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1. Definition of function : Domain, mapping rule, co-domain and range.

A function : $x \xmapsto{f} f(x)$

①. x: points that where the function f is defined, called domain. $\underline{\underline{D(f)}}$

② f: mapping rule, define the value of this function f at x, write as $f(x)$.

③. $f(x)$: If we collect all values of f, we would get a new set:

$$R(f) = \{f(x) \mid x \in D(f)\}$$, the range of $f(x)$

so we can see $R(f)$ is determinated by the $D(f)$ and mapping rule f.

Remark : the difference between co-domain and range:

$A \xmapsto{f} B. \Rightarrow$

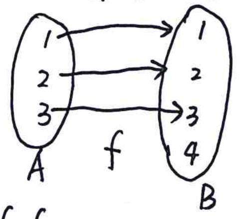


fig-1

$A = D(f)$: domain of f

always belongs to the max-domain of f.

B = co-domain : A set that always larger than the range of f, of course can be equal.

Ex. 1. In fig-1, $A = \{1, 2, 3\}$, so $D(f) = A = \{1, 2, 3\}$

And we define $\begin{cases} f(1) = 1 \\ f(2) = 2 \\ f(3) = 3 \end{cases} \Rightarrow R(f) = \{f(x) \mid x \in D(f)\} = \{1, 2, 3\}$

while our co-domain $B = \{1, 2, 3, 4\}$ so $R(f) \subsetneqq B$.

Ex. 2 Give the max-domain and range of following functions.

$$f(x) = \frac{1}{\sqrt{x^2-4}}$$

$D(f) = \{x \mid \sqrt{x^2-4} \neq 0\} \cap \{x \mid x^2-4 \geq 0\}$

$= \{x \mid x \neq \pm 2\} \cap \{x \mid x \geq 2 \text{ or } x \leq -2\}$

$= \{x \mid x > 2 \text{ or } x < -2\}$.

$R(f) = (0, +\infty)$

why? For we choose any $t \in (0, +\infty)$

let $f(x) = \frac{1}{\sqrt{x^2-4}} = t \Rightarrow x^2 = 4 + \frac{1}{t^2} \Rightarrow x_0 = \pm\sqrt{4+\frac{1}{t^2}} \in D($

which means we can find some x_0 make $f(x_0) = t$

this is the definition of range.

$$f(x) = \frac{1}{\cos x + \sin x}$$

$$D(f) = \{x \mid \cos x + \sin x \neq 0\}$$
$$= \{x \mid \sqrt{2}\sin(x+\tfrac{\pi}{4}) \neq 0\}$$
$$= \{x \mid x \neq k\pi - \tfrac{\pi}{4},\ k \in \mathbb{Z}\}.$$

for $\cos x+\sin x = \sqrt{2}\sin(x+\frac{\pi}{4}) \in [-\sqrt{2}, \sqrt{2}]$

and $\cos x + \sin x \neq 0$. so:

$$f(x) = \frac{1}{\cos x+\sin x} \in (-\infty, -\tfrac{\sqrt{2}}{2}] \cup [\tfrac{\sqrt{2}}{2}, +\infty)$$

$$\therefore R(f) = (-\infty, -\tfrac{\sqrt{2}}{2}] \cup [\tfrac{\sqrt{2}}{2}, +\infty)$$

$$f(x) = \sqrt{2 - |\ln(1-x)|}$$

$$D(f) = \{x \mid 2 - |\ln(1-x)| \geq 0\} \cap \{x \mid 1-x > 0\}$$
$$= \{x \mid -2 \leq \ln(1-x) \leq 2\} \cap \{x \mid x < 1\}$$
$$= \{x \mid 1-e^2 \leq x \leq 1-e^{-2}\} \cap \{x \mid x < 1\}$$
$$= [1-e^2,\ 1-e^{-2}]$$

$R(f) = [0, \sqrt{2}]$ for $0 \leq 2 - |\ln(1-x)| \leq 2$

Remark: domains and ranges of some important functions:

1). $D(\frac{1}{x}) = (-\infty, 0) \cup (0, +\infty) = \{x \mid x \neq 0\}$, $R(f) = (-\infty, 0) \cup (0, +\infty)$

2) $f(x) = \cos x$. $D(f) = \mathbb{R} = (-\infty, +\infty)$, $R(f) = [-1, 1]$

(3) $f(x) = \sqrt{x}$ $D(f) = [0, +\infty) = \{x \mid x \geq 0\}$. $R(f) = [0, +\infty)$

(4) $f(x) = \ln x$ $D(f) = (0, +\infty)$, $R(f) = (0, +\infty)$

2. Polynomial function and trigonometric function.

A polynomial of degree n has the form:

$P_n(x) = a_n x^n + \cdots + a_1 x + a_0$

with a restriction that $a_n \neq 0$, for if $a_n = 0$, it would reduce to a simpler polynomial, like degree $\leq n-1$.

If we want to analyse the behavior of $P_n(x)$ when $x \to +\infty$ or $x \to -\infty$, we need to know a little "limit" like:

$P_n(x) = a_n(x^n)\left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \cdots + \frac{a_1}{a_n}\cdot\frac{1}{x^{n-1}} + \frac{a_0}{x^n}\right)$

we have $n+1$ terms in bracket. all terms like $\frac{a_i}{a_n}\frac{1}{x^{n-i}}$ would tend to 0 when $x \to +\infty$ for $a_n \neq 0$ so $\frac{a_i}{a_n}$ remains finite. which means when x becomes sufficient large, we can say $1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \cdots + \frac{a_1}{a_n}\cdot\frac{1}{x^{n-1}} + \frac{a_0}{x_n} > 0$ (when $x > M$)

no matter what symbols $\frac{a_i}{a_n}$ be.

so the behavior of $P_n(x)$ just can be determined by the highest order term: $\underline{\underline{a_n x^n}}$

it implies the symbol and the principal magnitude of $P_n(x)$.

it's the same to analyse the case $x \to -\infty$.

the "Fundamental Theorem of Algebra" tells us that:

$P_n(x) = 0$ has (at most) n solutions means n roots.

"at most" depends on the domain we discuss Real number or complex number. i.e.

$x^2 + 1 = 0$, no roots in R, but 2 roots $x = \pm i$ (imaginary number $i^2 = -1$).

when we know x_0 is a root of $P_n(x)$, we can have:

$P_n(x) = (x - x_0)\,Q_{n-1}(x)$ which is called factorize. and we get a simpler polynomial $Q_{n-1}(x)$.

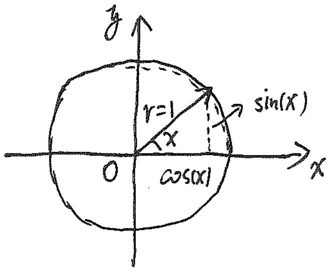
Another useful conclusion from "Fundamental Theorem" is:

if we know $P_n(x)$ has more than $n+1$ roots, the only possibility is $P_n(x) \equiv 0$

means reduced to the trival case.

sin(x) and cos(x).

One of definitions of sin(x) and cos(x) is unit circle:



it's very easy to see that sin(x), cos(x) are periodic functions like:

$\sin(x) = \sin(x+2\pi)$, $\cos(x) = \cos(x+2\pi)$

for 2π just means we take a round and back to the origin point.

Another is the boundness property:

$|\sin(x)| \leq 1$, $|\cos(x)| \leq 1$ for the radius of circle is 1.

Another definition is use the series:

$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$

$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$

this 2 formulas can be deduced from Euler's formula:

$e^{ix} = \cos(x) + \sin(x)\,i.$

it's useful to use the above formula to prove:

$$\begin{cases} \sin(x+y) = \sin(x)\cdot\cos y + \cos x\cdot\sin y \\ \cos(x+y) = \cos x\cdot\cos y - \sin x\cdot\sin y \end{cases} \quad (*)$$

for. $e^{ix}\cdot e^{iy} = e^{i(x+y)}$

The RHS $= \cos(x+y) + \sin(x+y)\cdot i$

LHS $= (\cos x + \sin x\cdot i)\cdot(\cos y + \sin y\cdot i)$

$= (\cos x\cdot\cos y - \sin x\cdot\sin y) + (\sin x\cdot\cos y + \cos x\cdot\sin y)\,i.$

this "−" comes from $i^2 = -1$

LHS = RHS, we get (*)

there are so many formulas on trigonometric function:

$\sin^2 x + \cos^2 x = 1$ from unit circle

$$\left.\begin{aligned} \sin 2x &= 2\sin x\cdot\cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}\right\} \text{from } (*)$$

please google to know more!