Tutorial (Week 13)

Properties of Def. Int., Derivative under Integral Sign

Things leanred

(Abbreviation: In the following, all functions to be integrated are $\boxed{\text{cont.}}$ functions on $[a, b]$. We also abbreviate $\int_a^b f(x)dx$ as $\int_a^b$ (Similar for $\int_a^b g(x)dx$ etc.

- $\int_a^b = -\int_b^a$ • $\int_a^b = \int_a^c + \int_c^b$ • $\int_a^a = 0$
- If $f(x) \le g(x)\ \forall x \in [a, b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$
- Integral Mean Value Theorem:
  Theorem:
  Assumption: Let $f : [a, b] \to \mathbb{R}$ be continuous, then
  Conclusion: $\exists \xi \in \underbrace{[a, b]}_{closed\ interval}$ such that $\left(\frac{1}{b-a}\right) \int_a^b f(x)dx = f(\xi)$.
- How to compute the derivative of a function defined by an integral.

Assignments

1. (Properties of Def. Int.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is periodic (i.e. like "sine", "cosine", "tangent", its curves repeats itself.), or mathematically: There is a certain some positive real number T (called the $\boxed{\text{period}}$), such that for each real number x, the following formula holds

   $$f(x + T) = f(x)$$

   (e.g. $\sin(x + T) = \sin(x)$, where $T = 2\pi$)

   (Question:) Let f be as described above with period T, then

   $$\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$$

2. Let g be a continuous function defined on $[0, 1]$, show that

   $$\int_0^{\pi/2} g(\sin x)dx = \int_0^{\pi/2} g(\cos x)dx$$

3. In the preceding question, why did we choose the domain of g to be $[0, 1]$?

4. Let p be a continuous function defined on $[0, 1]$, show that

   $$\int_0^{\pi} x \cdot p(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} p(\cos x)dx$$

5. (Differentiation under Integral Sign)
   Evaluate for any $x > 1$ the following:

$$\frac{d}{dx}\left(\int_1^{xe^x} \frac{\ln t}{t} dt\right)$$

6. By using the substitution

$$x = \pi - t$$

   evaluate

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$$