarrow$ $P = \{x : x = 0 \text{ or } x = 2^k, k \in \mathbb{Z}^+\}$.

b) $g_1(13) = 6$, $g_2(x) \equiv x \pmod 8$, $g_3(x) = x$.

$g_2(12) = 4$, $g_3(7) = 7$.

$g(13, 12, 7) = g_1(13) \oplus g_2(12) \oplus g_3(7) = 6 \oplus 4 \oplus 7 = 101_2 = 5.$

$$\begin{array}{r} 110 \\ 100 \\ \oplus\ 111 \\ \hline 101_2 \end{array}$$

c) $110_2 \to 011_2$, $g_1(x) = 3. \Rightarrow x = 7.$

$100_2 \to 001_2$, $g_2(x) = 1 \Rightarrow x = 9.$

$111_2 \to 010_2$ $g_3(x) = 2 \Rightarrow x = 2,$

$\Rightarrow$ Winning moves are $(7, 12, 7)$, $(13, 9, 7)$, $(13, 12, 2)$.

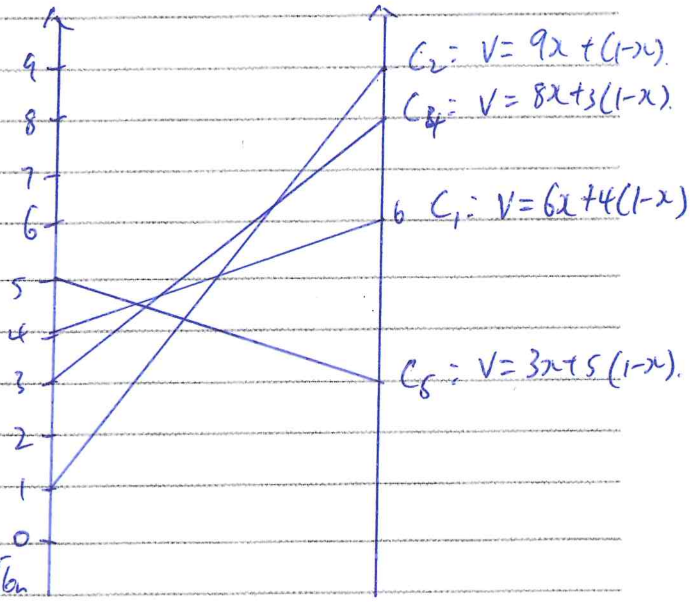
4a) $A = \begin{pmatrix} 6 & 9 & 4 & 8 & 3 \\ 5 & 3 & 7 & 6 & 2 \\ 4 & 1 & 6 & 3 & 5 \end{pmatrix}$

C_3

$\Rightarrow \begin{pmatrix} 6 & 9 & 8 & 3 \\ 5 & 3 & 6 & 2 \\ 4 & 1 & 3 & 5 \end{pmatrix}$

R_2

$\Rightarrow A' = \begin{pmatrix} 6 & 9 & 8 & 3 \\ 4 & 1 & 3 & 5 \end{pmatrix}$.



b) By drawing the lower envelope, the maximum point of it is the intersection point of C_2 and C_5.

C_2: $V = 8x+1$

C_5: $V = -2x+5$

$x = \frac{2}{5}$, $V = \frac{21}{5}$.

For the minimax strategy: $\begin{pmatrix} 9 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} y_2 \\ y_5 \end{pmatrix} = \begin{pmatrix} \frac{21}{5} \\ \frac{21}{5} \end{pmatrix}$.

$\Rightarrow y_2 = \frac{1}{5}$, $y_5 = \frac{4}{5}$.

Hence the value of the game is $V = \frac{21}{5}$, maximin strategy $= (\frac{2}{5}, 0, \frac{3}{5})$

minimax strategy: $(0 \; 0.2 \; 0 \; 0 \; 0.8)$.

Add $k=3$

5a)

	y_1	y_2	y_3	-1
x_1	6^*	2	2	1
x_2	5	3	1	1
x_3	0	4	6	1
-1	1	1	1	0

$\rightarrow$

	x_1	y_2	y_3	-1
y_1	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
x_2	$-\frac{5}{6}$	$\frac{4}{3}^*$	$-\frac{2}{3}$	$\frac{1}{6}$
x_3	0	4	6	1
-1	$-\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{6}$

$\rightarrow$

	x_1	x_2	y_3	-1
y_1	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
y_2	$-\frac{5}{8}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{8}$
x_3	$\frac{5}{2}$	-3	8^*	$\frac{1}{2}$
-1	$\frac{1}{4}$	$-\frac{1}{2}$	1	$-\frac{1}{4}$

$\rightarrow$

	x_1	x_2	x_3	-1
y_1	$\frac{7}{32}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$\frac{3}{32}$
y_2	$\frac{15}{32}$	$\frac{9}{16}$	$\frac{1}{16}$	$\frac{5}{32}$
y_3	$\frac{5}{16}$	$-\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{16}$
-1	$-\frac{1}{16}$	$-\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{5}{16}$

Value $= \frac{1}{d} - k = \frac{16}{5} - 3 = \frac{1}{5}$.

$p = \frac{1}{d}(x_1\ x_2\ x_3) = \frac{16}{5}(\frac{1}{16}\ \frac{1}{8}\ \frac{1}{8}) = (\frac{1}{5}\ \frac{2}{5}\ \frac{2}{5})$.

$q = \frac{1}{d}(y_1\ y_2\ y_3) = \frac{16}{5}(\frac{3}{32}\ \frac{5}{32}\ \frac{1}{16}) = (\frac{3}{10}\ \frac{1}{2}\ \frac{1}{8})$.

6(a) ① $Ay^T = v\mathbf{1}^T$

② $yA = yA^T = (Ay^T)^T = (v\mathbf{1}^T)^T = v\mathbf{1}$

③ $yAy^T = y(v\mathbf{1}^T) = v$, $\because y \in P^n$.

By Minimax Thm, v is the value.

(i). Let

$$A = \begin{array}{c c} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} & \begin{array}{c} \text{min} \\ -1 \\ -1 \\ -1 \end{array} \\ \begin{array}{ccc} 1 & 0 & -1 \end{array} & \end{array}$$

max $1 \quad 0 \quad -1$

Since maximin and minimax are -1, so the value of A is -1.

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad 0 \neq 1.$$

b(i)

$1 - x_2 = 0 \Rightarrow x_2 = 1$

$-1 - 1 + x_3 = 0 \Rightarrow x_3 = 2$.

$1 + 2 - x_4 = 0 \Rightarrow x_4 = 3$

$-2 - 3 + x_5 = 0 \Rightarrow x_5 = 5$.

$3 - 5 = a \Rightarrow a = -2$.

(ii)

$1 - y_2 = 1 \Rightarrow y_2 = 0$.

$-1 + y_3 = 1 \Rightarrow y_3 = 2$,

$2 - y_4 = 1 \Rightarrow y_4 = 1$.

$-2 - 1 + y_5 = 1 \Rightarrow y_5 = 4$.

$1 - 4 = b \Rightarrow b = -3$.

(cont'd) $\Rightarrow \beta = v$.

$-2\alpha - 3\beta = v$.

$\Rightarrow \alpha = -2v$.

Also, since $q \in P^5$, so.

$\alpha(x_1 + \cdots + x_5) + \beta(y_1 + \cdots + y_5) = 1$.

$12\alpha + 8\beta = 1$

$-24v + 8v = 1$

$-16v = 1$

$v = -\frac{1}{16}$

$\alpha = \frac{1}{8}$, $\beta = -\frac{1}{16}$.

$q = \frac{1}{8}(1\ 1\ 2\ 3\ 5) + (-\frac{1}{16})(1\ 0\ 2\ 1\ 4)$.

$= \frac{1}{16}(1\ 2\ 2\ 5\ 6)$.

(iii).

$$\alpha \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} v \\ v \\ v \\ v \\ v \end{pmatrix}$$

From (a), since A is symmetric and $Aq^T = -\frac{1}{16}\mathbf{1}^T$,

So the maximin strategy is q, minimax strategy is q and $v = -\frac{1}{16}$.