The Concept of Derivative

Keywords:
The following keywords are important, when you come to 'compute' derivatives from First Principle. They are conceptually important too.

1. Difference Quotient (差商)
2. Tangent line
3. Derivative = $\boxed{\text{limiting value}}$ of Difference Quotient$\boxed{\text{s}}$ [1]

In our previous notes, we mentioned that it is useful to know whether the derivative of a function is always positive (or negative or not, so as to decide whether the function has an inverse function or not.

This is one of the motivations of studying the concept of derivative.

1 Definition of Derivative

Question:
What is the derivative of a function, say f, at a point c in its domain?

To answer this question, you may have seen a picture like this trying to explain what the derivative of f at c is:

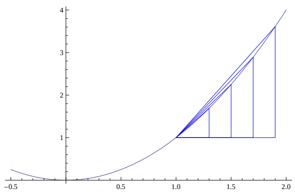

Figure 1: right-angled triangles whose slopes approximate the derivative

What the above picture aims to illustrate is this: to understand the $\boxed{\text{slope}}$ of the tangent line to a curve $y = f(x)$ at a point (in this case, the

[1] The word limiting value is usually called 'limit' in mathematics texts.

'common' left endpoint' of all the right-angled triangles in the picture!) one consider the 'collection' of this kind of right-angled triangle, compute the limiting values of the slope (if it exists!)

The above discussion leads to the following

Def.
Let $f : (a, b) \to \mathbb{R}$ and c be a point in (a, b). $\boxed{\text{If}}$ the following $\boxed{\text{difference quotients}}$ have a limit, i.e.

Notation: In the following lines, '$\to$' means 'tends to' or 'approaching' (it doesn't mean 'function' here!)

> As $h \to 0$ (from any direction, left or right, or sometimes left, sometimes right), the difference quotients (or simply 'quotients') $\frac{f(c+h)-f(c)}{h} \to$ some fixed 'finite' number L.

then we denote this 'limit' (i.e. L) by the symbol

$$\left.\frac{df}{dx}\right|_{x=c} \text{ (or } f'(c)\text{)}$$

Terminology.
$f'(c)$ is called

$\boxed{\text{The derivative of } f \text{ at } x = c}$ and f is said to be differentiable at $x = c$.

1.1 Geometric Meaning of $f'(c)$

> The derivative, i.e. $f'(c)$ is the $\boxed{\text{slope}}$ of the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$ (often times, people just write $x = c$.)

Comments:

1. The concept of slope is usually used only for a straight line.
2. The concept of tangent line is not easy to make clear, as the 2 pictures below show.

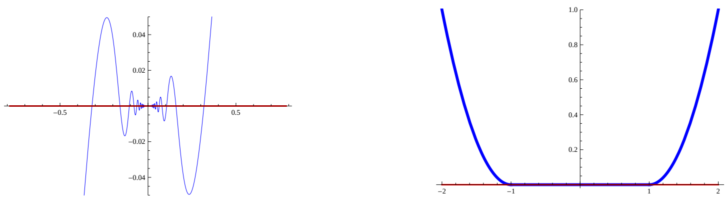

Figure 2: tg. line intersecting $y = f(x)$ in many points

1.2 Knowing Derivative without knowing Tangent Line

The important point is that: We can 'compute' the derivative $\boxed{\text{without}}$ knowing what the tangent line at $x = c$ is. An example will show this:

E.g.
Compute (from First Principle) the derivative of $f(x) = x^n$, $n = 1, 2, 3, \cdots$ at the point $x = c$ (c being any real number).

Solution: Consider the difference quotient

$$\frac{f(c+h) - f(c)}{h}, \text{where } h \neq 0.$$

In the future, we abbreviate this by the symbol

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=c}.$$

In this example, $f(x) = x^n$, so we obtain

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^n - c^n}{h}$$

After expanding and simplifying the numerator, we arrive at

$$= \frac{c^n + nc^{n-1}h + \boxed{\text{sum of terms like } c^{n-2}h^2, c^{n-3}h^3, \cdots, h^n} - c^n}{h}$$

$$= nc^{n-1} + h \cdot \boxed{\text{sum of terms like } c^{n-2}, c^{n-3}h, \cdots, h^{n-2}}$$

Finally, we take limit and let $h \to 0$ (meaning: h goes to zero 'from any direction') to obtain

$$\lim_{h\to 0}\left(nc^{n-1} + h \cdot \boxed{\text{sum of terms like } c^{n-2}, c^{n-3}h, \cdots, h^{n-2}}\right)$$

$$\begin{aligned}
&= \lim_{h\to 0} nc^{n-1} + \lim_{h\to 0}\left(h \cdot \boxed{\text{sum of terms like } c^{n-2}, c^{n-3}h, \cdots, h^{n-2}}\right) \\
&= nc^{n-1} + \lim_{h\to 0} h \cdot \lim_{h\to 0}\left(\boxed{\text{sum of terms like } c^{n-2}, c^{n-3}h, \cdots, h^{n-2}}\right) \\
&= nc^{n-1}
\end{aligned}$$

□

Comment:
The term

$$\boxed{\text{sum of terms like } c^{n-2}, c^{n-3}h, \cdots, h^{n-2}}$$

is a finite number because the sum is a 'finite sum'.

1.3 A Second Example

Next we consider how one can use 'First Principle' to compute the derivative of $f(x) = e^x$ at the point $x = 0$. This example shall demonstrate two things:

- how some kind of 'Sandwich Principle' can help us to compute limit
- that the sum-to-product rule $\exp(x + y) = \exp(x)\exp(y)$ is useful in computing derivative

E.g.

In the following, we will use the notations $\boxed{\exp(x)}$ and $\boxed{e^x}$ interchangeably.

We will show that if $f(x) = e^x$ then $f'(0) = e^0$.

Consider

$$\frac{f(0+h) - f(0)}{h}$$

and use the fact (Prove it yourself!) that:

$$1 + x \le e^x \le \frac{1}{1-x}, \text{ if } |x| < 1$$

Assuming the above inequalities are true, we can put $x = h$ in the above to obtain

$$1 = \frac{(1+h)-1}{h} \le \frac{e^{0+h}-e^0}{h} \le \frac{\frac{1}{1-h}-1}{h} \le \frac{\frac{1-(1-h)}{1-h}}{h} = \frac{h}{(1-h)h} = \frac{1}{1-h}$$

Letting $h \to 0$ in the above 'chains' of inequalities gives

$$\lim_{h\to 0} 1 \le \lim_{h\to 0} \frac{e^{0+h}-e^0}{h} \le \lim_{h\to 0} \frac{1}{1-h} = \frac{1}{\lim_{h\to 0}(1-h)} = 1$$

Hence the 'limit' of the middle term is $\boxed{\text{at the same time}}$ '≥ 1' and '≤ 1', forcing it to be 'equal to 1'.

<u>Conclusion:</u> $f'(0) = 1 = e^0$ □

<u>Comment:</u>
Using the formula $e^{c+x} = e^c e^x$, one can prove that if $f(x) = e^x$ and $x = c$ any real no., then

$$f'(c) = e^c.$$