Lect04-0121-Base

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Recall that for a given $\mathcal{S} \subset \mathcal{P}(X)$, one may create $\mathcal{B} = \{\bigcap \mathcal{F} : \text{finite } \mathcal{F} \subset \mathcal{S}\}$

$$= \{S_1 \cap \cdots \cap S_n : S_k \in \mathcal{S}\}$$

Then $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\} = \{\bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B}\}$

is the topology generated by $\mathcal{S}$ and $\mathcal{S}$ is a subbase of $\mathcal{T}$.

Definition. Let $\mathcal{T} \subset \mathcal{P}(X)$ be a topology and $\mathcal{B} \subset \mathcal{T}$. If $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$ then $\mathcal{B}$ is called a base (or basis) of $\mathcal{T}$.

For the standard topology $\mathcal{T}_{std}$ of $\mathbb{R}$

$\mathcal{B}_1 = \{\phi\} \cup \{(a,b) : a < b \in \mathbb{R}\}$ is a base

$\mathcal{B}_2 = \{\phi\} \cup \{(q_1, q_2) : q_1 < q_2 \in \mathbb{Q}\}$ is also a base.

Qu Given $\mathcal{S} \subset \mathcal{P}(X)$, we know that it always generates a topology $\mathcal{T}$. How do we know that $\mathcal{S}$ itself is indeed a base of $\mathcal{T}$.

Of course, if $\mathcal{S}$ is closed under finite intersection then it is already a base.

Qu Is there another way to express the condition?

Theorem. $\mathcal{S} \subset \mathcal{P}(X)$ is a base for a topology if

(i) $\phi, X \in \mathcal{S}$

(ii) For each $U, V \in \mathcal{S}$ and $x \in U \cap V$, $\exists\, W \in \mathcal{S}$ such that $x \in W \subset U \cap V$.

Exercise Given an example of $\mathcal{S}$ that satisfies (ii) but $\phi, X \notin \mathcal{S}$.

Idea of proof. Define

$\mathcal{J} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{S}\}$ and see if it is a topology. The crucial condition is finite intersections.

Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{S}$, we need

$$(\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2) \in \mathcal{S}$$
$$\|$$
$$\bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta) \text{ where } P_\alpha \in \mathcal{A}_1, Q_\beta \in \mathcal{A}_2$$

The problem is that no guarantee of $P_\alpha \cap Q_\beta \in \mathcal{S}$

By assumption (ii) in the Theorem, for $x \in P_\alpha \cap Q_\beta$ $\exists$ $W(x) \in \mathcal{S}$, depending on x, such that

$$x \in W(x) \subset P_\alpha \cap Q_\beta$$

Consider $x \in (\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2) \longmapsto W(x) \in \mathcal{S}$

Then
$$\bigcup_x W(x) \subset \bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta) = (\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2)$$
$$\bigcup$$
$$\{x : x \in (\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2)\}$$

Thus, $(\bigcup \mathcal{A}_1) \cap (\bigcup \mathcal{A}_2) \in \mathcal{S}$.

This shows that $\mathcal{S}$ is closed under finite intersection, and obviously also under arbitrary union.

A similar variation of the question

Qu. Given a topology $\mathcal{T}$ and a subset $\mathcal{B}$, how do we know that $\mathcal{B}$ is a base.

Theorem. $\mathcal{B} \subset \mathcal{T}$ is a base $\iff$

$\forall\ G \in \mathcal{T}$ and $x \in G$, $\exists\ U \in \mathcal{B}$, $x \in U \subset G$

Exercise. Prove the above.

Definition. Let $x \in X$. A local base (or nbhd base) at x is a set $\mathcal{U}_x \subset \mathcal{P}(X)$ such that $\forall$ nbhd N of x (i.e., $x \in \mathring{N}$) $\exists\ U \in \mathcal{U}_x$, $x \in U \subset N$

Example. In a metric space

$\mathcal{U}_x = \{B(x, \frac{1}{n}) : 1 \leq n \in \mathbb{Z}\}$ is a local base

A topological space $(X, \mathcal{T})$ is 2^{nd} countable if $\mathcal{T}$ has a countable base. It is 1^{st} countable if at every $x \in X$, it has a countable local base.