Solutions to Test 2

Q1. (a) $\{A \subset X \mid A \ni \{\infty\}\}$

Pf. • If $\{\infty\} \in A$

$\forall$ non-empty O open

$\Rightarrow O = \{\infty\} \cup U$ for $U \in \mathcal{T}_{cf}$

$\Rightarrow \{\infty\} \in O \cap A \neq \emptyset$

$\Rightarrow \bar{A} = X$

• If $\{\infty\} \notin A$

$\{\infty\}$ is open & $\{\infty\} \cap A = \emptyset$

$\Rightarrow \{\infty\} \notin \bar{A}$

$\Rightarrow \bar{A} \neq X$

b). second cat.

If $X = \bigcup_{n=1}^{+\infty} N_n$

w/ N_n no where dense

Then $\infty \in N_k$ for a k

Then $\{\infty\} = (\{\infty\})^\circ \subset \left(\overline{\{\infty\}}\right)^\circ \subset \left(\overline{N_k}\right)^\circ = \emptyset$

Contradiction !

<u>(c)</u>. In general NOT true.

$(\mathbb{Q}, \mathcal{T}_{cf}) = \bigcup_{q \in \mathbb{Q}} q$ is 1st cat.

$(X, \mathcal{T}_\infty)$ is 2nd cat by (b).

Q2. a) Suppose $\{x_n\}$ is Cauchy seq.

$\Rightarrow \exists. N_1$ s.t. $m, n \geq N_1 \Rightarrow d(x_n, x_m) < \frac{1}{1}$

$\exists. N_2 > N_1$ s.t. $m, n \geq N_2 \Rightarrow d(x_n, x_m) < \frac{1}{2}$

$\vdots$

$\exists. N_k > N_{k-1}$ s.t. $m, n \geq N_k \Rightarrow d(x_n, x_m) < \frac{1}{k}$

Def $F_k \triangleq \{x_{N_k}, x_{N_k+1}, \cdots\}$

Then:

- $F_k \supset F_{k+1} \supset \cdots$
- $\operatorname{diam}(F_k) \leq \frac{2}{k}$

$\left(\text{Pf: } d(x_m, x_n) \leq d(x_m, x_{N_k}) + d(x_n, x_{N_k}) \leq \frac{2}{k} \right)$

$\Rightarrow \bigcap F_k = \{x\}$

Claim: $x_n \to x$

Pf: For $\forall \varepsilon$

$\exists k$ s.t. $\frac{3}{k} < \varepsilon$

When $n > N_k$

we have $x_n \in F_k$

But $x \in F_k$

So $d(x_n, x) \leq \operatorname{diam}(F_k) \leq \frac{2}{k} \leq \frac{2}{3}\varepsilon < \varepsilon$

for $\forall\ n > N_k$

So $x_n \longrightarrow x$ $\square$

(b) This follows from 2 claims:

<u>Clam 1</u>: $d(x, X \setminus G) \neq 0$ for $\forall\ x \in G$

Pf: If $\exists\, x \in G$ s.t. $d(x, X\setminus G) = 0$

Then for $\forall\ n \in \mathbb{Z}$

$\exists\, x_n \in X \setminus G$

s.t. $d(x_n, x) < \frac{1}{n}$

$\Longrightarrow$ $x_n \longrightarrow x$

But $X \setminus G$ is closed, $x \in X \setminus G$

$\downarrow$

$\uparrow$

$x \in G$

Claim 2 : $d(x, A)$ is cts for $\forall$ subset A

Pf: Let $z \in A$

$$d(x, z) \le d(y, z) + d(x, y)$$
$$\le d(y, A) + d(x, y)$$

$$\Rightarrow \quad d(x, A) = \sup_{z} d(x, z)$$
$$\le d(y, A) + d(x, y)$$

Similarly $d(y, A) \le d(x, A) + d(x, y)$

So $|d(x, A) - d(y, A)| \le d(x, y)$

$\Rightarrow$ $d(x, A)$ is cts function on X.

c) $D \triangleq \{ (x, \varphi(x)) \in X \times \mathbb{R} \mid x \in G \}$

Claim: D is closed in $X \times \mathbb{R}$

Pf: If $(x, y) \notin D$

Then $\varphi(x) \neq y$

So $\exists \, \varepsilon < 0$

s.t. $(\varphi(x) - \varepsilon, \varphi(x) + \varepsilon) \cap (y - \varepsilon, y + \varepsilon)$

$= \emptyset$

φ is cts $\Rightarrow$

$\exists$. U open nbd of x

s.t. $\varphi(U) \subset (\varphi(x)-\varepsilon,\ \varphi(x)+\varepsilon)$

Then $O \triangleq U \times (y-\varepsilon,\ y+\varepsilon)$ is open

nbd of (x, y) satisfying

$$(x, y) \in O \subset D^c$$

$\Rightarrow$ D^c is open

$\Rightarrow$ D is closed $\square$

X, $\mathbb{R}$ are complete metric space

$\Rightarrow$ $X \times \mathbb{R}$ is complete space

(exercise)

So closed D is complete metric space

Def a map

$$f: G \longrightarrow D$$
$$x \longmapsto (x, \varphi(x))$$

It is easy to show it is homemorphism.

Q3. (a) $\{\frac{1}{n}\} = (\frac{1}{n} - \frac{1}{4n^2}, \frac{1}{n} + \frac{1}{4n^2}) \cap J$

$\Rightarrow$ $\{\frac{1}{n}\}$ is open for $\forall n$

$\Rightarrow$ J has discrete topo.

(b) see tutorial

(c) By tutorial

$$(A \times B)^\circ = A^\circ \times B^\circ$$

$$\Rightarrow \mathrm{Frt}(A \times B) = \overline{(A \times B)} \setminus (A \times B)^\circ$$

$$= (\bar{A} \setminus A^\circ \times \bar{B}) \cup (\bar{A} \times \bar{B} \setminus B^\circ)$$

$$= (\mathrm{Frt}(A) \times \bar{B}) \cup (\bar{A} \times \mathrm{Frt}(B))$$

Q4. (a) Let $q \triangleq p|_{\mathbb{Q}} : \mathbb{Q} \to (\mathbb{R}, \mathcal{T}_q)$

Def a map:

$$f : (\mathbb{R}, \mathcal{T}_q) \longrightarrow (\mathbb{R}, \mathcal{T}_{std})$$

$$x \longmapsto x$$

- f is bij ✓

- f is cts

For $\forall$ $(a,b) \in \mathcal{T}_{std}$

$$f^{-1}((a,b)) = (a,b) \in \mathcal{T}_q$$

since $q^{-1}(a,b) = \underbrace{(a,b)\times\mathbb{R}}_{\text{open in } \mathbb{R}^2} \cap Q$

is open in Q

- f is open

Suppose $O \in \mathcal{T}_q$ & $x \in O$

$\Rightarrow q^{-1}(O)$ is open in Q

$\Rightarrow \exists\, U$ open in $\mathbb{R}^2$

s.t. $q^{-1}(O) = U \cap Q$

Note that $(x,0) \in q^{-1}(O) = U \cap Q$

$\Rightarrow (x,0) \in U$

$\Rightarrow \exists\, \varepsilon > 0$

s.t. $(x,0) \in B_{(x,0)}(\varepsilon) \subset U$

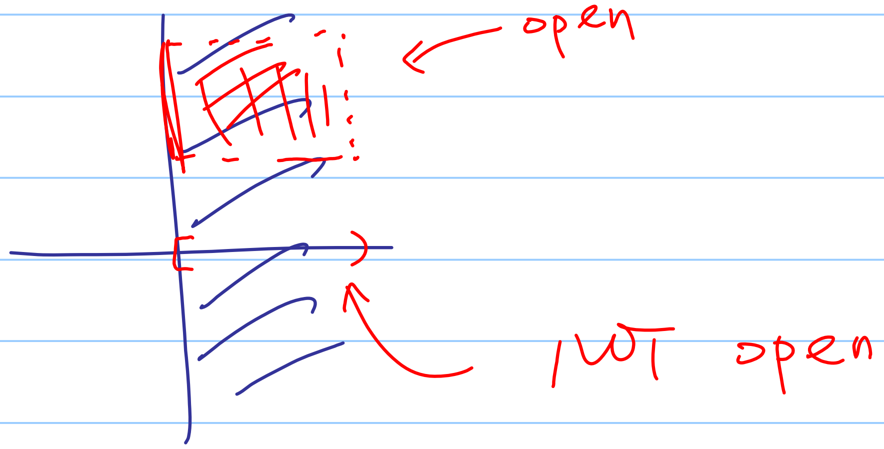
$$\Rightarrow (x-\varepsilon, x+\varepsilon)\times\{0\} \subset B_{(x,0)}(\varepsilon)\cap Q \subset U\cap Q = q^{-1}(O)$$

$$\Rightarrow (x-\varepsilon, x+\varepsilon) \subset O$$

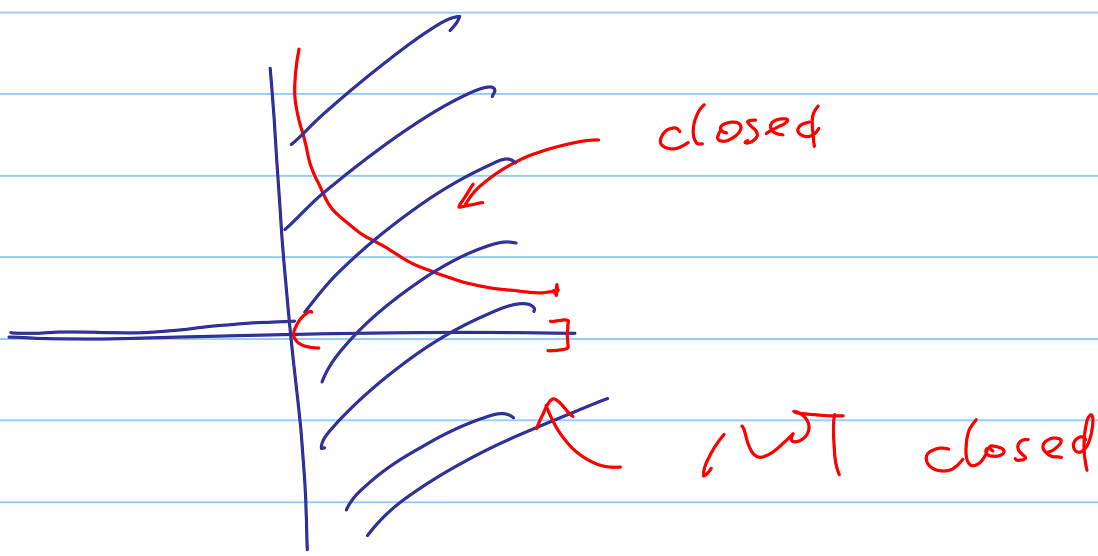
$$\Rightarrow x \in (x-\varepsilon, x+\varepsilon) \subset O$$

$$\Rightarrow O \in \mathcal{T}_{std}$$

b). NOT open



NOT closed



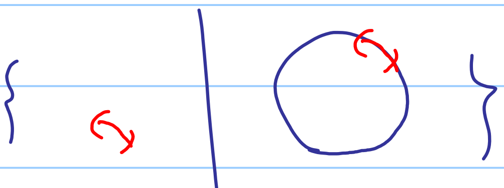
Q5 (a) Follows from : (See last tutorial)

<u>Claim:</u> $f : (\mathbb{R}^1, \mathcal{T}_{std}) \longrightarrow (S^1, \mathcal{T}_{std})$

$$x \longmapsto e^{2\pi i x}$$

is quotient map

- suj ✓
- cts

$\mathcal{B} = \{$  $\}$ is a base of $(S^1, \mathcal{T}_{std})$

$f^{-1}(\mathcal{B}) \ni$  open in $\mathcal{T}_{std}$

- $f^{-1}(O)$ open $\Longrightarrow$ O open

In fact f is open map :

$$\mathcal{B}' = \left\{ (a, b) \;\middle|\; \begin{array}{l} a, b \in \mathbb{R} \\ b - a < \pi \end{array} \right\}$$

is a base of $(\mathbb{R}^1, \mathcal{T}_{std})$

b). Follows from: (See last tutorial)

Claim:

$$f : X \longrightarrow (S^1, \mathcal{T}_{std})$$

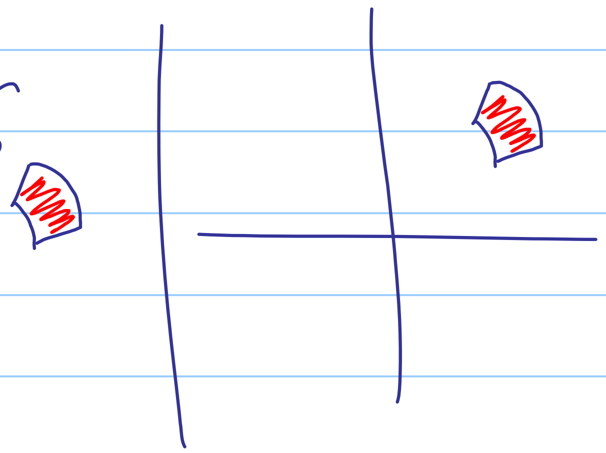
$$x+iy = z \longmapsto \frac{z}{|z|}$$

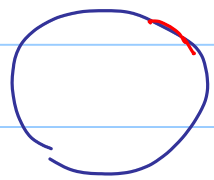
is quotient map

- surj ✓
- cts

  z, $|z|$ are cts

- open

$\mathcal{B} = \Big\{$  $\Big\}$ is a base of X

$f(\mathcal{B}) =$ 

c). Follows from : (See last tutorial)

Claim: $f: X \longrightarrow (S^1, \mathcal{T}_{std})$

$$\underset{\substack{x+iy \\ \shortparallel \\ z}}{} \longmapsto \left(\frac{z}{|z|}\right)^2$$

is quotient map

Pf: Similar to b)