Lect06-0128-Continuity

Monday, February 1, 2016 10:16 AM

For the mapping $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$,
$\qquad\qquad\qquad\qquad\quad \underset{x_0}{\cup} \qquad\qquad\qquad\qquad \underset{f(x_0)}{\cup}$

it is continuous at x_0 if

$\forall\ V \in \mathcal{T}_Y$ with $f(x_0) \in V$,

$\exists\ U \in \mathcal{T}_X$ with $x_0 \in U$, $\quad f(U) \subset V$;

it is continuous (everywhere) if

$\forall\ V \in \mathcal{T}_Y, \quad f^{-1}(V) \in \mathcal{T}_X$

Example. $f : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

* As $(\mathbb{R}, \mathcal{T}_{std}) \longrightarrow (\mathbb{R}, \mathcal{T}_{std})$, it is not continuous everywhere

* As $(\mathbb{R}, \text{discrete}) \longrightarrow (\mathbb{R}, \mathcal{T})$, it is continuous (everywhere) because

$f^{-1}(V) \in \mathcal{P}(\mathbb{R})$ in any case

* As $(\mathbb{R}, \mathcal{T}) \longrightarrow (\mathbb{R}, \text{indiscrete})$, it is continuous (everywhere) because

$\mathcal{T}_Y = \{\emptyset, \mathbb{R}\}$ and $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\mathbb{R}) = \mathbb{R}$

This shows that continuity is not only about the mapping, it is about the topologies.

Example.

Let $X = \{$ continuous functions on $[a,b]\}$

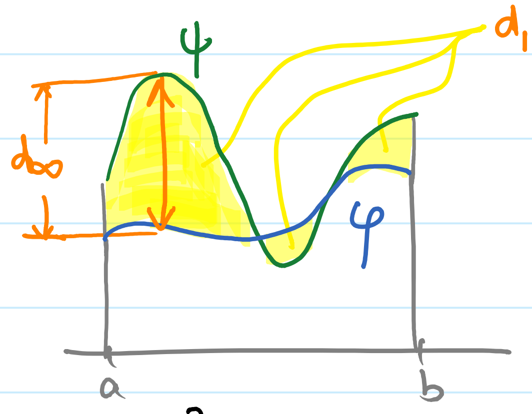
There may be different topologies on X

① L_1-topology $\mathcal{T}_1$ determined by the metric

$$d_1(\varphi, \psi) = \int_a^b |\varphi(t) - \psi(t)|\, dt$$

It is measuring the "area" between the functions

② Uniform Topology $\mathcal{T}_\infty$ determined by



$$d_\infty(\varphi, \psi) = \sup\{|\varphi(t) - \psi(t)| : t \in [a,b]\}$$

③ $id: (X, \mathcal{T}_\infty) \longrightarrow (X, \mathcal{T}_1)$ is continuous

Given any $\varepsilon > 0$ and any $\varphi \in X$

Take $\delta = \dfrac{\varepsilon}{b-a}$. Then for $\psi \in X$

satisfying $d_\infty(\psi, \varphi) < \delta = \dfrac{\varepsilon}{b-a}$

$$d_1(\psi, \varphi) = \int_a^b |\psi(t) - \varphi(t)|\, dt < \frac{\varepsilon}{b-a} \int_a^b dt = \varepsilon$$

④ Bad Situation

$id : (X, \mathcal{T}_1) \longrightarrow (X, \mathcal{T}_\infty)$ is not continuous

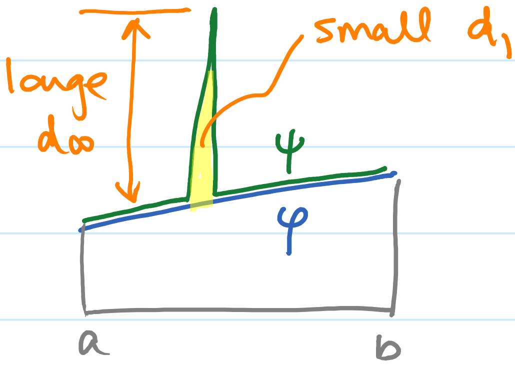
$\exists\ \varepsilon = 1$ such that

$\forall\ \delta > 0$, take $\frac{1}{n} < \delta$

and choose $\psi_n \in X$ with

$$\int_a^b |\psi_n(t) - \varphi(t)|\, dt < \frac{1}{n} < \delta$$

but $d_\infty(\psi_n, \varphi) > 1 = \varepsilon$



In terms of open sets, let

$$V = \{\psi \in X : d_\infty(\psi, \underbrace{\varphi}_{id(\varphi)}) < 1\} \in \mathcal{T}_\infty$$

For any $U \in \mathcal{T}_1$

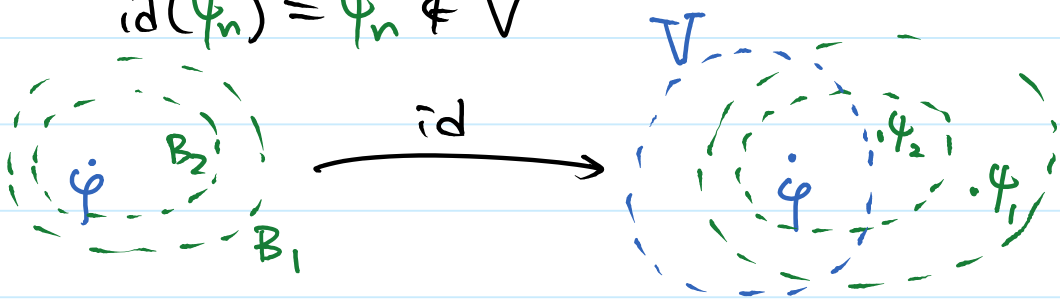
with $\varphi \in U$, we can insert a base set,

$$B_n = \{\psi \in X : d_1(\psi, \varphi) < \frac{1}{n}\} \quad \text{i.e.}$$

$$\varphi \in B_n \subset U$$

Also, we can construct $\psi_n \in B_n$ but

$$id(\psi_n) = \psi_n \notin V$$



Theorem. The following statements are equivalent.

① $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ is continuous at $x\in X$ $\forall x$.

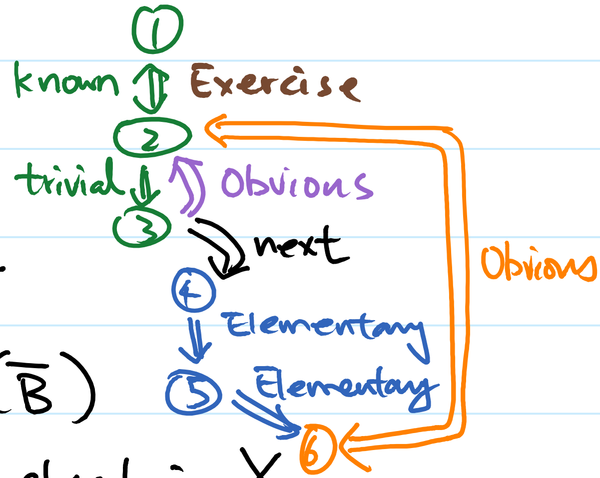
② $\forall V\in\mathcal{T}_Y$, $f^{-1}(V)\in\mathcal{T}_X$

③ $\forall B\in\mathcal{B}_Y$, $f^{-1}(V)\in\mathcal{T}_X$

④ $\forall A\subset X$, $f(\overline{A})\subset\overline{f(A)}$

⑤ $\forall B\subset Y$, $\overline{f^{-1}(B)}\subset f^{-1}(\overline{B})$

⑥ $\forall$ closed $H\subset Y$, $f^{-1}(H)$ is closed in X



③ $\Rightarrow$ ② Let $V\in\mathcal{T}_Y$. Then $V=\bigcup_\alpha B_\alpha$, $B_\alpha\in\mathcal{B}_Y$

$$f^{-1}(V)=f^{-1}\Big(\bigcup_\alpha B_\alpha\Big)=\bigcup_\alpha \underbrace{f^{-1}(B_\alpha)}_{\text{each in } \mathcal{T}_Y}$$

② $\Leftrightarrow$ ⑥ Take complement

④ $\Rightarrow$ ⑤ Take $B=f(A)$

⑤ $\Rightarrow$ ⑥ Take $H=B=\overline{B}$

} Exercises