# Chapter 1

# Introduction

## Outline of the Course

Topics to be discussed in this course include (as collected from survey of students' preferences)

- Philosophy of mathematics
- History of mathematics
- Interesting math topics

From these the topic "Number" has been singled out.

## Introduction

Before going into the individual topics related to "Number", let's ask ourselves one question.

## The First Question:

*What is mathematics?*

### Various Answers

Various answers were given to this (maybe "not well-formulated" question.) They can be broadly classified under two titles:

(i) A description of mathematical topics you have heard so far, e.g. arithmetic, geometry, algebra, …. ;

(ii) An attempt to "define" what the noun "mathematics" mean.

One "perhaps interesting" answer:

*"Mathematics is the language of Nature."*

### Moral of the Story

This simple question leads us to think about the following phenomenon – namely when one is asked a question of the form

*What is A ?*

Then usually there are two strategies to answer this question, (i) either you <u>try</u> to "describe" the collection of all "objects" which can be understood to be "A"; or (ii) you try to "define" A by some "properties".

Let us illustrate this phenomenon by yet another example.

**Example:**
Suppose we ask the question

"*What is a typical super-rich man in Hongkong?*"

Then the following may be answers:

(i) A rich man is a man like K.S. Li, P.K. Kwok, … ; or (ii) A rich man is a man whose income exceeds so and so much dollars per month.

**Moral of the Story (related to today's math.):**
The aforementioned question and its answers are related to modern mathematics in a certain way.

In what way? In the way that modern mathematics (by this we mean mathematics in the $20^{th}$ and $21^{st}$ centuries) are written down in a specific way, i.e. by using the language of

Set Theory

What we are saying is similar to saying that "English" is the Lingua Franca of the modern internet world (i.e. most people use English as their common language of communication.) In pretty much the same way, when people in the $20^{th}$ century write mathematics proofs or theorems, they write it down in a special format, using something called Set Theory.

**Summary**
Set Theory is the modern language of mathematics. By the word "language" we mean nowadays most people (not all, though) like to <u>describe</u> mathematical objects, or write mathematics, using Set Theory.

## The Meaning of the Word "Set"

The word "set" is just another name for "collection" (of "objects"), if we try to understand them intuivtively.

**Set Theory and "How Mathematicians use Sets to answer "What-Questions" "**

Let us define a "What-question" to be a question which start with the word "What".

Using this language, when a "What-question" is asked, e.g.

"What is a rich man in HK?"

Mathematicians immediately think of the following:

"Can you describe the Set of all rich men in HK?"

| Question for you: Will you react the same way to a <u>What-question</u>? |
|---|

Mathematicians do it in two ways:

(i) **(List-all-rich-persons-Method)** The Set of rich men in HK = {K.S. Li, P.K. Kwok, …} (i.e. you try to list all the names of these rich guys and enclose them in the bracket "{" and the bracket "}".)

**(ii)** **(Definition Method)** The Set of all rich men in HK = { a guy : this guy's monthly income exceeds  xyzw dollars}

**Remark:**

In the second method, we have 2 things inside the brackets. (i) a guy <u>in HK</u>, (ii) a "condition" this guy needs to satisfy (in order that) he can be "counted" as "rich".

**Remark:**

(a) The method (i) has a name. It is called the "extensional" description of a set. The word "extension" is related to the verb "extend", and describes "those objects that are included in the Set.

(b) The method (ii) has also a name. It is called "intensional" description of a set. The way of writing it is typically:

{ **object** :  **Property** which this **object** must satisfy }

(c) What is surprising is that **(a)** is not always the same as **(b)**. (if you are interested

**Summary of the Above Discussion**

**<<“What is Mathematics?” – a question without an “answer”.>>**

A lot of questions are like the above-mentioned. You can ask for an answer, and if you suppose the answer is a kind of a “definition”, then there is no such thing. But instead of a definition, you can sometimes “list” the “things” that are/should be included in the concept you think is “mathematics”.

But this brings up a very important point.

Let’s illustrate this point using a story from $20^{th}$ century Chinese Literature..

*Once somebody got asked as to what a human being is, upon this he got as answer the following one. “A human being is a featherless two-legged animal”.*

*But then some other guy challenged him by bringing to him a chicken with all its feathers shaved…*

柏拉圖為人類下定義云：“人者，無羽毛之兩足動物也。”可謂客觀极了！但是按照希腊來阿鐵斯(Diogeneslaertius)《哲學言行論》六卷二章所載，偏有人拿著一只拔了毛的雞向柏拉圖去質問。

From: 錢鐘書-->寫在人生邊上-->一個偏見

(URL: www.bwsk.net/mj/q/qianzhongshu/xzrs/010.htm)

**BUT …. What does the above story has to do with Math? This has to do with MATH because since the $20^{th}$ century, set theory became the “language of mathematics”.**

## A slightly simpler Question

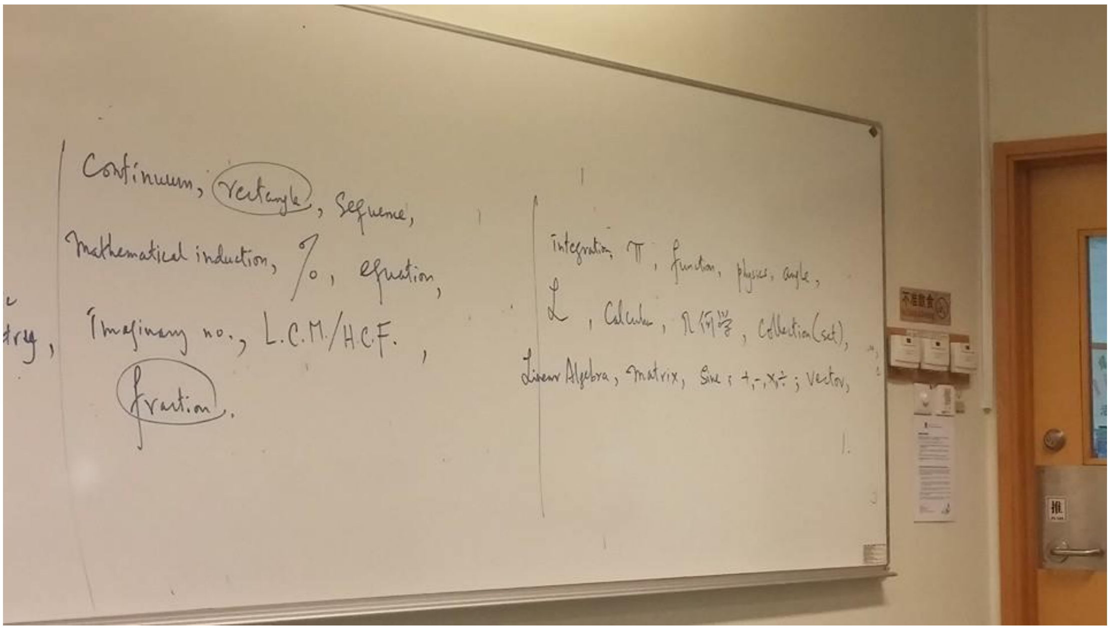
If we now change the question to the more concrete (and therefore, simpler) one:

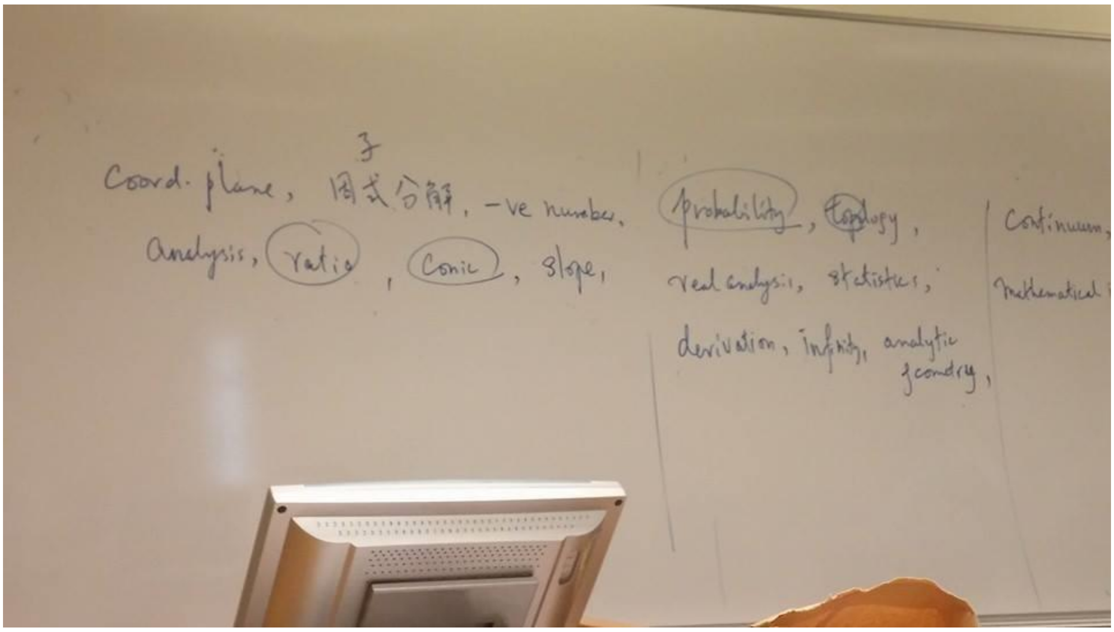
“what are the objects (i.e. mathematical objects) you have encountered so far ? Can you name them ?”

Then it may seem to be much easier to answer. In fact, in the first lecture, students in the class gave diversified list of very intriguing answers:

### Answers from the audience

From the photos the tutor made, I got the following (which of course contains many “vocabularies” for students. It doesn’t matter. At the end of the day, this is a General Education course “about Math”, it is not a Math. Course.) on the attached photos.




## A question not discussed

Related to this question is the following very "philosophical" question:

*What do you think, is the difference between "mathematical objects" and other objects ?*

(Traditional Answer): According to Greek philosophers, e.g. Plato, mathematical objects exist **<u>for ever</u>** in an ideal universe, i.e. in the universe of "**ideas**".

(More modern answer): Mathematical objects are just things our minds created, just like characters in fictions.

## Planned Outline of the Course

Math objects, i.e. (numbers, figures, functions), i.e. algebra versus geometry versus analysis ***("Analysis" is a new word here! Doesn't matter if you don't know what it means. Most people don't know the meaning of this word either!)***

**History:**

In ancient Greek mathematics, numbers and certain geometric figures co-existed were closely related, as the Pythagoreans school used "geometry" to study (or "to define" whole numbers (or "natural numbers"). These two objects, i.e. Numbers, Geometric Objects such as right-angles triangles, regular pentagons, regular polyhedral (singular: polyhedron) were probably the oldest geometric objects people got interested in. Much later mathematicians came to study new mathematical objects known as "functions", around the time of Newton.

**Examples of Functions**

In school we learned things like ***f(x)=sin x, g(x)=***$2x^2 + 3x - 7$ ***or h(x)=***$\frac{2x^3-x+1}{x^2+1}$

which are all examples of something called "functions". (We will not go deeper into this kind of mathematical objects.)

**Explanation of the Planned Outline**

1. From the list of mathematical objects students in the class mentioned, I want to choose 2 simple ones, which should be familiar to everyone, to discuss.
   They are: "numbers, figures"

2. Later on, we may (or "may not") discuss the more "modern" object called "function"
3. We mentioned these 3 kinds of math. objects because they are related to 3 main branches of mathematics, i.e. Algebra, Geometry and Analysis.

| |
|---|
| It doesn't matter if you don't know the meaning of the word "function" or the word "analysis". If it is useful, we will describe them later. If not, we won't use them again. |

In the first lecture, I also asked some of the following questions (though not all of them) which may (or may not) be interesting to some of the audience.

## Simple (?) Questions related to Numbers

The following questions may be worth thinking about. Some of them are simple, some of them rather involved.

*What is the purpose of inventing numbers?*

*What kinds of numbers do you know?*

*How are numbers related to geometric figures?*

*What do you think a "Prime Number" is?*

*What do you think an "Irrational Number" is?*

Some of the above questions will be discussed in Lecture 2.

# Lecture 2

# More on Numbers

Most, if not all, ancient cultures have the concept of numbers, be they Babylonian, Chinese, Egytian, Roman, …

It is therefore tempting to think that the concept of "numbers" is universal. But it may be not so, as we will see in the sequel.

## Greek concept of "Numbers"

Greek mathematics can be dated back to the Pythagoreans, which is a school of mathematics (or "mathematicians") who were active in the period around 5 B.C..

Their work has been recorded in the voluminous work of Euclid.

One main feature of their work is to base everything on Geometry.

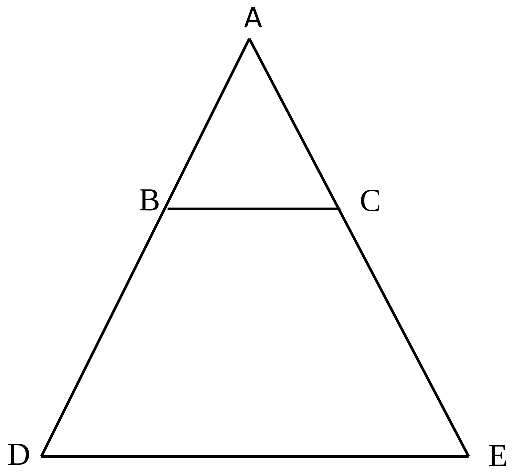
In their work, it is proposed to build everything, e.g. numbers, geometric figures based on two geometric tools, i.e. straightedge ( = a ruler without markings ) and compass.

Using this, one can (i) form a "unit" ( = the natural number "one" )

By repeatedly cutting and pasting this "unit", one can construct any natural number, 1,2,3, … etc.

One can also "add", "subtract" two natural numbers.

One can also, using Thales' Theorem on the proportionality of sides of similar triangles, "multiply" and "divide" two natural numbers.




(Thales' Theorem: If BC is parallel to DE, then AB/AD=AC/AE).

One application of Thales' Theorem is in the construction of *a* times *b*, where *a*, *b* are natural numbers (which are "constructible" using compass and straightedge). To construct $a \times b$, use the follow steps:

- Draw any angle, e.g. ∠BAC', then
- extend the line segments AB and AC, after that
- mark a length 1 on the extended line segment AB,
- mark a length *a* on the same extended line segment, call it AD;
- mark a length *b* on the extended line segment AC', call it AC, then
- draw the line segment BC, and the line segment DE parallel to it.

Then we have AB = 1, AD = *a*, AC=*b*. Thales' Theorem then gives AB/AD = AC/AE, i.e. 1/a=b/AE, which in turn gives AE= *ab*.

(Similarly, one can find *a/b* using straightedge and compass).

## The First Contradiction Proof

One thing which astonished Greek mathematicians is the square root of 2. They discovered that while one can easily construct it as the hypotenuse of a right-angled triangle of sides 1, this number square root of 2 is not "commensurable" (i.e. writable using straightedge and ruler), hence leading to a contradiction.

Later on, people called this kind of number "irrational".

**(Please refer to the short essay in Hardy's "A Mathematician's Apology" in the Appendix).**

The discovery of the above fact forms the first "contradiction proof" in human history. Having said that, it should be mentioned that in Ancient China and other ancient cultures, there were similar contradiction proofs, but a really systematic one was only, to the best of my knowledge, known to be attributable to the Greeks.

**Moral**

Ancient Greek geometers had as point of departure two measuring tools, i.e. the straightedge and the compass. Using these they defined natural numbers, rational numbers (commensurable numbers).

Then they encountered the "first mathematical crisis", namely the existence of numbers which they could not account for using their rules-of-game.

## Number Systems

Up to this point, we can draw (or "define") natural numbers starting from 1 using straightedge and compass. We denote the collection of these numbers by the notation ℕ. In modern textbooks, we describe this collection by "writing the symbol ℕ on the left-hand side, followed by an = sign, then put the "collection" on the right-hand side, viz. ℕ={1,2,3,…}.

**Notes:**

1. Note that we always enclose the "elements" in the "collection" in two "curly" brackets (i.e. " { " and " } " ).
2. Note also that we use 3 dots, i.e. …, to illustrate that our process of putting in elements is on-going. I.e. we can keep on doing this.

## Extensional versus Intentional Definition of a Set

The word "collection" in mathematics texts is usually replaced by the word "set".

Because of this, we can say that ℕ is the "collection" or the "set" of all natural numbers.

To specify the "elements" (or "members") of a set, there are at least two ways to proceed.

1. We list all the elements in it, like what we have done above.

2. We list one “typical” element, put a vertical line (i.e. “ | ”) or a colon (i.e. “ : “). Then we write down the “defining property” of this element on the right-hand side of the “vertical line” or the “colon”), e.g. ℕ={x | x is a natural number} or ℕ={x: x is a natural number}.

**Comment**

In the above two points, the first way of defining a set is known as “extensional definition”, i.e., we define the set by “listing out all its elements” (“all its elements” is the “extension” of the set.)

On the other hand, the second way of defining a set is known as “intentional definition”. It defines using the “property” of a “typical” element in the set.

Before Bertrand Russel, the great English philosopher, most people believed that these 2 ways of defining a set are the same. In fact, the German mathematician and logician Gottlob Frege published books in which he attempted to define 1,2,3,… etc. using “sets”.

In this work, he had to put down several assumptions, one of which is related to the question “extensional definition of a set = intentional definition of a set”. In everyday English, his Axiom V says

Every predicate (i.e. “property”) defines a set.

That this is not true was pointed out by B. Russel. Russel asks us to consider the following predicate (let us give it a symbol and call it $p$)

$$x \notin x$$

He also asks whether the following is a “set” or not.

$$\{x : x \notin x\}$$

Let us call this object $S$. then $S = \{x : x \notin x\}$. If $S$ is a set, then we can go on and ask “is $S$ an element of itself or not?”

Now we have two cases,

(i) if $S \in S$, then this "phrase" $S \in S$ doesn't satisfy the predicate $p$. That is, for this $S$, the predicate $p$ isn't true.

Because of the above, $S$ cannot be an element of the set $S$.

That is, $S \notin S$.

(ii) On the other hand, if $S \notin S$, then $S$ satisfies the predicate $p$, i.e. $S$ is an element of $S$ itself. But then, $S$ will be an element of $S = \{x : x \notin x\}$. Or put in another way, $S \in S$.

Either way, it's kind of like if we say "yes", then we get "no", and if we say "no", then we get "yes". Hence we get contradictions.

Hence such "intentional definition" of $S$ leads to paradox. In conclusion, we say that such definition is nonsensical, hence doesn't lead to anywhere. As a result, no such "set" can exist.

## Size Numbers versus Counting Numbers

### Size Numbers

It is worth mentioning that even very young children have the concept of two types of numbers, namely the "size numbers" and the "counting numbers".

Size Numbers = number denoting the "size" of a set (or "collection") of objects. It is interesting to see that one doesn't really need to know the exact size of two collections, before one can say which of two given collections has larger "size". Let's explain this using a simple example.

Suppose we have two collections of sweets, e.g Collection 1={candy A, candy B, candy C}, Collection 2 = {chocolate p, chocolate q, chocolate r}.
Now, you can ask a kid to check which of these is larger. For this, he doesn't have to know that the "size" of the first collection is 3 and also the second one. All he has to do is to pick one candy from Collection 1 and one chocolate from collection 2. If it is true that after some time, both of them have been picked up, then the two collections are of the same size.

If, however, it happens that when all candies in Collection 1 have been picked up, but there are still chocolates remaining in Collection 2, then Collection 2 has greater “size” than Collection 1. Similarly, if after certain amount of candies (but not ALL!) in Collection 1 have been picked up, ALL chocolates in Collection 2 have been picked up, then Collection 1 has more objects than Collection 2.

**Summary:** This kind of numbers measuring the size of a collection is usually known under the name “cardinal number”.

The important point is that you can compare the cardinal numbers (“sizes”) of two collections without knowing the exact numbers.

**Counting Numbers**

In many cultures, there are (not only) numbers that measure sizes of sets but also numbers that “counts”, like *first, second, third, fourth, etc.*
These numbers are “counting numbers” or “ordinal numbers”.

Just to mention a few more examples, they are e.g.

(English)
Cardinal Numbers
One, two, three, four, five, …

Ordinal Numbers
First, second, third, fourth, fifth, …

(Italian)
Cardinal Numbers
Uno, due, tre, quattro, cinque, …

Ordinal Numbers
Primo, second, terzo, …

(German)
Cardinal Numbers
Eins, zwei, drei, vier, …

Ordinal Numbers

Erste, zweite, dritte, …

(Chinese)
一，二，三，…

第一，第二，第三，…

**Indo-Arabic Number System**
Historically, it took human beings a long time before they invented the present day system of representing ONE, TWO, THREE, FOUR, FIVE, …, ELEVEN, … TWENTYTHREE, … , by (respectively) 1, 2,3, 4, 5, …, 11, … , 23

In Roman times, they used another (more cumbersome) system of representation, namely
1 = I, 2 = II, 3 = III, 4 = IV, 5 = V
And 23 = XXIII, …

We know that it was Leonardo of Pisa (or Fibonacci) who brought this idea of representing numbers using the symbols 1,2,3,4,5,6,7,8,9,0 to the Western world.

This system is much more superior than the Roman numeral system and other systems.

**Remark:** Very ancient cultures used something like I, II, III, IIII, IIIII to represent 1,2,3,4,5, etc. This was very inconvenient. Later on, various new ways of representing numbers appeared, but it was the Indo-Arabic numeral that appeared to be more effective and compact.

**Please refer to the pages from Brown's book on the Philosophy of Mathematics in the Appendix**

The Indo-Arabic numeral system isn't only a way of recording numbers, it also includes a way of calculating.

For example 237 means $2 \times 100 + 3 \times 10 + 7 \times 1$

**Numerals versus Numbers**
One last point that has to be mentioned is the following. In some older textbooks on

mathematics, some authors were so careful as to distinguish between a “number” and its name. The name of a number is called a “numeral”.

E.g. The “numeral” of the number “one” is 1 (pronounced as “one” in English).

**Some Special Whole Numbers**

The natural numbers are also called “whole numbers”. Historically, some whole numbers have attracted a lot of people’s attentions. Let us mention two of these, one is the Fibonacci numbers which is linked to Fibonacci.

The Fibonacci numbers have to do with the size of “rabbit population”. It goes like this. One starts with the two numbers 1 and 1, then one get the third “Fibonacci number” by taking the sum of these two numbers, i.e. $F_3$
(meaning “the third Fibonacci number”) $F_1 + F_2$ the “first” Fibonacci number plus the “second” Fibonacci number = 1+1 = 2.
In the same way, one gets $F_{n+2} = F_n + F_{n+1}$. That is, the *n*+2 th Fibonacci number = the sum of the *n* th Fibonacci number and the *n*+1 th Fibonacci number.

The Fibonacci numbers are closely linked to another interesting number, known as the Golden Section. It is the “limit” of the ratio of two consecutive Fibonacci numbers, e.g 1/1, 2/3, 3/5, …

More precisely, one computes the ratios of the form $F_n/F_{n+1}$ or $F_{n+1}/F_n$ for different values of *n*. It turns out that as *n* becomes larger and larger, the ratios will go nearer and nearer to the quantities $\frac{2}{\sqrt{5}-1}$ or $\frac{\sqrt{5}+1}{2}$.

**Stories about Golden Section**

There are innumerable stories about the use of Golden Section and the occurrence of Golden Section in nature. However, the correctness of most of these stories are dubious, i.e., it is hard to verify or refute (i.e. show that it is “wrong”) these assertions.

Let us just mention one such story. Many people claim that many ancient architectural works have Golden Section in them, such as the great pyramid in Giza. This is however doubtful because it is difficult to determine from where to where both in “depth” and in “width” one should measure.

**Prime Numbers**

Prime numbers are the "atoms" of numbers, because one can always represent a whole number as the products of prime numbers.

Here it is worth mentioning that only the Greeks invented "Prime numbers".

**Please refer to Mo Shaokwei's paper on the Non-existence of Prime Number concept in ancient Chinese mathematics in the Appendix (will be sent to you.)**

# Lecture 3

# Some Ideas of Euclid

## Introduction

In a very important book written by Galileo, he said the following (translated from Italian. If you want the original text, you can wiki under “il saggiatore”):

> *Philosophy [i.e. physics] is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.*

From the above, we know that scientists of all ages were equally enthusiastic and excited about the use of mathematics to explain the nature.

Euclid was one of the pioneers in using mathematics to explain the nature.

In this notes, we mention some ideas from Euclid.

## Euclid’s Deductive Geometry

Rational number, irrational number are both studied intensively by Euclid and his followers.

It is therefore worth looking into the way Euclid did Geometry. Euclid works out his Geometry in a systematic, deductive way, which had great impact on later generations of mathematicians.

Euclid did the following first, he made the following

- Definitions (23)
- Postulates (5)
- Common Notions (5)
- Propositions (48)

Let us look at some of the definitons which Euclid made.

In Book 1, he wrote (we select only some of them to illustrate what he did.)

Definitions

Definition 1. A point is that which has no part.

Definition 2. A line is breadthless length.

Definition 3. The ends of a line are points.

Definition 4. A straight line is a line which lies evenly with the points on itself.

Definition 5. A surface is that which has length and breadth only.

Definition 6. The edges of a surface are lines.

...

Definition 12. An acute angle is an angle less than a right angle.

...

Definition 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

The he made the following

**Postulates**

Let the following be postulated:

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

And he made the following

**Common Notions**

Common notion 1. Things which equal the same thing also equal one another.
Common notion 2. If equals are added to equals, then the wholes are equal.
Common notion 3. If equals are subtracted from equals, then the remainders are equal.
Common notion 4. Things which coincide with one another equal one another.
Common notion 5. The whole is greater than the part.

Equipped with these definitions, postulates and common notions, he started to prove theorems. Among these we have the following simple examplars.

**Propositions**

Proposition 1. To construct an equilateral triangle on a given finite straight line.
Proposition 2. To place a straight line equal to a given straight line with one end at a given point.

**Proposition 3. To cut off from the greater of two given unequal straight lines a straight line equal to the less.**

…

Proposition 5. In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.

Proposition 9. To bisect a given rectilinear angle.

Proposition 10. To bisect a given finite straight line.     Etc.

# Lecture 4

# The Golden Number

Keywords: Fibonacci numbers, Golden Number, relation to Geometry, Triangulation;
Activities: DVD film “Mesh” by Jantzen and Polthier; DVD film “To Open a Cube: How many Edges do I need” by C. Lewis.

## More on Numbers

In one previous lecture, we ended by mentioning, in particular, the special types of whole numbers, viz. the Fibonacci Numbers and the Golden Number (or “Golden Section”).

Let us go more deeply into the use of Golden Number in Geometry.

Rather than discussing examples mentioned in most popular science books, which (as mentioned) are usually quite controversial, let us concentrate on some facts which can be proved using elementary geometry and are rigorous.

**Example**

Construction of the regular pentagon

As we know, Greek mathematicians such as the Pythagoreans claimed that the universe is made out of five elementary building blocks, namely the five Platonic solids.

Platonic solids are convex bodies whose faces are all congruent, i.e. they can be transformed to one another via a motion which does not change the lengths or angles.

One more point is worth mentioning, at each vertex (i.e. corner of such a solid, the number of faces meeting at the vertex should be the same).

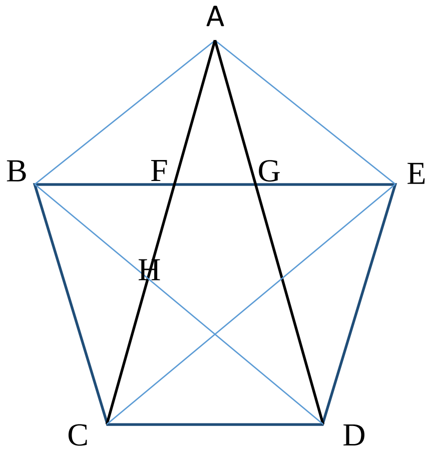
And as we all know, one can prove that there are five and only five such Platonic solids. They are the tetrahedron, cube, octahedron, dodecahedron and the icosahedron.

Tetrahedron is made of equilateral triangles, so are octahedron and icosahedron.

Cube is made of squares.
The most interesting one is the dodecahedron, which is made of 12 regular pentagons.

**Question: How does one construct the regular pentagon?**

**Hint: In a regular pentagon, there is the Golden Section, i.e.** $\frac{\sqrt{5}+1}{2}$**.**




Using elementary geometry, one can show that if AB = BC = CD = DE = EA =1, then the triangles $\Delta\mathrm{ACD}$ and $\Delta\mathrm{DHC}$ are "similar".

Indeed, $\Delta\mathrm{ACD}$ is a "Golden" triangle, that is it is similar to $\Delta\mathrm{HCD}$, the triangle formed by removing a certain triangle (which in this case, is the isosceles triangle $\Delta\mathrm{ADH}$).

By some considerations (which we will not write down the details), one can show that AH=1. Since, as mentioned, the triangles $\Delta\mathrm{ACD}$ and $\Delta\mathrm{DHC}$ are similar, we have the following equality of "ratios".

$$\frac{\mathrm{AD}}{1} = \frac{1}{HC} = \frac{1}{AC - AH}$$

Let now AD = *y*, then since AH=1, so we have $\frac{y}{1} = \frac{1}{y-1}$ i.e. $y^2 - y = 1$ giving $y = \frac{\sqrt{5}+1}{2}$.

Conclusion: The Golden section is inside the pentagon.

Using this, and the Pythagoras theorem, one can construct regular pentagon using straightedge and compass.

## More on Platonic Solids

In the DVD “Mesh”, the authors K. Pothier and B. Jantzen mention that in an old archaeological site in Italy, some archaeologists discovered old stones which look like the five Platoic solids. The authors also asked why people wanted to carve things like that in the ancient world.

Subsequently, the authors mention that actually the concept of “symmetry” is universal. It exists already in the crystal structure of things as simple as sodium chloride or diamond.

And in the time around 5BC, the Pythagoreans school of mathematicians had the theory that the world is made out of the five Platonic solids.

Since at that time humanity did not have the concept of coordinates, they described these solids using right-angled triangles. By counting the number of right-angled triangles inside these solids, they were able to classify them.

They also had the concept of “dual” Platonic solids. For example, the tetrahedron is dual itself, the cube is dual to the octahedron.

Almost one thousand years later, the astronomer Kepler tried to use these solids to build his model of the solar system (at that time, they knew only FIVE planets!). This model is a model build upon philosophical considerations which later proved to be wrong because it did not agree with the observed data.

As a side remark, in the book “Timaeus” of Plato, the author had the idea that everything in the world is made out of these five solids. Since each of these five solids had faces made out of right-angled triangles, we can say that Plato had the idea of "periodic table", because he tried to build everything out of some elementary building blocks, which in this case, are the "right-angled triangles".

This was the humanity's first attempt to show that everything in the world could be "triangulated".

Please refer to the pamphlet of "Mesh" in the Appendix.

# Lecture 5

# From Platonic Solids to Algebraic Topology

# (or "Rubber Sheet Geometry")

## More on Platonic Solids

In the video "Mesh", it was mentioned that there are five (and only "five") Platonic solids.

**Question:** How to show it?

(Idea): One way is to use the formula $\mathrm{V}-\mathrm{E}+\mathrm{F}=2-2\mathrm{g}$,
where g is the "genus" of the polyhedron, which is the "number" of holes.

For a "convex" polyhedron, the above formula takes a simple form, i.e.

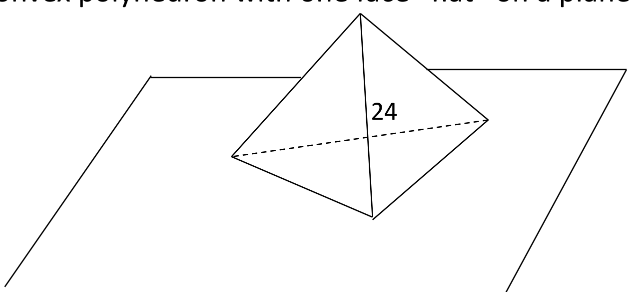
$$\mathrm{V}-\mathrm{E}+\mathrm{F}=2.$$

Here are some questions, which we have discussed in the class.

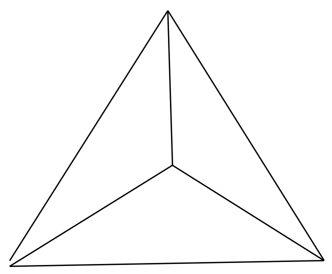
1. What is according to you "convex"?
2. Any other definition/suggestions?

Here is the "proof" outlined in the book by Imre Lakatos, "Proofs and Refutations".

1. Put the convex polyhedron with one face "flat" on a plane, as shown



2. Remove the "base" face.
3. "Flatten" the convex polyhedral.



1. Remove the edges one by one, in the following ways:
   (i) $V' = V_{old}$
   (ii) $E' = E_{old}$
   (iii) $F' = F_{old} - 1$ (Reason for 3. Is that the bottom face is now removed.)
2. Repeating the above, and removing all such edges, we arrive finally at 1 triangle which has 3 vertices, 3 edges and 1 face. That is, $V_{final} - E_{final} + F_{final} = 1$.
3. Going back the steps, we find that the "old" one satisfies
   $V_{old} - E_{old} + F_{old} = 1 + 1 = 2$.

## Criticism of this "Proof"

In Lakatos' book, it is mentioned that there are several drawbacks of this "proof".

1. It is not clear that one can "flatten" the convex polyhedron as outlined in step 1.
2. The removal of edges method doesn't seem to be clear enough.

Indeed, Euler didn't prove it this way. In his time there was no "concept" of <u>continuous change of an object without tearing it</u>. Euler proved it using other methods, but his proof is not correct.

You can find this in the paper "How Euler did it" on the web.

## Euler's Theorem and Exterior Angle Sum

Euler's theorem is actually related to many "modern" developments in 20th century mathematics. One of these is the so-called Gauss-Bonnet Theorem in "Differential Geometry" which relates "number of holes" of a surface to "curvature of a surface.

The formulas goes like this (you don't need to know the precise meaning of the expressions in the formula. You only need to notice that on the left-hand side there is something called "curvature", and on the right-hand side there is something called "genus" which counts the number of holes).

$$\iint K dM = 2\pi(2 - 2g)$$

The expression *K* is known as the Gaussian curvature; the expression *g* is known as the genus.

**Remark.** This formula is thought to be "wonderful", because it "relates" two different types of mathematical concepts, on one hand we have curvature (which we can only measure using some concepts like "distance" or "angles"). On the other hand, we have the concept of genus, which relies only on counting the number of "holes".

**Example.** If you look at a sphere of radius 1, you can convince yourself that its curvature is everywhere the same (though we haven't defined curvature yet!) Suppose the curvature is 1 everywhere, then the left-hand side of the formula gives the area of the sphere, which is known to be $4\pi$, while on the right-hand side we have the expression $2\pi(2 - 2 \times 0) = 4\pi$.

## Gauss-Bonnet Formula $\approx$ Exterior Angle Sum of Polygon

While the formula we mentioned in the preceding paragraph seems very complicated, it can actually be done in a very easy way if we assume that the surface is just a polyhedron (i.e. something like the Platonic solids).

And also, the idea is related to the idea that the "exterior angle sum of a polygon is

360 degrees".

## Angle defect for a corner of a polygon

The idea comes from copying the "exterior angle sum of a triangle = 360 degrees" proof. There, if we rethink the proofs carefully, we can see that actually we can define the curvature of a corner by either the corner angle itself, or by the number 180 degrees minus that corner angle.

Let's think about these two definitions one by one. If we use the first one, then the smaller the corner angle, the more curved the corner is. So we have to do something (like taking 1/corner angle) to get a good definition. If we use the second one, then, we get the following facts:

- If the corner is a point on a straight line, then the curvature is zero, because by our definition, the "curvature" is equal to "180 degrees minus that angle, which is now 180 degrees", i.e. we get a zero degree "angle defect".
- The sharper the corner is, the smaller the angle it subtends, hence 180 degrees minus that angle becomes larger, in line with our intuition.

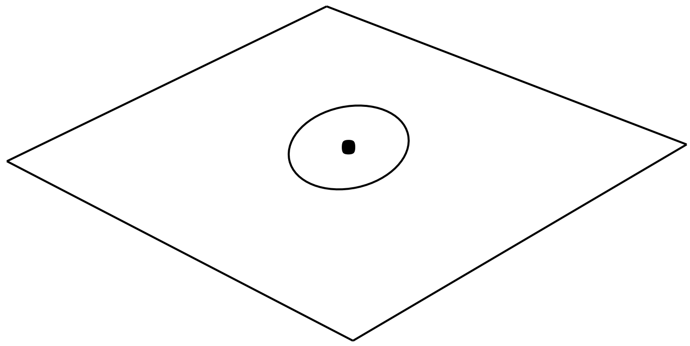
As we mentioned already, a good way to define "curvature" at a corner is to use the formula

**curvature = angle defect = 180 degrees minus corner angle**

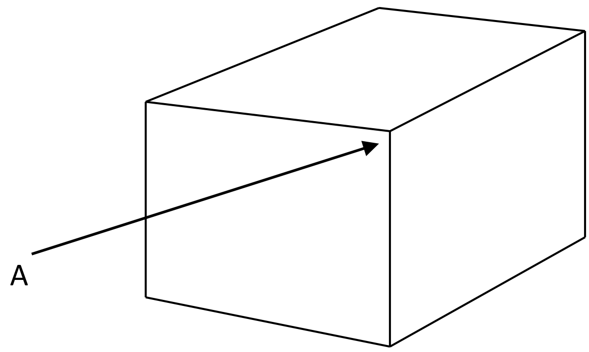
which can rightly be given the name "angle defect" of the corner.

## Angle defect for a vertex on a surface

Having more or less successfully defined "angle defect for a corner" on a polygon, we can go on to copy this idea to do similar thing on a polyhedron.
First, we note that if the polyhedron is just a plane, than every point on it is a "flat point", which has 360 degrees surrounding it (see figure below).



Now if we have a vertex (or corner), then we have the following situation:




Take for example the vertex A, there three right-angles meet, so vertex A contains altogether 3 × 90 degees = 270 degrees.
To get the angle defect, we imitate the polygon case and define the angle defect at the vertex to be

Angle defect = 360 degrees − angles meeting at the vertex

In this example, the angle defect (which is basically the "curvature") at A is then

360 degrees − 270 degrees = 90 degrees

Now this rectangular box has 8 such vertices, so the <u>total angle defect</u> is

8 x 90 degrees = 720 degrees

which is equal to

2x 180 degrees x (2− 2x number of holes)

=360 degrees x (2 − 2x 0 hole)

$$=720 \text{ degrees}$$

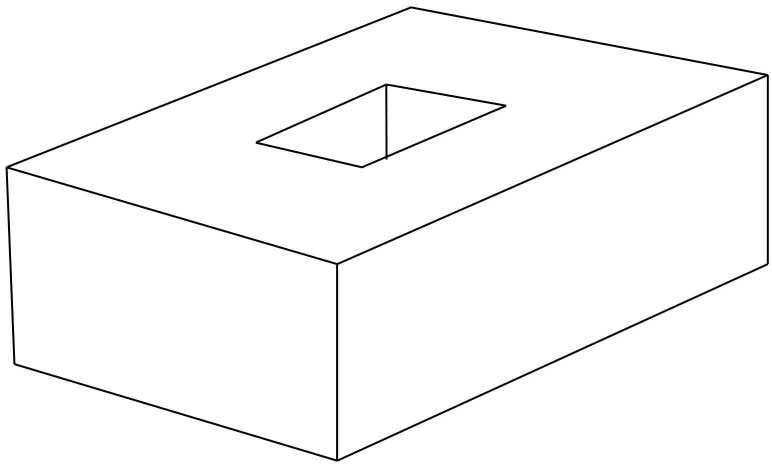
In agreement with the formula

$$\text{sum of angle defects} = 2\pi\,(2 - 2 \times \text{number of holes})$$

**Exercise.** One can easily verify that each outside vertex in the “torus” we saw before has “angle defect” equal to 90 degrees, whereas each inside vertex has “angle defect” equal to $-90$ degrees, resulting again in

$$0 = \text{sum of angle defects} = 2\pi\,(2 - 2 \times 1)$$

because now there is one hole.



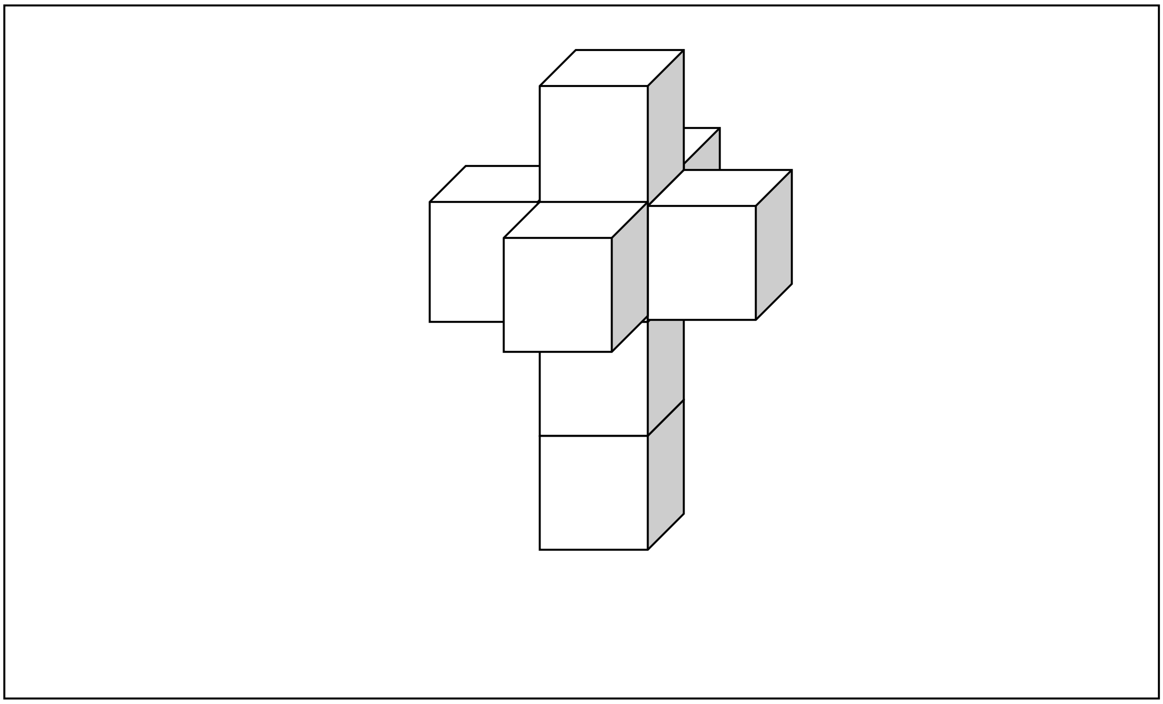
# Lecture 6

# The Fourth Dimension.

This lecture is about how one can visualize the “fourth dimension” (in short “4D”) in a more concrete way.

## 4D Cube (or “Hybercube”)

For most people, the fourth dimension seems “mysterious”, but indeed it is not so mysterious, and in fact one of the most famous 20th century surrealist Spanish painter Salvador Dali painted a 4D cube (better known as “hypercube”) in his picture of Jesus on the cross. In this picture, the cross onto which Jesus is nailed is actually a “hypercube”, i.e. a “cube-analogue” in the 4-dimensional space. (see the picture below for the “hypercube”).



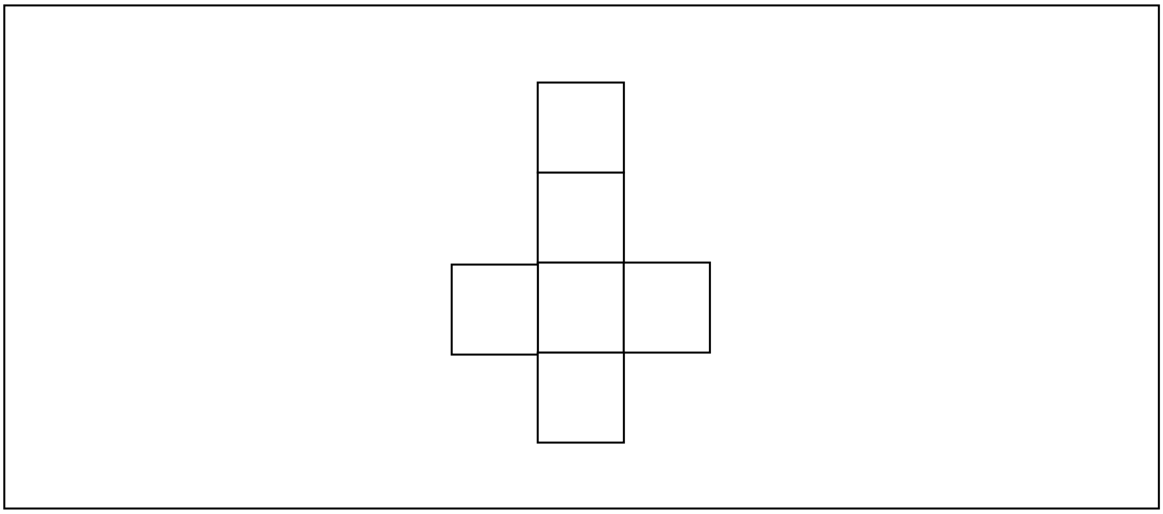
**Rationale behind the Dali 4D cube**

What is the idea behind this figure? More precisely, what justifications do we have to guarantee that this “collection” of cubes represents the 4D cube (i.e. “hypercube”)?

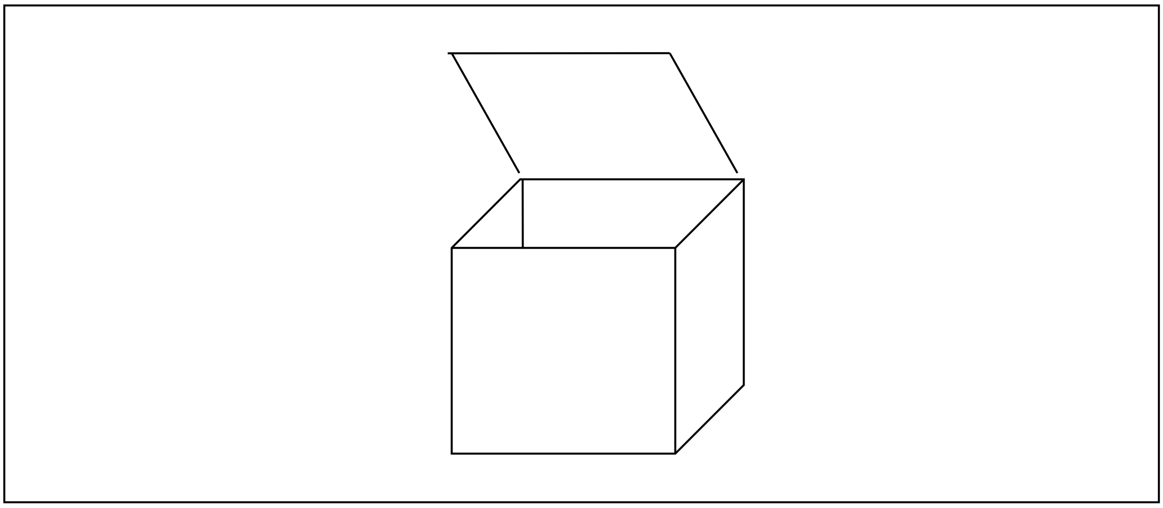
To get the right idea, we first try to understand how one can tell a person from the two-dimensional world (in short “2D world” or “flatland”) what a 3D cube (the usual cube is). The idea is simple:

Show them the blueprint of a 3D cube

By “blueprint”, we mean a floor-plan (or “assembling plan”), with the help of which one can build a cube.



Now this is a geometric object in the plan, hence can be understood by people living in the 2D world. But if one is living in the 3D world, one can get back the 3D cube by gluing sides.



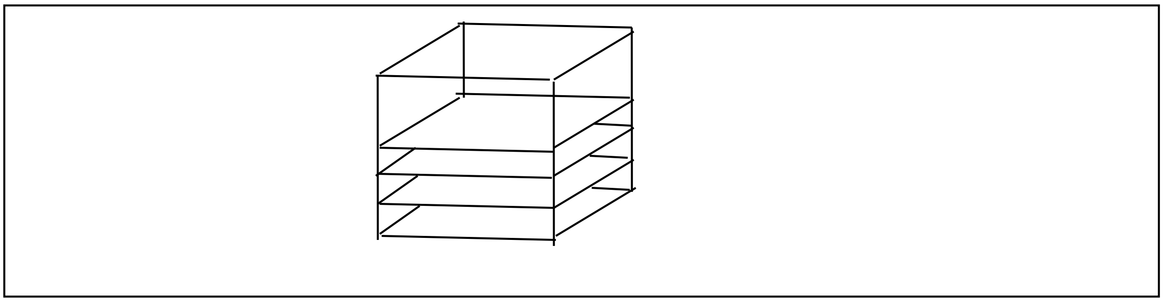
So gluing together various edges, one obtains the 3D cube even when one is living in the 2D world.

In much the same way, we glue various faces to arrive at a 4D cube.

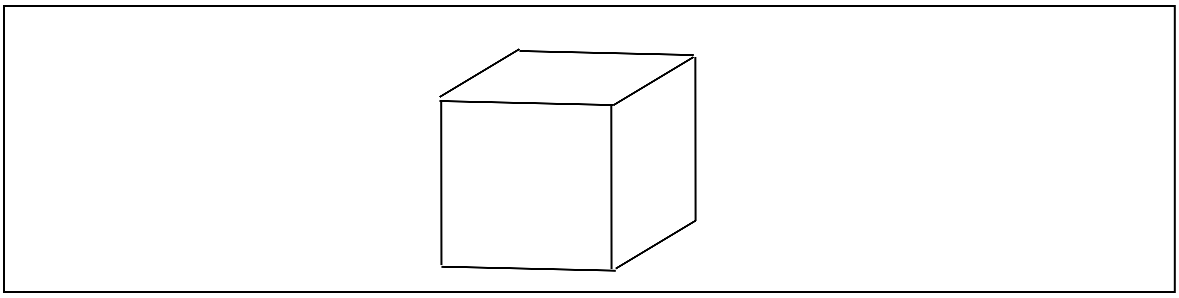
## Other Means of Representing the Hypercube

Having mentioned Dali's hypercube, we want to explore other ways to represent the 4D cube.

Again the idea comes from "thinking about the 3D cube from the perspective of the flatlanders (those people living in the flatland)" (See the picture below!)

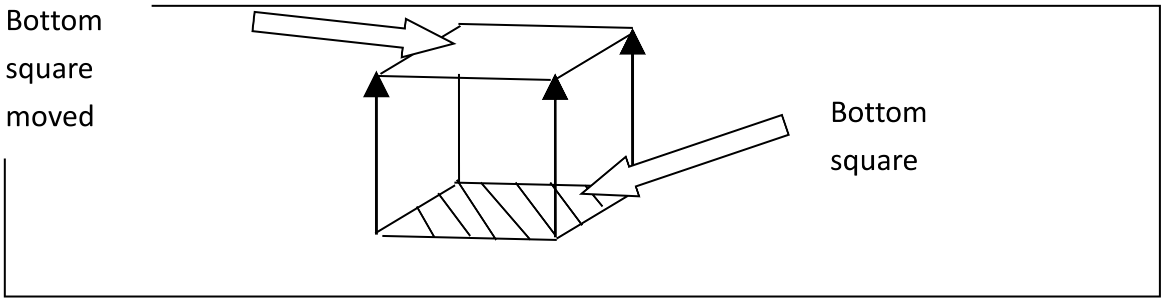


To obtain a cube in the 3D space, one can "stack" squares of lengths and widths 1 unit. We stack as many of these squares as we can, until their height become 1 unit. In that way, we obtain a cube. To explain this to a flatlander, one can for example show them a picture of the form

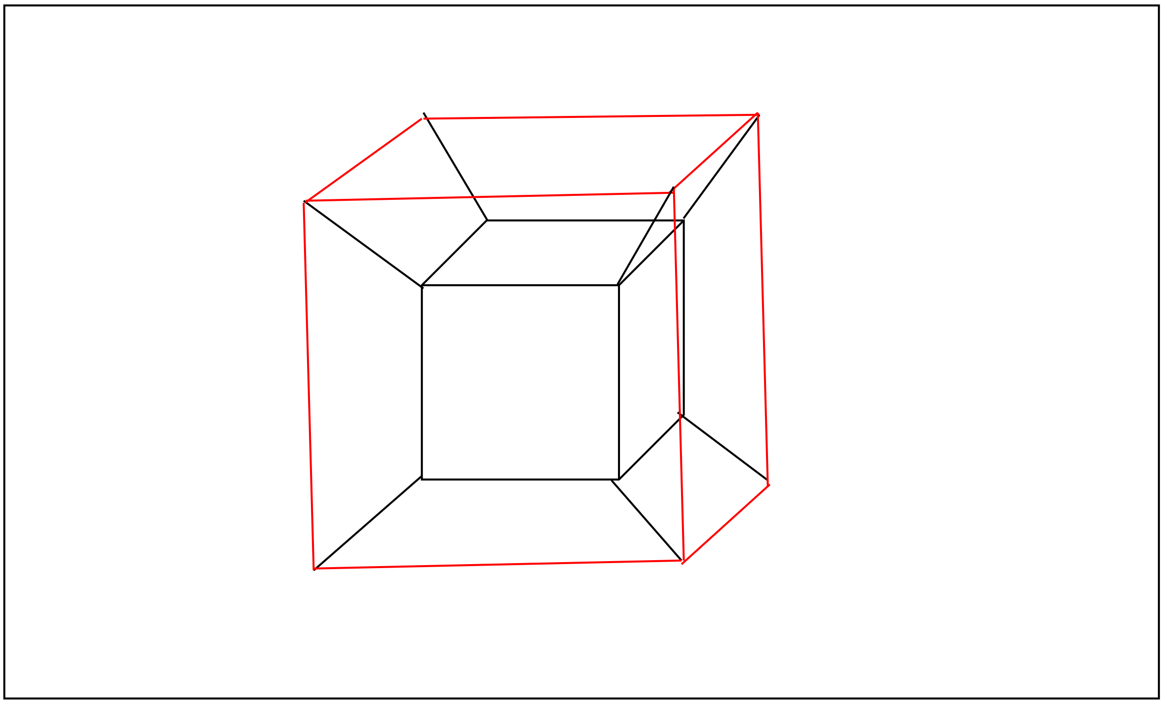


which is actually a drawing in the 2D plane (this is obvious, because we indeed have successfully drawn it on this sheet of paper, which is a 2D plane!) of an object in the 3D space.

Anther way of thinking about the above "stacking squares" method is to think of these stacks as "many squares" obtained by "moving up" the base square 1 unit (see the picture below!)




In the same way, one can make a 3D picture of a 4D cube (or “hypercube”) by “moving cubes” (rather than “squares”) in the following way (you can really make a 3D model of it!):



As one can easy see, here we are moving the center cube 1 unit up in the fourth dimension (which can of course not be seen in the 3D world!). So to draw such an “impossible” figure, we distort the lengths and obtain the moved red cube outside.

Of course, these cubes don’t look like a cube at all, because we are representing things in the 4D space (or “hyperspace”) in the plane, hence everything is distorted.

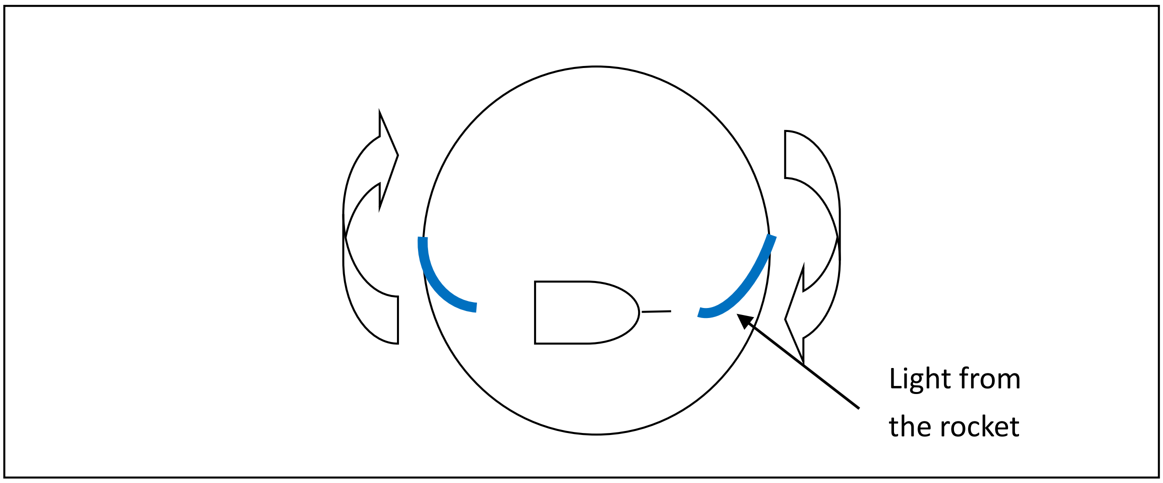
**Remark:** We have not put down the details here, but if you are interested you can take a look at the book "Flatland" and other books on how to visualize the fourth dimension.

**Remark:** At this point, you might have noticed that when thinking about things like the fourth dimension, mathematicians think using "analogy" or "metaphors", starting from their experience in working with similar (but simpler) objects in 3D or 2D.

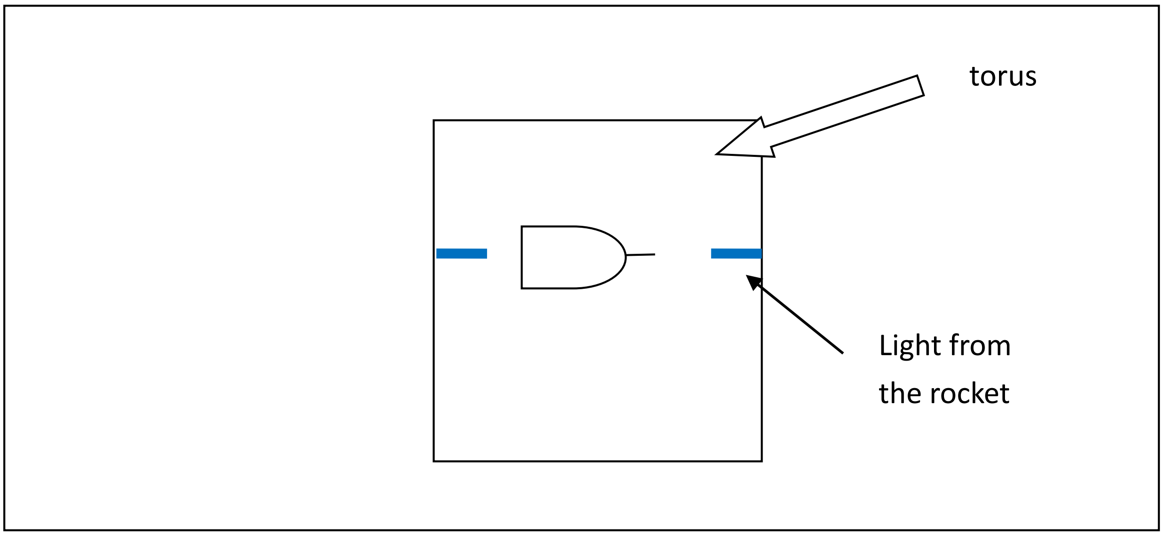
## Application to Physics

There are two rather simple-to-read and interesting books on the 2D plane and about the 4D space. The title of the former book is "Flatland", and the title of the second one is "The Shape of Space".

The book "The Shape of Space" is about one interesting thought experiment, namely the thought experiment that our universe, though seemingly infinite, may ultimately be finite, just like a sphere, or a cylinder. They only looks "infinite", because the light emitted from the back of a spaceship may be observed from the same space ship after the light has turned one full circle and reaches the front of the spaceship.




Similarly, if the space were a torus (i.e something like a donut), then similar phenomenon may happen. To represent a torus, one can use a square (the "fundamental domain" or "building block") and think of gluing together the top edge and the bottom edge, as well as the right-hand edge with the left-hand edge. This way, the light coming from the back of a spaceship may just turn around and reach the front of the same spaceship.

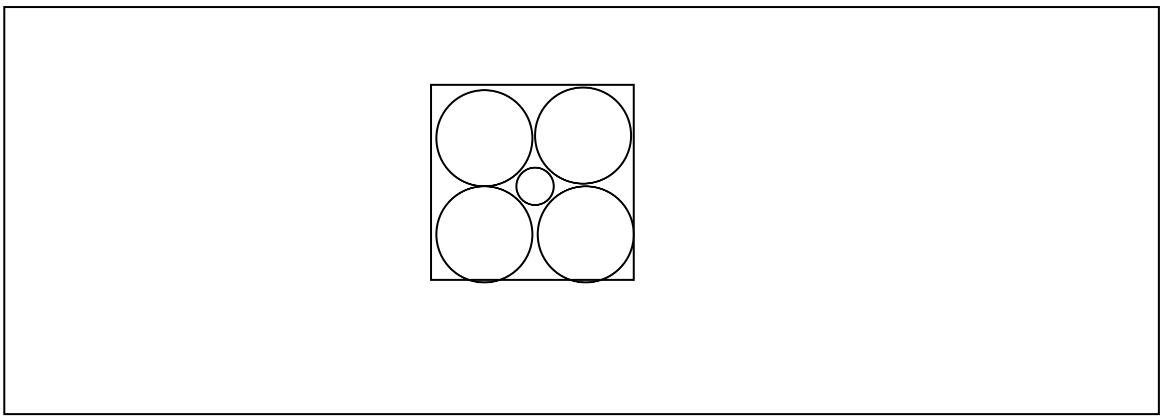



**Conlcusion:**

Such universes looks “infinite” but are actually finite.
In similar way, one can visualize a 3D torus, which is formed by taking a cube as a fundamental domain, then gluing the front face with the back face, the left face with the right face etc.
Doing this, one obtains something known as the “torus” in the 4D space.

**A mysterious phenomenon in the n-dimensional space**

Suppose now *n=2* and consider the following



**Question:**

Take a square of side lengths 4. Take 4 identical balls each of radius 1 in it. Next consider the gap in the middle of the square and fill it with a ball of largest radius. This ball has radius $\sqrt{2}-1$.

In a completely analogous way, one can show that in a 3D cube of side lengths 4, one can pack 8 identical balls of radii 1. In the gap in the center one can then put a ball of largest radius equal to $\sqrt{3}-1$.

Similarly, in an *n*-dimensional space, we can put an *n*-dimensional cube of side lengths 1, place $2^n$ identical *n*-dimensional balls of radii 1 in it. In the gap in the center, one can put an *n*-dimensional ball of largest radius $\sqrt{n}-1$.

Now comes the puzzling point, as *n* becomes larger than a certain number, the expression $\sqrt{n}-1$ will be larger than 2 (i.e half the size of the side lengths), therefore we have to conclude that the center *n*-dimensional ball has to pop out of the *n*-dimensional cube, counter to our intuition!

**Conclusion:**

Sometimes our intuition may mislead us to an incorrect conclusion, when it comes to imagining something like the four dimension or higher dimensions. One Novel Laureate has written a book related to this (not a math book though) called "Thinking: Fast and Slow".

# Lecture 7

# Symmetry

After describing some 4D and *n*-dimensional phenomena, we change the topic and talk about symmetry.

**Question:** What is symmetry? How can one define it?

For most people, when they are asked about the definition of "symmetry", they may think of "symmetry axis", "left hand versus right hand", "isosceles triangles", "circle" etc.

It is true that all the above are in one way or other related to the concept of symmetry, but can one really define "symmetry" in a more satisfactory way.

In this lecture, we will outline one way in which mathematicians managed to do so. And when we look at how they did it, we somehow may grasp the idea how mathematicians work out "definition" of some natural phenomena, just as "symmetry".

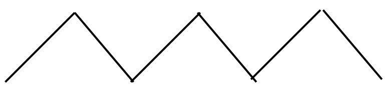
## Some Simple Examples

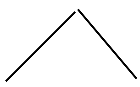
In the opening chapter of the book "Symmetry—a brief introduction", the author describes the symmetry in objects like "waves", "rainbow", "wheel" or the game "paper-scissors-rock".

For simplicity, let us just consider two out of these three examples.

**Waves**

Just take a wave which oscillates like the curve in the first picture below. One sees that here the pattern repeats itself after a certain time, i.e. the entire wave is just a repeated cutting and pasting of the fundamental shape in the second picture below.





**Wheels**

As all of us know, a wheel is of circular shape, and circles have rotational symmetry. What this means is that there is something called a "center", and every point on the circle can be rotated to another point on this circle.

But …when we replace a circle by a rectangle, this is no longer the case, so what symmetry does a rectangle possess?

By looking closely at the symmetry of rectangle and how mathematicians manage to bookkeep this symmetry using mathematical tools, we may grasp how mathematicians worked out their definition of symmetry of a mathematical object.

## Math in the Bedroom

"Mathematics in the Bedroom" is the title of a popular science article which describes in layman's language something known as "groups".
Groups are mathematical objects which mathematicians invented to help defining symmetry.

Consider a mattress.

A mattress is a rectangular object with four corners on the top and four at the bottom.

Let us label the four corners on the top face as A,B,C,D (counting clockwise from the top-left to the bottom-left).

Let us also label the four corners on the bottom faces by similar letters A',B',C' and D'.

So initially below A, we have A', below B, we have B', etc..

Now consider how you move the mattress so that after the motion, the mattress overlaps with the original position ("congruent" to the original mattress) We can bookkeep each of these transformations (=movement) by looking at something like

$$\begin{matrix} A & B & C & D \\ A' & B' & C' & D' \end{matrix}$$

These two rows of letters simply means the “initial configuration”.

If we now flip it along the long axis, we get

$$\begin{matrix} A & B & C & D \\ B' & A' & D' & C' \end{matrix}$$

To better record these transformations, let’s put brackets around the letters to form

$$\begin{pmatrix} A & B & C & D \\ B' & A' & D' & C' \end{pmatrix}$$

‘We can also give a symbol to each of these transformation, e.g. the first one,

$$\begin{pmatrix} A & B & C & D \\ B' & A' & D' & C' \end{pmatrix}$$

since it changes nothing, we call it the “identity transformation” and give it the notation

$$I$$

After giving notations, we can form a “multiplication table” of these symbols.

In conclusion, we have done the following:

- Consider a rectangle
- Consider each of the transformations which moves the rectangle *ABCD* to a position which is congruent to the initial position
- Bookkeep each such transformation by symbols like *I*, *R*, *M1*, *M2* (*I* = identity, *R*= rotation by 180 degrees, *M1*, *M2* denote respectively reflection about the long axis, reflection about the short axis
- Form a “multiplication table”

The above process is a typical process of building up the mathematical object known as “group”, an object used to describe symmetry.

Looking back, we see that in a "group", we have three things:

- Some symbols
- Some operations *, such as *M1*R*, meaning "rotate first, then reflect about the long axis"
- Performing one transformation after the other will not give a completely new transformation (in technical language, we say " it is closed under the operation *")
- The identity transformation does nothing new to each given transformation, i.e. for each transformation *T,* the following holds

$$T*I=T=I*T$$

- To each transformation, there is a backward (or "reverse") transformation, with which one can "undo" what one has just done, technically this is written in the form (for each transformation, say *T*, there is a reverse transformation, denoted by the symbol *Rev(T)*, such that the following holds:

$$T*Rev(T)=I$$

- Some other technical requirements, which are not important to us (e.g. (*M1*M2*)**R*=*M1**(*M2*R*)

## Comments

In the above example, we have considered a "group" consisting of "some kind of transformations" of the rectangle *ABCD*. We have also mentioned that a group has two ingredients, the elements in it (i.e. *R,M1,M2,I* in the above example) and an operation * between each pair of elements.

Also, the operation has to satisfy some rules as listed above.

### What benefits do we get from this?

- By using group and its "multiplication table", we can understand symmetry of a mathematical object.
- One can also "interpret" a group as a new kind of Number System, in which the "multiplication" is different from the usual one.
- One can also consider other groups, not immediately related to symmetry. For example, one can consider the following set of elements given by $S = \{e,o\}$ , where *e* means "even" number, *o* means "odd"

number. Let * means the usual addition (i.e. $+$) you learned in school, then we have the following "multiplication" table (which is also a group "multiplication" table).

| * | e | o |
|---|---|---|
| e | e | o |
| o | o | e |

## Other Examples

Instead of considering the symmetries (more technically, "symmetry group") of the rectangle, we can consider the symmetry group of an equilateral triangle and obtain another multiplication table, different from the one for the rectangle.

For each regular polygon (正多邊形), we can likewise obtain its symmetry group.

**Why is this important?**

In mathematics, symmetry considerations has to led to the successful proof of a theorem which says that "Not all fifth order polynomial equations can have its solutions representable in radical form. Fifth order polynomial equations are equations of the form

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

By "radical form" we mean the solutions are representable by formulas of finite length using $+, -, \times, \div, \sqrt[k]{\ }$ .

In school math, we know that this can be done for quadratic equations, i.e. the roots of the equation

$$ax^2 + bx + c = 0$$

are given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

Similar formulas (though much more complicated) exist for third order and fourth order polynomial equations, but not for fifth order ones.

This was shown by the French mathematician, Evariste Galois, whose used symmetry to show the impossibility of such formulas.

## More on Symmetry

In a city known as Granada in Spain, there is an ancient fortress called Alhambra (阿蘭布拉宮)(see the wiki page: en.wikipedia.org/wiki/**Alhambra** ) In there one can see wall painting by Islamic artists. Such wall painting design is also related to symmetry and groups.

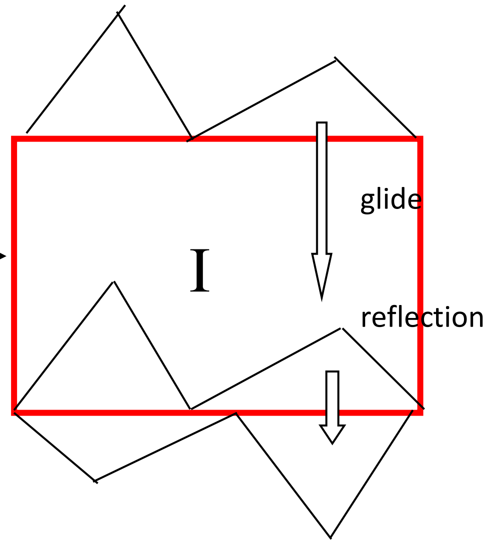
## Escher's Artwork

Related to the above painting in Alhambra is the Dutch painter M.C. Escher. Here is one of his works

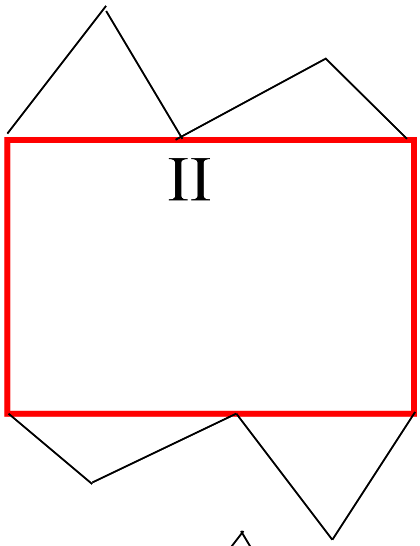
In Escher’s artwork, there are a lot of “symmetry” considerations. He might have started from some “fundamental building block” (let’s call it “fundamental domain”), such as a rectangle, draw some figure on it, move (or “glide”) it and reflect it, thus giving new shapes. E.g.

(Initial Figure)

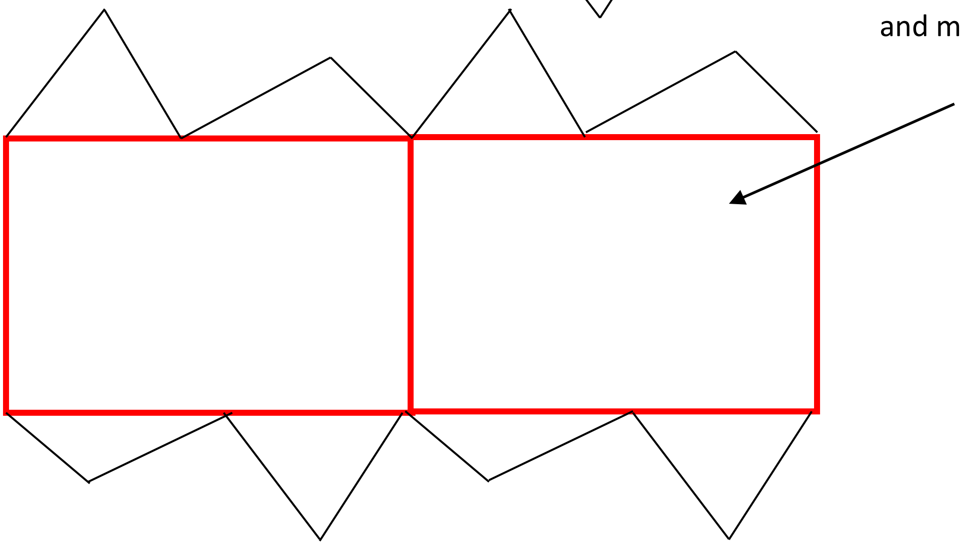
Fundamental Domain




(Figure after gliding and reflecting)




Cutting and pasting II lead to this figure and more.



If you are interested in Escher's Artwork, you can Google it and find a lot of materials.

One of the most interesting work of Escher is the following known as the "Print Gallery":

Some mathematicians have tried to figure out why there is a white dot in the middle of this picture. (You can find more about it by searching for the words "Escher Print Gallery" in the web.

## Penrose Tiling

Symmetry consideration as well as the Golden Number "Phi" are closely related to a wonderful discovery of Roger Penrose, known as "Penrose Tiling".

Before going into this, we have to explain the word "tiling". A tiling of the plane is a way to cover the plane completely by "tiles". A tile is a figure, such as a rectangle, which when cut-and-pasted, can cover the entire plane (no overlapping is allowed!)

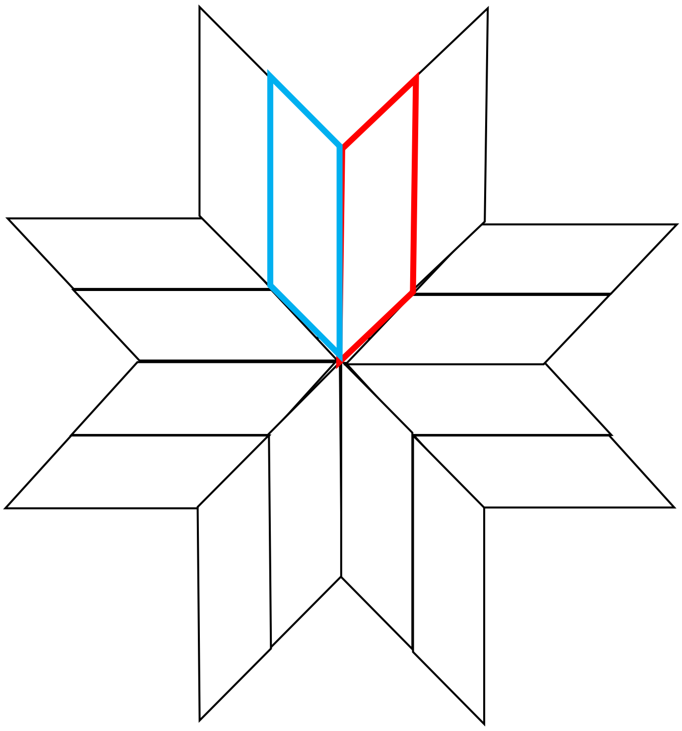
If we use a rectangle to tile the plane, we can cut and paste horizontally as well as vertically. So we say it has two periods.

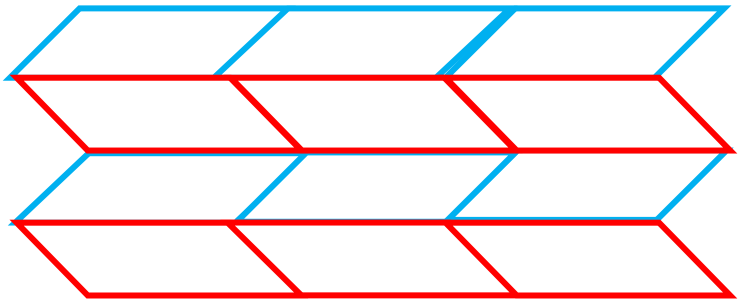
**Question:** Does there exist a way of tiling the entire plane by some number of tiles, so that it is non-periodic?

**Example:**

The figure here shows two parallelograms (the red one and the blue one), which together can form a non-periodic tiling of the plane.

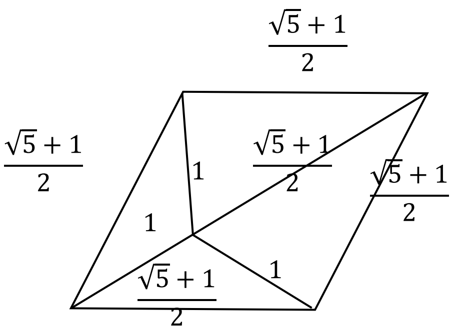
However, these two parallelograms can also form a periodic tiling as shown in the lower figure here.





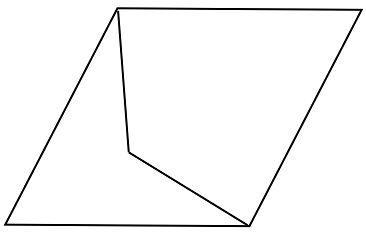
**Partial Answer:**

Before Penrose, some scientists managed to use a tile consisting of many many pieces to achieve this. Then came Penrose, who thought of the following tile consisting of two pieces, which can ONLY non-periodically tile the entire plane.




He calls one of this "kite", and the other "dart".

Here again, the Golden Number appears.



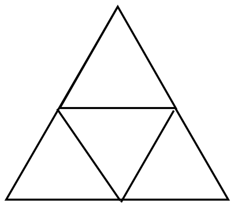
You can find more details in the attached appendix.

## Final Words about Symmetry – Fractals

One more application of symmetry is in fractal, a kind of mathematical object which might be related to the Golden Rectangle.

A fractal is easily created by transformations such as (i) scaling (ii) rotating (iii) translating.

One simple example is the following picture:



One can repeat the process and get "finer" and "finer" equivalteral triangles inside, one nested inside another.

### An Application

One really useful application of fractals is in computer art, one can generate pictures of trees, hairy animals, or natural landscapes using this idea of fractals.

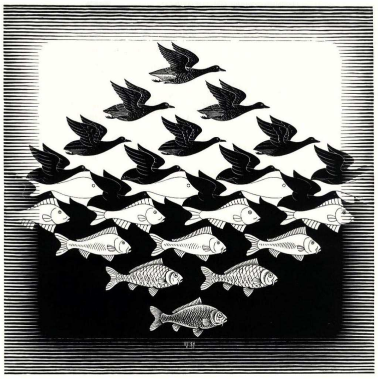
### Photo of "fractal" in nature

**Summary of Lecture 5, 6 & 7**

# Golden Section and Penrose Tiling

## 1 Alhambra and Escher's paintings

We mentioned at some point Escher's paintings and tiling of the plane. In addition to this, we mentioned that in the palace Alhambra in Granada, Spain, one can find Islamic patterns worked out by Islamic artists a long time ago. It is often said that all 17 patterns related to wall-paper designs (one can show that there only 17 such patterns, though the proof is complicated!) can be found in Alhambra. The following two quotes are related to Alhambra.



"Ornamental patterns are important in all cultures. Among the most famous are the Islamic patterns at Alhambra. This brings in symmetry groups $\cdots$" (by H. Aslaksen, NUS)

"In mathematics M. C. Escher's visit in 1922 and study of the Moorish use of symmetry in the Alhambra tiles inspired his subsequent work on regular divisions of the plane. These symmetric patterns are studied to find all seventeen possible symmetrical wallpaper tilings." (wikipedia, "Alhambra")

**Remark.** If you are interested in Prof. Aslaksen's work, the webpage is:
http://www.math.nus.edu.sg/aslaksen/teaching/math-art-arch.shtml#Symmetry

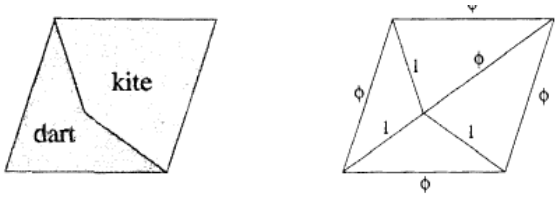
## 2 Penrose tiling

As mentioned last time, a key question in Recreational Mathematics was the following following :

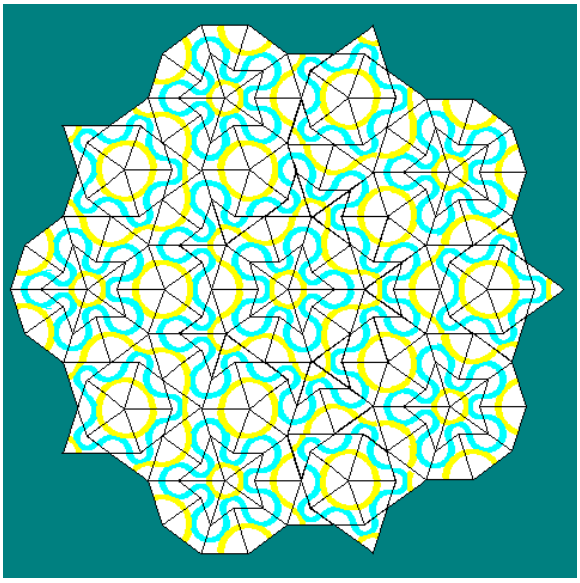
- Is there a collection of polygons, with which one can tile the plane only in a non-periodic way?

This question has been attempted by several mathematicians, first using computer. The best answer was given by Roger Penrose, who showed that it is possible to tile the plane that way using the Kite and the Dart, both being figures built using the Golden Numbers.

Here below is the pictures of (i) a Kite and (ii) a dart. Using them, one can construct polygons, which admits no periodic tiling, such as shown in the next picture (which is from the webpage http://www.uwgb.edu/dutchs/symmetry/penrose.htm).




The rules for attaching kites and darts to each other is rather complicated (see the notes given to you! **Note that the details is not important!**)



As mentioned, Penrose's construction uses the Golden Number. Can we use other irrational numbers to obtain similar results? This may be a question worth investigating.

# 3 Applications

Penrose tiling turns out to be related to something called 'quasicrystals' in chemistry.

**Remarks:**

1. The penrose tiling has a "pentagonal" symmetry.
2. Its fundamental building block consists of two objects, a kite and a dart.
3. It is related to something called "quasi-crystals" in Chemistry.

**References on Penrose tilings**

1. Wikipedia notes on 'aperiodic tiling'. http://en.wikipedia.org/wiki/Aperiodic_tiling
2. http://en.wikipedia.org/wiki/Aperiodic_tiling

# Lecture 10

# From Matrices to other Number-like Objects

In our previous lecture, we described matrices. Square matrices behave pretty much like numbers because one can add, subtract and multiply them. So it is fair to say that they are number-like objects.

One more thing which is useful is that one can use for example $2 \times 2$ matrices to represent imaginary number *I* in the following way.

Just think of each real number $a$ as a $2 \times 2$ matrix of the form $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the same way, we can think of each imaginary number *b* as a $2 \times 2$ matrix of the form $b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then one can represent the complex number *a+bi* by a $2 \times 2$ matrix.

## Quaternions and Matrices

Similar to complex numbers, Rowen Hamilton invented the quaternions, which are numbers of the form

$$a + bi + cj + dk$$

Where $i^2 = j^2 = k^2 = -1$ and $ij = k,\ jk = i, ki = j$ and if we compute $ji, kj, ik$ we get the $-k, -i, -j$.

A good thought-experiment is to find matrices which can represent these objects.

**Conclusion**

Starting from 1,2,3,… , then to rational numbers, irrational numbers, imaginary numbers, all the way to groups, matrices and quaternions, we've encountered different form of "numbers", from the most concrete ones defined by ancient civilizations to the modern ones.

Numbers are the most fundamental objects in mathematics, and even the oldest numbers, i.e. the natural (or "whole") numbers have deep and mysterious properties, such as the simple theorem which says "each whole number can be written as a product of prime numbers in a unique way".

Numbers are also related to Geometry in some intriguing ways, such as the relation between regular pentagon and the Golden Number, between pi and some infinite sums as given by the following formula dating back to Euler:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

## Some Further Thoughts

Here are some questions which might be interesting to some of the audience.

1. What are 1,2,3,… etc., are they "nouns", "adjectives" or …?
2. What are their nature?
3. Are there other ways to define them?
4. Are the ancient definitions of numbers a "must", or are they culture-dependent?
5. Why are numbers related to Geometry?
6. Why does mathematics work? Why is 1+1=2, 1+1000=1001, and it can go on for ever?
7. Is mathematics, just as Gallilei said in his "The Assayer" the language of nature?
8. Is it just accidental that some math works?

**Partial Answers**

It seems that 1,2,3, … were originally words referring to some concrete items. Later on, people abstracted them and gave them the status of "pure" numbers, thus leading to the philosophical puzzle we have today.

But still, whole numbers are mysterious, and it took many people like Dedekind, Frege etc. a lot of effort to try to pin them down.

One approach to do it is to use Set Theory. The philosophy here is the motto: "Everything is a set".

But this is problematic.

Another way is to build the language of mathematics is use the assumption that "everything is a function".

Yet another way is to say that “everything is a loop”. I.e. “1” is a loop which turns once, “2” is a loop that turns twice, etc.

This is the newest proposal to establish the foundation of mathematics and has applications in computer languages such as Coq.

Each of these ways has its own merit.

**Relation to Mathematical Way of Thinking**

Many people have thought about “mathematical ways of thinking”. One thing is clear, in math, one usually try to crystallize the most important properties of a subject matter, using the concepts at hand (i.e. relevant to that age in which the speaker is at), then derive consequence from them. Then one use “some” of these consequence as the new starting points …

In any case, mathematics is linked in a mysterious way to nature, and many natural phenomena can be modelled using mathematics.