Tutorial 2015/1/20

Q1. Let (X, d) be a metric space & $A \subset X$. Show that the derived set A' is closed.

Pf: Let $a \notin A'$

$\Rightarrow \exists\, \varepsilon > 0$

s.t $B(a, \varepsilon) - \{a\} \cap A = \emptyset$

Claim: $a \in B(a, \varepsilon) \subset (A')^c$

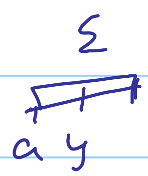
$\Rightarrow (A')^c$ is open $\Rightarrow A'$ is closed

Pf of claim:

Consider $y \in B(a, \varepsilon)$

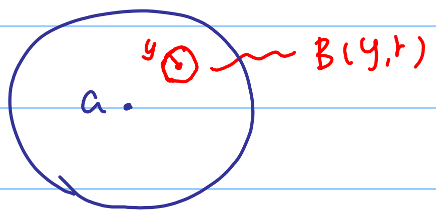
Case 1: $y = a \notin A'$

Case 2: $y \neq a$



$$(B(y, r) - \{y\}) \cap A = \emptyset$$

where $r \triangleq \min\left\{\dfrac{d(a,y)}{2}, \dfrac{\varepsilon - d(a,y)}{2}\right\}$



$\square$

Q2. $\mathcal{T}_{std} \triangleq$ standard topo on $\mathbb{R}$.

$$\mathcal{T} \triangleq \{ G \cup A \mid G \in \mathcal{T}_{std},\ A \subset \mathbb{R} \setminus \mathbb{Q} \}$$

(a) Show $\mathcal{T}$ is topo on $\mathbb{R}$

(b). Is $(\mathbb{R}, \mathcal{T})$ 2nd countable?

(a) • $\emptyset, X \in \mathcal{T}$

• If $G_\alpha \cup A_\alpha \in \mathcal{T}$

then $\bigcup_\alpha (G_\alpha \cup A_\alpha) = (\bigcup_\alpha G_\alpha) \cup (\bigcup_\alpha A_\alpha) \in \mathcal{T}$

• If $G_1 \cup A_1 \in \mathcal{T}$, $G_2 \cup A_2 \in \mathcal{T}$

then $(G_1 \cup A_1) \cap (G_2 \cup A_2)$

$$= \underbrace{(G_1 \cap G_2)}_{\substack{\cap \\ \mathcal{T}_{std}}} \cup \underbrace{\left[(G_1 \cap A_2) \cup (G_2 \cap A_1) \cup (A_1 \cap A_2)\right]}_{\substack{\cap \\ \mathbb{R} \setminus \mathbb{Q}}}$$

$\Longrightarrow (G_1 \cup A_1) \cap (G_2 \cup A_2) \in \mathcal{T}$

(d) No.

<u>Pf</u>: If $\mathcal{B} = \{B_1, B_2, \cdots\}$ is a base of $\mathcal{T}$

then $\forall\ t \in \mathbb{R} \setminus \mathbb{Q}$

$\exists\ n_t$

s.t. $t \in B_{n_t} \subseteq \{t\}$

i.e. $\forall\ t \in \mathbb{R} \setminus \mathbb{Q}$ $\exists.\ n_t$ s.t. $B_{n_t} = \{t\}$

$\Longrightarrow$ we have a injective map

$$\{\mathbb{R} \setminus \mathbb{Q}\} \hookrightarrow \{B_1, B_2, B_3, \cdots\}$$

$$t \longmapsto B_{n_t}$$

uncountable $\longrightarrow \longleftarrow$ countable

Q3. $\mathcal{T}_f$: cofinite topo on $\mathbb{R}$. Show that:

(a). $(\mathbb{R}, \mathcal{T}_f)$ is separable

(b). $(\mathbb{R}, \mathcal{T}_f)$ is NOT 1st countable

Pf: (a) Claim: $\overline{\mathbb{Z}} = \mathbb{R}$ in $(\mathbb{R}, \mathcal{T}_f)$

let $U \in \mathcal{T}_f$ s.t. $U \neq \emptyset$

$\Rightarrow \# U^c < +\infty$

$\overset{\# \mathbb{Z} = +\infty}{\Longrightarrow} \mathbb{Z} \not\subset U^c$

$\Rightarrow U \cap \mathbb{Z} \neq \emptyset$

(b). Suppose $(\mathbb{R}, \mathcal{T}_f)$ is 1st countable

let $\mathcal{B}_{x_0}$ = countable local base of x_0

$\Rightarrow \forall, x \neq x_0$

$\exists. U_x \in \mathcal{B}_{x_0}$

s.t. $x_0 \in U_x \subset \mathbb{R} \setminus \{x\}$

$$\Rightarrow \{x_0\} \subset \bigcap_{x \neq x_0} U_x \subset \bigcap_{x \neq x_0} \mathbb{R} \setminus \{x\} = \{x_0\}$$

$$\Rightarrow \mathbb{R} \setminus \{x_0\} = \bigcup_{x \neq x_0} U_x^c \subset \bigcup_{B \in \mathcal{B}_{x_0}} B^c$$

uncountable $\longrightarrow \longleftarrow$ countable

Q4. Let $(X, \mathcal{T})$ be a 1st countable space

s.t X is countable

Show: $(X, \mathcal{T})$ is 2nd countable

Pf: let $X = \{x_1, x_2, x_3, \cdots\}$

let $\mathcal{B}_i$ be countable local base of x_i

let $\mathcal{B} = \bigcup_{i=1}^{+\infty} \mathcal{B}_i$

Easy to show:

- $\mathcal{B}$ is countable
- $\mathcal{B}$ is a base of $(X, \mathcal{T})$