Lect03-0119-OpenClosed

Tuesday, January 19, 2016 8:55 PM

Let $\mathcal{T}$ be a topology for X, the space is denoted $(X,\mathcal{T})$. Any set $G\in\mathcal{T}$ is open.

For $x\in X$ and $N\subset X$, if $\exists\, U\in\mathcal{T}$ such that

$$x\in U\subset N$$

N is called a neighborhood of x, while x is an interior point of N, denoted $x\in\mathring{N}$

Let $A\subset X$. The set $\mathring{A}$ or $\mathrm{Int}(A)$ is called the interior of A, containing all interior points of A

Theorem $\mathring{A}$ is the largest open subset of A.

The essential fact,

$$\mathring{A}=\bigcup\underbrace{\{G\subset A : G\in\mathcal{T}\}}$$

1. each $G\subset A$, $\therefore$ union $\subset A$
2. each $G\in\mathcal{T}$, $\therefore$ union is open
3. Any open subset of A is used, $\therefore$ union is the largest

Theorem. A set $G\subset X$ is open

$\overset{\text{by above}}{\iff}$ Every $x\in G$ is an interior point of G

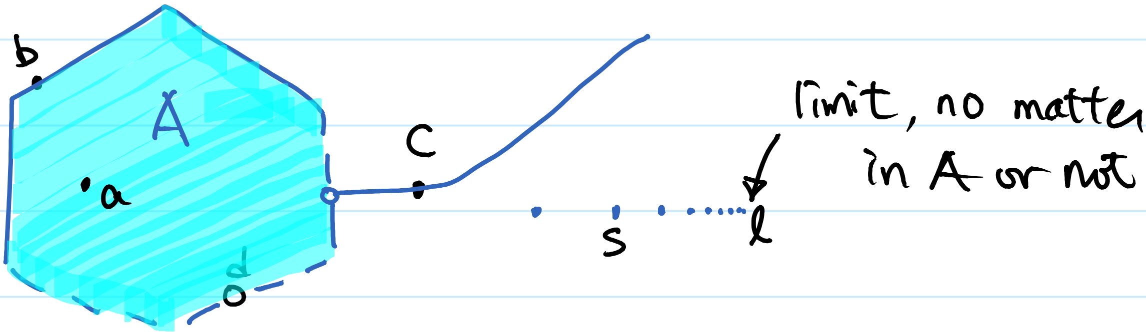
$\iff G\subset\mathring{G}\iff G=\mathring{G}$

by def of interior $\qquad$ $G\supset\mathring{G}$ is known

Qu. What is the purpose of discussing open sets towards the aim of limits, convergence, and approximation?

Targets of topology

Let us draw a set A in $\mathbb{R}^2$



There are different types of point

$a, b, c, s \in A$, $d \notin A$, $\ell \in X = \mathbb{R}^2$

For any $x \in X$, it is called a cluster point or accumulation point (or limit point) if for each $\underbrace{U \in \mathcal{J} \text{ with } x \in U}_{\text{or nbhd of } x \text{, i.e., } x \in \mathring{U}}$, $U \cap A \setminus \{x\} \neq \phi$

Qu. In the above picture of $\mathbb{R}^2$, which points are cluster points of A?

Answer. a, b, c, d, ℓ are, s is not.

Definition. $A' = \{\text{all cluster points of } A\}$ is called the derived set of A. Moreover, $\bar{A}$ or $Cl(A) = A \cup A'$ is called the closure of A.

Exercise. Convince yourself that

$$x \in \bar{A} \iff \forall\, U \in \mathfrak{J} \text{ with } x \in U,\ U \cap A \neq \emptyset$$

In layman's words, $\bar{A}$ contains points in A and points "stick to" A.

Then, the points "separating" A and its outside is defined below.

A point $x \in A$ is called a Frontier (or boundary) point of A if $x \in \bar{A} \cap \overline{(X \setminus A)}$. The frontier is denoted $\mathrm{Frt}(A)$. Obviously, $x \in \mathrm{Frt}(A)$ $\iff \forall\, U \in \mathfrak{J}$ with $x \in U$,

$$U \cap A \neq \emptyset \quad \text{and} \quad U \cap (X \setminus A) \neq \emptyset$$

We do not use "boundary" here because it causes confusion when manifold is involved.

For example, $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ and

$$\mathbb{D}^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$$

In $X = \mathbb{R}^2$, $\mathrm{Frt}(S^1) = S^1$ and $\mathrm{Frt}(\mathbb{D}^2) = S^1$

But, as manifolds,

S^1 has no boundary (notation $\partial S^1 = \emptyset$)

and $\mathbb{D}^2$ has boundary S^1 (notation $\partial \mathbb{D}^2 = S^1$)

A subset $A \subset X$ is closed if its complement $X \setminus A$ is open, i.e. $X \setminus A \in \mathfrak{J}$

The first observation:

ϕ, X are closed as $X = X\setminus\phi$, $\phi = X\setminus X \in \mathfrak{J}$

Thus, open and closed are not negation to each other. In fact, there are other both open and closed subsets.

Qu. Give an easy example of neither open nor closed subsets.

Theorem. A set $F \subset X$ is closed

$$\Leftrightarrow F = \bar{F} \Leftrightarrow F \supset \bar{F} \Leftrightarrow F \supset F'$$

easy because $\bar{F} \supset F$, $\bar{F} = F \cup F'$

Since $\bar{F} \supset F$ is known, the issue lies at $x \notin \bar{F}$.

Logically, $x \notin \bar{F} \Leftrightarrow \exists\, U \in \mathfrak{J}$ with $x \in U$, $\underbrace{U \cap F = \phi}_{U \subset X\setminus F}$

$$\Leftrightarrow x \in U \subset X\setminus F \text{ and } U \in \mathfrak{J}$$

$$\Leftrightarrow x \in (X\setminus F)^\circ$$

Thus, $X\setminus\bar{F} = (X\setminus F)^\circ$, $\therefore \bar{F} = X\setminus(X\setminus F)^\circ$

F is closed $\Leftrightarrow X\setminus F$ is open

$$\Leftrightarrow X\setminus F = (X\setminus F)^\circ$$

$$\Leftrightarrow \bar{F} = X\setminus(X\setminus F) = F$$ Done.

Theorem. $\bar{F}$ is the smallest closed set containing F.

Obviously, due to $(X\setminus F)^\circ$ is the largest open subset of $X\setminus F$.

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Think about the possible topology for X

The largest : $\mathcal{P}(X)$ Discrete

The smallest : $\{\phi, X\}$ Indiscrete

There may be others in between

① Suppose there is $\phi \neq A \subsetneqq X$, then

$\{\phi, A, X\}$ clearly satisfies the conditions

② Assume we have $\phi \neq A \neq B \subsetneqq X$, then

$\{\phi, A, B, X\}$ is not enough, we need

$\{\phi, A\cap B, A, B, A\cup B, X\}$

Qu. Given a subset $\mathcal{S} \subset \mathcal{P}(X)$, how to

get a minimal topology $\mathcal{T} \supset \mathcal{S}$?

, otherwise $\mathcal{P}(X) \supset \mathcal{S}$ always.

Naturally, by brute force, try all combinations of arbitrary union and finite intersection.

Qu. Is there a simple and systematic way?

Answer. Step 1 $\longrightarrow$ Get finite intersections of $\mathcal{S}$

Step 2 $\longrightarrow$ Get all unions of step 1 DONE

Theorem. Given any $\mathcal{S} \subset \mathcal{P}(X)$, let

$$\mathcal{B} = \{\cap \mathcal{F} : \text{finite } \mathcal{F} \subset \mathcal{S}\} = \{S_1 \cap \cdots \cap S_n : S_k \in \mathcal{S}\}$$

$$\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\} = \{\bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B}\}$$

Then $\mathcal{T}$ is the smallest topology containing $\mathcal{S}$;

or $\mathcal{T}$ is generated by $\mathcal{S}$ or

$\mathcal{S}$ is a subbase (subbasis) of $\mathcal{T}$.

Example. For $S = \{(-\infty, b) : b \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$

After taking finite intersections,

$\mathcal{B} = \{\phi\} \cup \{(a,b) : a < b\} \cup S$ and

the $\mathcal{T}$ generated by S is exactly the standard topology $\mathcal{T}_{std}$ of $\mathbb{R}$

Lower Limit Topology $\mathcal{T}_{ll}$ is the topology generated by $\{[a,b) : a < b \in \mathbb{R}\}$

Note that $\mathcal{T}_{std} \subsetneq \mathcal{T}_{ll}$ because

$$(a,b) = \bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b)$$

Exercise. Find a sequence $x_n \in \mathbb{R}$ satisfying the following,

(a) $x_n \to x$, i.e., $\forall\ \varepsilon > 0\ \exists$ integer N such that $\forall\ n \geq N$ $x_n \in (x - \varepsilon, x + \varepsilon)$, but

(b) $\exists\ \delta > 0$ such that $\forall$ integer N $\exists\ n \geq N$ where $x_n \notin [x, x + \delta)$

This example shows that $x_n \to x$ in $\mathcal{T}_{std}$ while $x_n \not\to x$ in $\mathcal{T}_{ll}$.