THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 11 Cauchy-Goursat Theorem

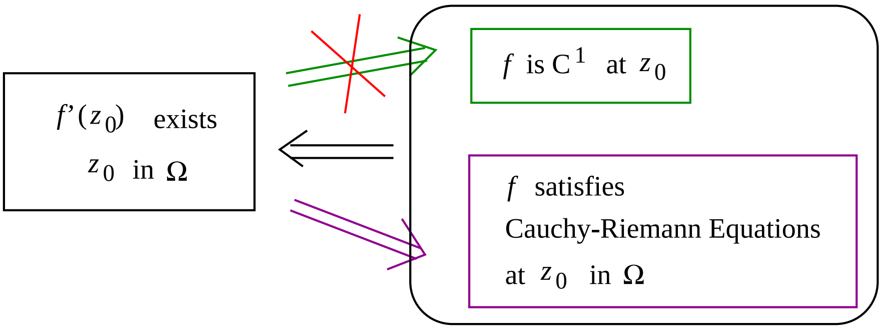
11.1 Zero Integral on Closed Contour

We continue to discuss the situations where $\int_\Gamma f = 0$ for a closed (usually simple) contour. Previously, we have two conditions; each one is sufficient to guarantee a zero integral.

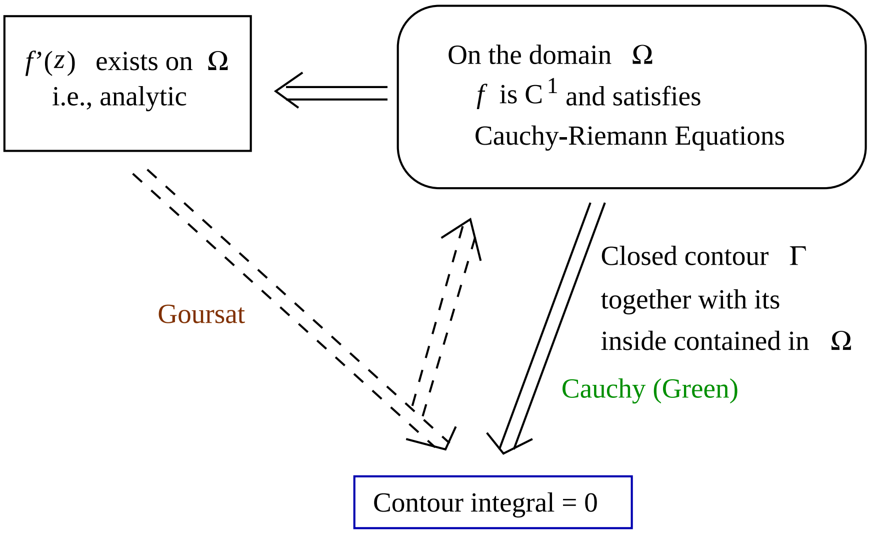
$$F'(z) = f(z) \quad \text{for } z \in \Omega \text{ and } \Omega \supset \Gamma. \tag{$**$}$$

$$f \text{ is of } C^1 \text{ and satisfies Cauchy-Riemann on } \Omega; \quad \text{also } \Gamma \cup S_b \subset \Omega, \tag{$\dagger\dagger$}$$

Let us focus on the second one ($\dagger\dagger$). At the beginning, we know that there is a close relation between complex differentiability and Cauchy-Riemann Equations.



The interesting and surprising part about complex functions is that the above diagram will be changed when it is true at every point $z_0 \in \Omega$. And the proof indeed goes through contour integration. More precisely, the dotted implications in the diagram below can be proved.



In this lesson, we will discuss the proof of Goursat. It involves very detail estimate and ε-δ argument. One is encouraged to understand the overall logic flow before studying the analysis.

11.1.1 Integrating analytic functions

Let us first understand the situation when the integrand f is analytic. Near a point z_0, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (\text{small error}), \qquad \lim_{z \to z_0} \frac{\text{small error}}{z - z_0} = 0\,.$$

Its integral along a small contour γ around z_0 is given by

$$\int_\gamma f = f(z_0) \int_\gamma dz + f'(z_0) \int_\gamma (z - z_0)\, dz + \int_\gamma (\text{small error})\, dz = \int_\gamma (\text{small error})\, dz\,,$$

because of existence of antiderivatives for the first two integrals. Thus, (small error) is the crucial content and we also expect that the third integral is small. In the example below, we will investigate the situation of integrating a (small error) along a contour in a small region. For this, we will use the notation

$$(\text{small error}) = \eta_0(z)\,(z - z_0)\,, \quad \text{where } \lim_{z \to z_0} \eta_0(z) = 0\,.$$

EXAMPLE 11.1. In this example, we deal with two cases of contours that occur over a small region. The first case is the contour $\partial\square = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ formed by the four sides of a square $\square$ with center z_0 and side length 2δ (left hand picture). The second case is a contour Γ_0 with an arc γ inside $\square$ and some parts $\sigma_1, \ldots, \sigma_4$ along the sides of $\square$ (right hand picture).

Let η be a function as above, i.e., defined on an open set containing $\square \cup \partial\square$ with $|\eta(z)| \le \varepsilon$ for all $z \in \square \cup \partial\square$. We are going to give an upper bound for the integrals of the above two cases, namely, $\int_{\partial\square} \eta(z)(z - z_0)\, dz$ and $\int_{\Gamma_0} \eta(z)(z - z_0)\, dz$.

Since σ_1 is from $z_0 + \delta(1 - \mathbf{i})$ to $z_0 + \delta(1 + \mathbf{i})$, it can be parametrized by

$$z(t) = (1 - t)\,[z_0 + \delta(1 - \mathbf{i})] + t\,[z_0 + \delta(1 + \mathbf{i})]\,, \quad t \in [0, 1]\,.$$

We have $z'(t) = \delta(1 + \mathbf{i}) - \delta(1 - \mathbf{i}) = 2\delta\mathbf{i}$. Together with $|\eta(z(t))| \le \varepsilon$, we have

$$\begin{aligned}\left| \int_{\sigma_1} \eta(z)(z - z_0)\, dz \right| &\le \int_0^1 |\eta(z(t))| \cdot |z(1) - z(0)| \cdot |2\delta\mathbf{i}|\ dt \\ &\le \int_0^1 \varepsilon \cdot (\sqrt{2}\delta) \cdot (2\delta)\, dt = 2\sqrt{2}\delta^2 \varepsilon\,.\end{aligned}$$

Thus $\quad \left| \int_{\partial\square} \eta(z)(z - z_0)\, dz \right| \le 8\sqrt{2}\delta^2\varepsilon = 2\sqrt{2}\,\text{Area}(\square)\varepsilon.$

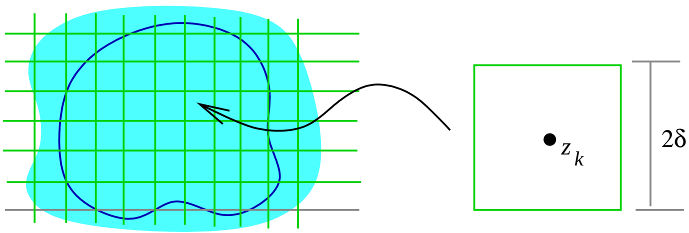
The contour Γ_0 may almost include all four sides $\partial\square$ of the square and a curve γ inside the square $\square$. Then,

$$\left|\int_{\Gamma_0} \eta(z)(z-z_0)\,dz\right| \le \sum_{k=1}^{4}\left|\int_{\sigma_k} \eta(z)(z-z_0)\,dz\right| + \left|\int_{\gamma} \eta(z)(z-z_0)\,dz\right|$$
$$\le 2\sqrt{2}\,\mathrm{Area}(\square)\varepsilon + \varepsilon\int_{\gamma} |(z-z_0)\,dz|$$
$$\le 2\sqrt{2}\,\mathrm{Area}(\square)\varepsilon + \sqrt{2}\delta\,\mathrm{Length}\,(\gamma)\,\varepsilon\,.$$

11.1.2 Cauchy-Goursat

Theorem 11.2. *Let Γ be a simple closed contour with bounded complement component S_b such that $\Gamma\cup S_b\subset\Omega$ and $f:\ \Omega\subset\mathbb{C}\to\mathbb{C}$ is analytic. Then* $\displaystyle\int_{\Gamma} f = 0$.

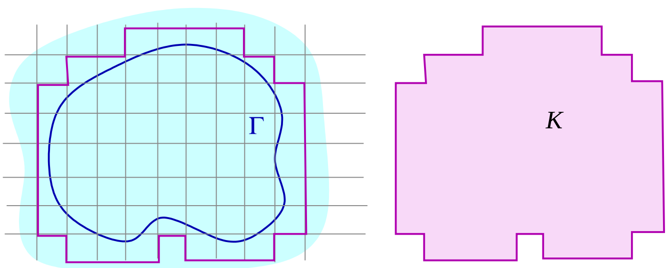
The result of Example 11.1 is very useful in understanding the proof of the theorem. We are going to divide $\Gamma\cup S_b$ into small squares $\square_k$ of side length 2δ as shown in the figure below.



Given any $\varepsilon>0$, by the analyticity of f on Ω, there exists a suitable $\delta>0$ (the compactness of $\Gamma\cup S_b$ is needed) such that if $z_0\in\Gamma\cup S_b$ and $z\in\Omega$ with $|z-z_0|<\sqrt{2}\delta$, then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta_0(z)\,(z-z_0) \quad \text{where} \quad |\eta_0(z)|\le\varepsilon\,.$$

Further reduce δ to be small enough such that there is a subset K such that $\Gamma\cup S_b\subset K\subset\Omega$. The set K is a union of squares $\square_k$, $k=1,\ldots,N$, such that each $\square_k$ either lies in S_b completely (case 1 in Example 11.1) or intersects Γ (case 2 above). Denote $\gamma_k=\square_k\cap\Gamma$, which plays the role of γ in Example 11.1.



It can be easily observed that

$$\sum_{k=1}^{N}\int_{\partial\square_k} f(z)\,dz = \int_{\partial K} f(z)\,dz \ne \int_{\Gamma} f(z)\,dz\,.$$

By the argument of the Example 11.1 (note that $\gamma_k = \emptyset$ if $\Box_k \subset S_b$),

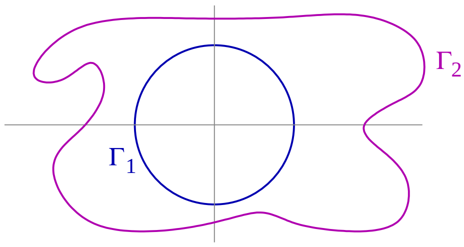
$$\begin{aligned}
\left|\int_\Gamma f(z)\,dz\right| \le \sum_{k=1}^N \left|\int_{(\partial\Box_k,\gamma_k)} f(z)\,dz\right| &= \sum_{k=1}^N \left|\int_{(\partial\Box_k,\gamma_k)} \eta_k(z)\,(z-z_k)\;dz\right| \\
&\le \sum_{k=1}^N \left|\int_{\partial\Box_k} \eta_k(z)\,(z-z_k)\;dz\right| + \left|\int_{\gamma_k} \eta_k(z)\,(z-z_k)\;dz\right| \\
&\le \sum_{k=1}^N \left[2\sqrt{2}\,\text{Area}(\Box_k)\varepsilon + \sqrt{2}\delta\,\text{Length}\,(\gamma_k)\,\varepsilon\right] \\
&\le \sqrt{2}\,[\,2\,\text{Area}(K) + \text{Length}\,(\partial K)\;\text{Length}\,(\Gamma)\,]\,\varepsilon \\
&\le \sqrt{2}\left[\,4\,\text{Area}(S_b) + 2\,\text{Length}\,(\Gamma)^2\,\right]\varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\left|\int_\Gamma f(z)\,dz\right| = 0$.

11.2 Further Results

In the statement of Cauchy-Goursat Theorem, the contour is *simple* closed. However, the result is useful in many other situations. Furthermore, the statement apparently is about analytic functions, it actually is often used on integrand that are not analytic.

EXAMPLE 11.3. Let Γ_1 be a circle of center 0 and radius R; Γ_2 be the contour as shown.



What are $\int_{\Gamma_1} g$ and $\int_{\Gamma_2} g$ in the cases that $g(z) = 1/z^3$ or $g(z) = 1/z$?

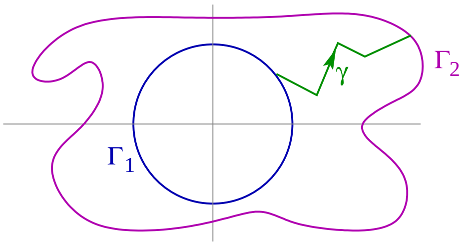
Since Γ_1 can be explicitly parametrized by $Re^{\mathbf{i}t}$ for $t \in [0, 2\pi]$, we may directly calculate that

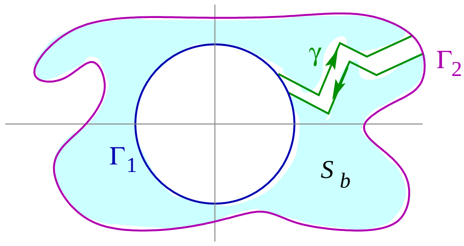
$$\int_{\Gamma_1} \frac{1}{z^3}\,dz = 0 \quad \text{and} \quad \int_{\Gamma_1} \frac{1}{z}\,dz = 2\pi\mathbf{i}\,.$$

Without explicit parametrization for Γ_2, can we use Cauchy-Goursat Theorem to get the result? The answer is no; because for both Γ_1 and Γ_2, the bounded complement component S_b contains 0 and thus g is ***not*** analytic on $\Gamma \cup S_b$. Still, we have

$$\int_{\Gamma_2} \frac{1}{z^3}\,dz = 0 \qquad \text{because } g(z) = 1/z^3 \text{ has an antiderivative on } \mathbb{C} \setminus \{0\} \supset \Gamma_2.$$

Obviously, for $g(z) = 1/z$, it does not have an antiderivative on $\mathbb{C} \setminus \{0\}$, one must look for another method. Let γ be a curve (not necessarily straight) joining Γ_1 and Γ_2.





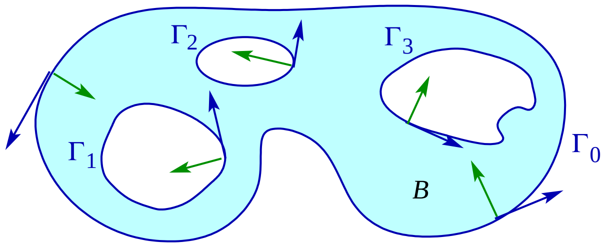
Then the contour $\Gamma = (-\Gamma_1, \gamma, \Gamma_2, -\gamma)$ actually satisfies $\partial S_b = \Gamma$. Moreover, the proof of Cauchy-Goursat Theorem is still valid in such a situation. Therefore, we have

$$0 = \int_\Gamma f(z)\, dz = \int_{\Gamma_2} f + \int_\gamma f - \int_\gamma f - \int_{\Gamma_1} f, \qquad \text{and thus} \quad \int_{\Gamma_2} \frac{1}{z}\, dz = \int_{\Gamma_1} \frac{1}{z}\, dz = 2\pi \mathbf{i}\,.$$

11.2.1 Regions with Holes

As in Example 11.3 above, we always need to deal with functions that are not completely analytic. Thus, it is convenient to have a more general version of Cauchy-Goursat Theorem.

Let Γ_0 be a simple closed contour and $\Gamma_1, \ldots, \Gamma_p$ be simple closed contours contained in the bounded complement component of Γ_0 (see an illustration below).



Let $S_{b,0}, S_{b,1}, \ldots, S_{b,p}$ be the bounded complement components of $\Gamma_0, \Gamma_1, \ldots, \Gamma_p$ respectively. Then $B = S_{b,0} \bigcap_{k=1}^{p} (\mathbb{C} \setminus S_{b,k})$ is the region *between* the contours.

THEOREM 11.4. *In the setting above, let Γ_0 and all Γ_k, $k = 1, \ldots, p$ be positively oriented. If $f:\ \Omega \subset \mathbb{C} \to \mathbb{C}$ is analytic on a domain $\Omega \supset B$, then*

$$\int_{\Gamma_0} f = \sum_{k=1}^{p} \int_{\Gamma_k} f\,.$$

Idea of Proof. Add short arcs $\gamma_1, \ldots, \gamma_p$ to connect each contour Γ_k from Γ_0 and then use the argument of Example 11.3. □

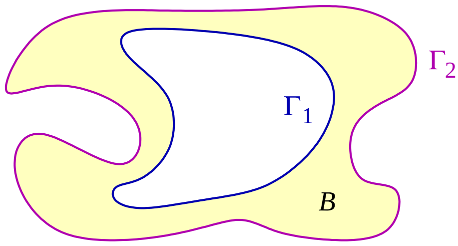
Remark. Note that if all the $\Gamma_1, \ldots, \Gamma_p$ take ***negative*** orientation, then their normals will behave the same as the normal of Γ_0 to point towards B. Corresponding to the above picture, this is written as

$$\partial B = \Gamma_0 \cup (-\Gamma_1) \cup \cdots \cup (-\Gamma_p)\,.$$

The situation of Example 11.3 is often expressed in the following form.

THEOREM 11.5 (Invariance of Deformation). *If Γ_1 and Γ_2 can be deformed smoothly to each other through a region B where f is analytic, then*

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f \,.$$



The function f may not be analytic at many places. However, as long as it is analytic on the yellow region *between* the two contours, its integrals along the contours are the same.

EXAMPLE 11.6. Let us consider the rational function $g(z) = \dfrac{z^3+z^2+z-2}{z^2-z}$. It is clearly analytic on $\mathbb{C}\setminus\{0,1\}$. The first observation is that the numerator is of degree 3 while the denominator of degree 2. We may use long division to get

$$g(z) = z+2+\frac{3z-2}{z^2-z}\,.$$

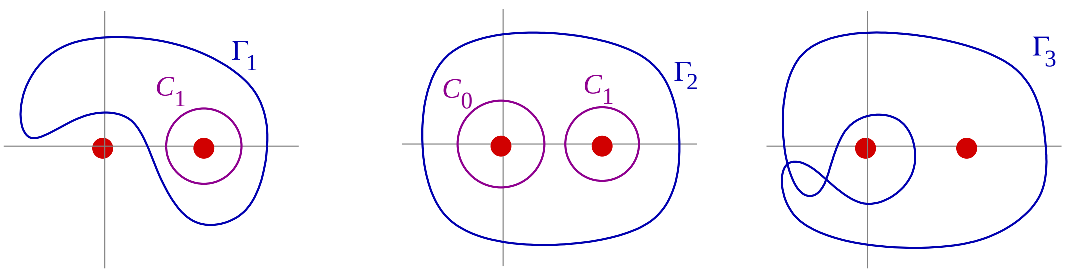
Thus, for any closed contour Γ, we can use Cauchy-Goursat Theorem to have

$$\int_\Gamma g(z)\,dz = \int_\Gamma (z+2)\,dz + \int_\Gamma \frac{3z-2}{z^2-z}dz = 0 + \int_\Gamma \frac{3z-2}{z^2-z}dz\,.$$

This illustrates that we can always throw away the analytic part of a function when doing integral along a closed contour. Next, we will use partial fraction to have

$$\int_\Gamma g(z)\,dz = \int_\Gamma \frac{3z-2}{z^2-z}dz = \int_\Gamma \left(\frac{2}{z}+\frac{1}{z-1}\right)\,dz\,.$$

For the following contours, we are able to reduce the calculation to the special contours C_0 and C_1, which are circles at centers 0 and 1 respectively. The radii of the circles are not important. We assume all contours are positively oriented



By Invariance of Deformation, g is analytic on the region *between* Γ_1 and C_1, so

$$\int_{\Gamma_1} g(z)\,dz = \int_{C_1} g(z)\,dz = \int_{C_1} \frac{2}{z}\,dz + \int_{C_1} \frac{1}{z-1}\,dz\,.$$

Note that the integrand $2/z$ is analytic on and inside C_1, therefore $\int_{C_1} \frac{2}{z}\, dz = 0$. Moreover, by the parametrization C_1 by $z(t) = 1 + Re^{\mathbf{i}t}$, $t \in [0, 2\pi]$, we may calculate that $\int_{C_1} \frac{1}{z-1}\, dz = 2\pi\mathbf{i}$. Similarly, we have

$$\int_{\Gamma_2} g(z)\, dz = 4\pi\mathbf{i} + 2\pi\mathbf{i} = 6\pi\mathbf{i}\,.$$

For the contour Γ_3, we have to break it into simple closed contours and obtain the answer $2\pi\mathbf{i}$. This example is easy because it is a fraction of two polynomial. What if $g(z) = \dfrac{\sin z}{z^2 - z}$? We can certainly use partial fraction to have $g(z) = \left(\dfrac{-1}{z} + \dfrac{1}{z-1}\right)\sin z$. From this, we see a major concern in finding contour integral, namely, the integrand is of the form

$$g(z) = \frac{f(z)}{z - z_0} \quad \text{where } f \text{ is an analytic function.}$$

This will be our next topic.