Tutorial notes - 2 Deng Yongzhe / whizkiddeng@gmail.com

1. The Euler's number "e"

The first definition of "e" is:

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots \qquad (1)$$

which is comes from:

$$e^x = 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+\cdots$$

the series definition where we set $x=1$ is ok.

(1) is easy to understand and compute, if we want to get a more accurate "e" we just need to add more terms like "$\frac{1}{n!}$", but the meaning of "e" hides in another formula:

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \qquad (2) \text{ the limit definition.}$$

Before we discuss the properties of (2), we introduce an example to see where this (2) comes from.

2. The compound interest.

Assume now we have P_0 dollars in bank as our principle. with the interest rate is $r\%$ per year.

In reality, we collect or summary the money at the end of every year, like:

1-st year $\underbrace{P_0}_{\text{principle}} + \underbrace{P_0\cdot\frac{r}{100}}_{\text{interest}} = \underline{P_0(1+\frac{r}{100})} = P_1$ (as principle of next year)

2-nd year $P_1+P_1\frac{r}{100} = P_1(1+\frac{r}{100}) = P_0(1+\frac{r}{100})^2 = P_2$

⋮ go on

n-th year: $\underbrace{P_n = P_0(1+\frac{r}{100})^n}_{\text{the whole money we get from bank.}}$

But if we summary the money every half year, it would be:

0.5 year: $P_0+P_0\cdot\frac{r}{100\cdot2} = P_0(1+\frac{r}{100\cdot2}) = P_1$

1 year: $P_1+P_1\cdot\frac{r}{100\cdot2} = \underline{P_0(1+\frac{r}{100\cdot2})^2} = P_2$

⋮

A simple computation we can show:

$$P_0\left(1+\frac{r}{100\cdot2}\right)^2 - P_0\left(1+\frac{r}{100}\right) = P_0\cdot\frac{r^2}{100^2\cdot2^2} > 0 \quad (*)$$

means we get more money from bank if we apply such way to compute our compound interest.

So in math, now we divide one year to n parts, then the money we get at the end of the 1-st year would be:

$$P_n = P_0 \left(1+\frac{r}{100\cdot n}\right)^n \longrightarrow P = \lim_{n\to+\infty} P_n = P_0 \lim_{n\to+\infty}\left(1+\frac{r}{100\cdot n}\right)^n \quad \text{(the limit money we can get)}$$

now we apply (2) here:

$$P = P_0 \lim_{n\to+\infty}\left(1+\frac{r}{100\cdot n}\right)^n \quad \left(\text{change variable } u=\frac{100n}{r}, \text{ so } u\to+\infty \text{ when } n\to+\infty\right)$$

$$= P_0 \lim_{u\to+\infty}\left[\left(1+\frac{1}{u}\right)^u\right]^{\frac{r}{100}}$$ (there is a little difference from (2) for u may not be integer, but it doesn't matter, for actually (2) can be generalised to $e=\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$ real number case)

$$= P_0 \cdot e^{\frac{r}{100}} \qquad (3)$$

So we see the "e" appears in such compound interest.

Actually (3) also tells us even we divide one year into infinite many parts, the money we can get is a finite number, we can't become a millionaire by using such method.

Next we try to discuss 2 properties of (2). we rewrite (2) as:

$e=\lim_{n\to+\infty} x_n$, where $x_n = \left(1+\frac{1}{n}\right)^n$ be a sequence.

we try to prove:

(i) $x_{n+1} > x_n$, $\{x_n\}$ is an increasing sequence

(ii) $x_n < M$. $\{x_n\}$ has a bound.

For (i)
$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left(1+\frac{1}{n+1}\right)\left(\frac{\frac{n+2}{n+1}}{\frac{n+1}{n}}\right)^n$$

$$= \frac{n+2}{n+1}\left(\frac{n(n+2)}{(n+1)^2}\right)^n = \frac{n+2}{n+1}\left(1-\frac{1}{(n+1)^2}\right)^n$$

Bernoulli inequlity: $(1+x)^n \geq 1+nx$, for $x>-1$.

$$\geq \frac{n+2}{n+1}\left(1-\frac{n}{(n+1)^2}\right) \quad \left(x=-\frac{1}{(n+1)^2}\right)$$

$$= \frac{(n+2)(n^2+n+1)}{(n+1)^3} = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1$$

$\Rightarrow x_{n+1} > x_n$. this one just explains (*), the more parts we divide, the more money we get

(ii) $x_n = (1+\frac{1}{n})^n$ (apply ~~[illegible]~~ binomial thm here $(a+b)^n = \cdots$)

$$= 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$\binom{n}{k} = \frac{(n-k+1)\cdots n}{k!}$$ combination number.

$$= 1 + \sum_{k=1}^{n} \frac{1}{k!} \underbrace{\frac{n-k+1}{n}}_{<1} \cdots \underbrace{\frac{n}{n}}_{\leq 1}.$$

$$\leq 1 + \sum_{k=1}^{n} \frac{1}{k!}$$ (this is just our (1)!)

And $k! > 2^k$ for $k \geq 4$

$$\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \sum_{k=4}^{n} \frac{1}{2^k}$$

$$\leq 1 + 1 + 0.5 + \frac{1}{6} + \frac{1}{2^3}$$

$$\leq 2.792\ldots = M.$$

$\underline{x_n \leq M}$, this explains why we can't have infinite money.

3. Two problems.

problem-1. function like $f(\theta) = a\sin\theta + b\cdot\cos\theta$

recall additional formula: $\sin\theta\cdot\cos\varphi + \cos\theta\cdot\sin\varphi = \sin(\theta+\varphi)$

rewrite: $f(\theta) = a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\,(\sin\theta\cdot\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{=\cos\varphi} + \cos\theta\cdot\underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{=\sin\varphi})$

$$= \sqrt{a^2+b^2}\,\sin(\theta+\varphi)$$

where $\tan\varphi = \frac{b}{a}$.

problem-2. For function with absolute value.

$$f(x) = \left| |x-2| - 4 \right|$$

Try to simplify it to some piece-wise function, means take off the absolute value.

$$f(x) = \begin{cases} \overset{|x-6|}{|x-2-4|} & x \geq 2 \\ \underset{\substack{|-x-2| \\ |x+2|}}{|2-x-4|} & x < 2 \end{cases} \Rightarrow \begin{cases} x-6 & x \geq 6 \\ 6-x & \boxed{2} \leq x < 6 \\ x+2. & -2 < x < 2 \\ -(x+2) & x \leq -2 \end{cases}$$