1. Let $f : (X, d_X) \to (Y, d_Y)$ be a continuous mapping ; let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X.

(a) Is $\{f(x_n)\}_{n\in\mathbb{N}}$ a Cauchy sequence in Y ?

(b) What additional condition on f and Y will guarantee that $\{f(x_n)\}$ is convergent?

(a) No. In general, $\{f(x_n)\}_{n\in\mathbb{N}}$ may not be Cauchy.

For example. $f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. by $f(x) = \tan x$ $\quad \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Let $x_n = \tan^{-1}(n)$, $\forall n \in \mathbb{N}$.

$x_n \to \frac{\pi}{2}$ in $(\mathbb{R}, \text{standard})$ $\Rightarrow$ $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. and $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$f(x_n) = n \to \infty$ as $n \to \infty$ $\Rightarrow$ $\{f(x_n)\}_{n\in\mathbb{N}}$ is not cauchy.

(b) f is uniformly cts. and Y is complete.

Pf. f is uniformly cts $\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $d_X(x_1, x_2) < \delta$, then $d_Y(f(x_1), f(x_2)) < \varepsilon$

$\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(X, d_X) \Rightarrow \exists N \in \mathbb{N}$. s.t. $\forall m, n \geq N$, $d_X(x_m, x_n) < \delta \Rightarrow d_Y(f(x_m), f(x_n)) < \varepsilon$

i.e. $\{f(x_n)\}_{n\in\mathbb{N}}$ is Cauchy in (Y, d_Y)

Y is complete $\Rightarrow$ $\{f(x_n)\}_{n\in\mathbb{N}}$ is convergent in (Y, d_Y).

2. a) U is open dense in X, show that U^c is nowhere dense in X.

b) Show that $\forall x \in \mathbb{R}^n$, $\{x\}$ is nowhere dense.

c) Show that $(\mathbb{Z}, d_{\mathbb{Z}})$ is complete, where $\mathbb{Z}$ is integer and $d_{\mathbb{Z}}$ is restriction of standard metric.

a) Suppose U^c is not nowhere dense

i.e. $\exists x \in \operatorname{int}(\overline{U^c}) = \operatorname{int}(U^c)$ $\quad (\because U$ is open$)$

$\Rightarrow \exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subset U^c \Rightarrow x \in B(x, \varepsilon) \neq \phi$, $B(x, \varepsilon) \cap U = \phi$.

Contradiction ! U^c is nowhere dense.

b) $\forall y \neq x \in \mathbb{R}^n$, take $\varepsilon = \frac{1}{2} d(x, y) > 0$, clearly. $B(y, \varepsilon) \cap \{x\} = \phi$

$\therefore \{x\}$ is closed $\Rightarrow$ $\overline{\{x\}} = \{x\}$

$\forall \varepsilon > 0$, $z = x + (\frac{\varepsilon}{2}, 0, \cdots, 0) \in B(x, \varepsilon)$, but $z \neq x$ $\Rightarrow$ $B(x, \varepsilon) \not\subset \{x\}$, $\forall \varepsilon > 0$

$\Rightarrow x \notin \operatorname{int}\{x\}$ $\Rightarrow$ $\phi = \operatorname{int}\{x\} = \operatorname{int}\overline{\{x\}}$

c) let $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{Z}, d_{\mathbb{Z}})$

i.e. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N$. $d_{\mathbb{Z}}(x_n, x_m) = |x_m - x_n| < \varepsilon$

Let $\varepsilon = \frac{1}{2}$, $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$ $d_{\mathbb{Z}}(x_n, x_{n_0}) = |x_n - x_{n_0}| < \frac{1}{2} < 1$

$\{x_n\}_{n \geq n_0} \subset \mathbb{Z} \Rightarrow x_n = x_{n_0} \in \mathbb{Z}$ $\forall n \geq n_0$ $\Rightarrow$ $x_n \to x_{n_0}$ as $n \to \infty$.

Remark: 3b) $\mathbb{Z} = \bigcup_{n\in\mathbb{Z}} \{n\}$ is of Cat-I. in $\mathbb{R}$.

3c) By Baire Category Theorem, $(\mathbb{Z}, d_{\mathbb{Z}})$ complete $\Rightarrow$ $\mathbb{Z}$ is of Cat-II of itself.

3 a) Show that $\mathbb{Q}$ is of Cat-I in $\mathbb{R}$, and $\mathbb{Q}^c$ is of Cat-II in $\mathbb{R}$

b) Show that any open ball in $\mathbb{R}^n$ is of Cat-II in $\mathbb{R}^n$.

c) Show that every nonempty open set in $\mathbb{R}^n$ is of Cat-II in $\mathbb{R}^n$.

a) $\because$ $\mathbb{Q}$ is countable and $\{r\}$ is nowhere dense in $\mathbb{R}$. $\forall r \in \mathbb{R}$.

$\therefore \mathbb{Q} = \bigcup_{r\in\mathbb{Q}} \{r\}$ is of Cat-I.

Suppose $\mathbb{Q}^c$ is of Cat-I. i.e. $\exists N_i$ be nowhere dense in $\mathbb{R}$ s.t. $\mathbb{Q}^c = \bigcup_{i=1}^{\infty} N_i$

$\therefore \mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ is of Cat-I of itself.

But $(\mathbb{R}, d)$ is complete $\Rightarrow$ $\mathbb{R}$ is of Cat-II of itself. Contradiction!

b). Suppose $\exists$ open ball $B = B(x_0, \varepsilon_0) \subset \mathbb{R}^n$ is of Cat-I in $\mathbb{R}^n$.

Let $T_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{x_0}(x) = x - x_0$. $\forall x \in \mathbb{R}^n$

$T_{x_0} \circ T_{-x_0} = T_{-x_0} \circ T_{x_0} = Id_{\mathbb{R}^n} \Rightarrow T_{x_0}$ is one to one, onto. $(*)$

Use $(*)$ to show ①, ② and ③ with ε-δ language.

① $T_{x_0}(B(x, \varepsilon)) = B(x - x_0, \varepsilon)$. See the third page.

② $T_{x_0}(\operatorname{int} A) = \operatorname{int}(T_{x_0}(A))$.

③ $T_{x_0}(\overline{A}) = \overline{T_{x_0}(A)}$

Using ①, ② and ③.

N_i is nowhere dense in $\mathbb{R}^n$, $\forall i \Rightarrow \operatorname{int}(\overline{T_{x_0}(N_i)}) = T_{x_0}(\operatorname{int}(\overline{N_i})) = \phi$

$\Rightarrow T_{x_0}(N_i)$ is nowhere dense in $\mathbb{R}^n$ $\forall i \Rightarrow B(0, \varepsilon_0)$ is of Cat-I in $\mathbb{R}^n$

Similarly, consider $f_c : \mathbb{R}^n \to \mathbb{R}^n$ by $x \mapsto cx$ $\forall c > 0$.

we have $B(0, \delta)$, $\forall \delta > 0$ are of Cat-I in $\mathbb{R}^n$

$\mathbb{R}^n = \bigcup_{i=1}^{\infty} B(0, i)$ is of Cat-I in $\mathbb{R}^n$

But $(\mathbb{R}^n, d)$ is complete $\Rightarrow$ $(\mathbb{R}^n, d)$ is of Cat-II of itself. contradiction!

c). Suppose Open $U \subset \mathbb{R}^n$ and $U \neq \phi$ is of Cat-I in $\mathbb{R}^n$

then $\exists$ open ball $B \subset U$ and $U = \bigcup_{i=1}^{\infty} N_i$, where N_i is nowhere dense, $\forall i$.

$B = U \cap B = \bigcup_{i=1}^{\infty} (N_i \cap B)$

$N_i \cap B \subset N_i \Rightarrow \operatorname{int}(\overline{N_i \cap B}) \subset \operatorname{int}(\overline{N_i}) = \phi \Rightarrow \forall i$, $N_i \cap B$ is nowhere dense in $\mathbb{R}^n$

$\Rightarrow B$ is of Cat-I in $\mathbb{R}^n$. Contradition!

4. Let $f : (X, d_X) \to (Y, d_Y)$ be a continous mapping ; $N, D \subset X$ are respective nowhere dense and dense subsets.

a) Is $f(N)$ nowhere dense in Y? b) In addition, assume that f is surjective. Prove that $\overline{f(D)} = Y$

a) No. e.g.: $f : (\mathbb{R}, d_{std}) \to (\{0\}, d_{\{0\}})$ $x \mapsto 0$ $\forall x \in \mathbb{R}$.

Note: $\{x\}$ is nouhere dense in $(\mathbb{R}, d_{std})$, but $f(\{x\}) = \{0\}$ is dense in $(\{0\}, d_{\{0\}})$

b) Let $y \in Y$. f is noto $\Rightarrow \exists x \in X$ s.t. $f(x) = y$

$\overline{D} = X \Rightarrow x \in \overline{D} \Rightarrow \exists \{x_n\}_{n\in\mathbb{N}} \subset D$ s.t $x_n \to x$ as $n \to \infty$

f is cts $\Rightarrow f(x_n) \to f(x) = y$ and $\{f(x_n)\}_{x\in\mathbb{N}} \subset f(D) \Rightarrow y = f(x) \in \overline{f(D)}$

$\Rightarrow Y \subset \overline{f(D)} \Rightarrow \overline{f(D)} = Y$.

$T_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{x_0}(x) = x - x_0$.

① $T_{x_0}(B(x,\varepsilon)) = B(x - x_0, \varepsilon)$

let $y \in B(x - x_0, \varepsilon)$

$\Leftrightarrow d(x - x_0, y) < \varepsilon$

$\Leftrightarrow \left(\sum_{i=1}^{n} (x^i - x_0^i - y^i)^2\right)^{\frac{1}{2}} < \varepsilon$

$\Leftrightarrow d(x, y + x_0) < \varepsilon$

$\Leftrightarrow \exists z = y + x_0 \in B(x,\varepsilon)$ s.t $T_{x_0}(z) = y$

$\Leftrightarrow y \in T_{x_0}(B(x,\varepsilon))$

② $T_{x_0}(\text{int}(A)) = \text{int}(T_{x_0}(A))$

Let $x \in \text{Int}(T_{x_0}(A))$

$\Leftrightarrow \exists \varepsilon > 0$ s.t. $B(x,\varepsilon) \subset T_{x_0}(A)$

$\Leftrightarrow \exists \varepsilon > 0$ s.t. $T_{x_0}(B(x + x_0), \varepsilon)) \subset T_{x_0}(A)$

$\Leftrightarrow \exists \varepsilon > 0$, s.t. $B(y = x + x_0, \varepsilon) \subset A$

$\Leftrightarrow \exists y \in \text{int} A$ s.t. $T_{x_0}(y) = x$

$\Leftrightarrow x \in T_{x_0}(\text{Int} A)$

③ $T_{x_0}(\bar{A}) = \overline{T_{x_0}(A)}$

Let $x \in \overline{T_{x_0}(A)}$

$\Leftrightarrow \forall \varepsilon > 0$, $B(x,\varepsilon) \cap T_{x_0}(A) \neq \phi$

$\Leftrightarrow \forall \varepsilon > 0$, $T_{x_0}(B(x + x_0, \varepsilon) \cap A) \neq \phi$

$\Leftrightarrow \forall \varepsilon > 0$, $B(x + x_0, \varepsilon) \cap A \neq \phi$

$\Leftrightarrow \exists y = x + x_0 \in \bar{A}$ s.t. $T_{x_0}(y) = x$

$\Leftrightarrow x \in T_{x_0}(\bar{A})$.