Lect23-20180416

Wednesday, 11 April 2018 11:23 AM

Contractible Spaces, $\mathbb{R}^n$, D^n, etc.

Pick $x_0 \in X$ and $c : [0,1] \to \{x_0\} \subset X$

Then $\pi_1(X, x_0) = \{[c]\}$.

Trivial group, denoted 1.

Circle S^1 and Punctured Plane $\mathbb{R}^2 \setminus \{0\}$

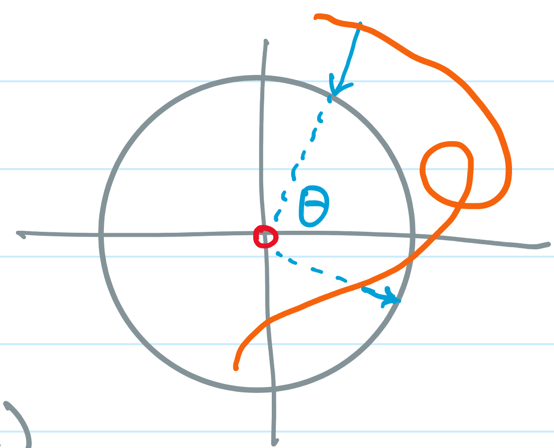
Let $x_0 \in S^1 \subset \mathbb{R}^2 \setminus \{0\}$.

$$\pi_1(S^1, x_0) = \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) = (\mathbb{Z}, +)$$

First, define a mapping

$$\begin{aligned} \varphi : \mathbb{R}^2 \setminus \{0\} &\longrightarrow S^1 \\ \text{\rotatebox{90}{$\in$}}\quad & \\ x &\longmapsto \varphi(x) = \frac{x}{\|x\|} \end{aligned}$$

$(\cos\theta, \sin\theta)$



We then have (prove later)

- $[\gamma] \in \pi_1(S^1, x_0) \longmapsto [\gamma] \in \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$
- $[\gamma] \in \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \longmapsto [\varphi \circ \gamma] \in \pi_1(S^1, x_0)$
- $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) = \pi_1(S^1, x_0)$
- Finally, $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) = (\mathbb{Z}, +) = \pi_1(S^1, x_0)$

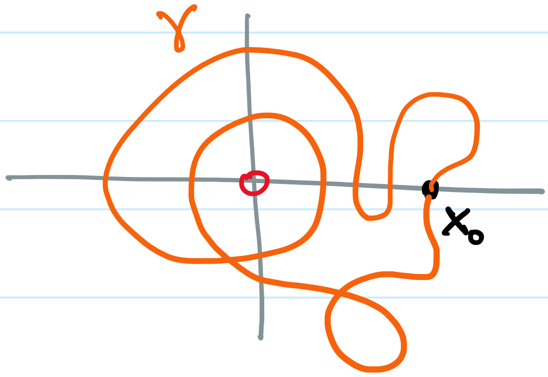
What steps are needed to have the above?

Try to outline it.

Calculate (not yet proof)

$$\pi_1(\mathbb{R}^2\setminus\{0\}, x_0) = \pi_1(\mathbb{C}\setminus\{0\}, z_0) \longrightarrow \mathbb{Z}$$

$$\begin{array}{ccc} \cup & & \\ [\gamma] & \longmapsto & ? \end{array}$$



winding number, $w(\gamma)$

How?

Winding Number

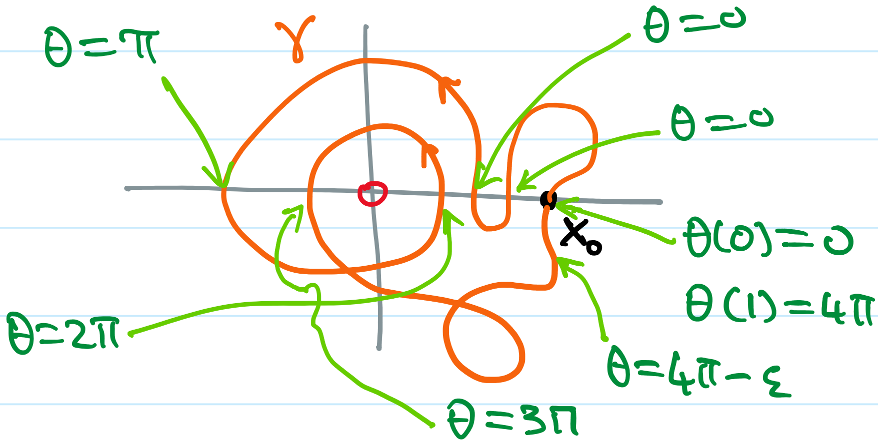
If $\gamma : [0,1] \longrightarrow \mathbb{C}\setminus\{0\}$ is piecewise differentiable with $\gamma(0)=\gamma(1)$, then

$$w(\gamma) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z} =$$

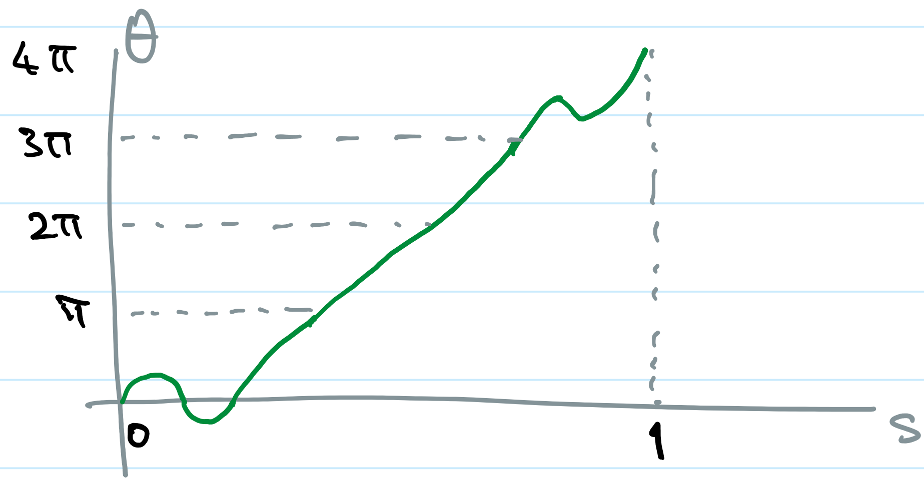
For $s \in [0,1]$, write $z = r(s)e^{i\theta(s)}$

$$= \cdots = \frac{\theta(1)-\theta(0)}{2\pi}$$

In the example, $\theta(s)$ varies as below.



The graph of $\theta : [0,1] \longrightarrow \mathbb{R}$ looks like

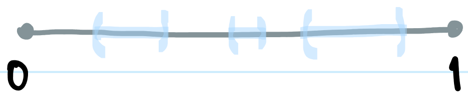


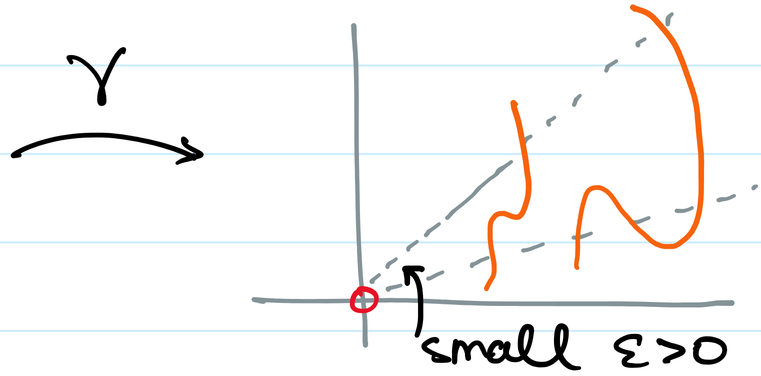
Theorem. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$ be continuous (not necessarily $\gamma(0) = \gamma(1)$).
Then there is a continuous polar form parametrization, i.e., continuous

$$r : [0,1] \longrightarrow (0,\infty) \quad \text{and}$$
$$\theta : [0,1] \longrightarrow \mathbb{R} \quad \text{such that}$$

$$\gamma(s) = \big(r(s)\cos\theta(s),\ r(s)\sin\theta(s) \big)$$

Idea of proof.





By compactness of $[0,1]$, it can be covered by finitely many such open intervals

Then $\theta(s)$ can be inductively defined.

Definition. In the case that $\boxed{\gamma(0)=\gamma(1)}$,

winding number $w(\gamma)=\frac{1}{2\pi}\big(\theta(1)-\theta(0)\big)\in\mathbb{Z}$

why independent of $\theta(s)$?

For two choices $\theta(s)$, $\hat\theta(s)$, consider

$$s\in[0,1] \xrightarrow{\text{continuous}} \frac{1}{2\pi}\left[\hat\theta(s)-\theta(s)\right]\in\mathbb{Z}$$

connected

only constant in s

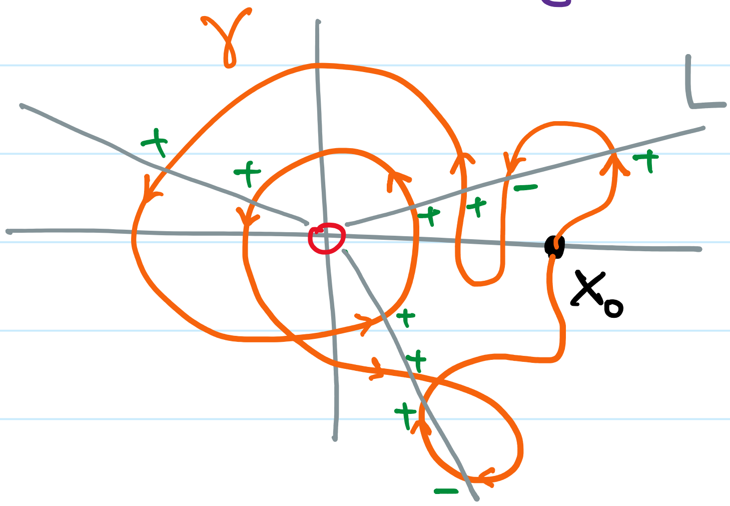
$$\therefore\ \frac{1}{2\pi}\left[\hat\theta(1)-\theta(1)\right]=\frac{1}{2\pi}\left[\hat\theta(0)-\theta(0)\right]$$

$$\therefore\ \frac{1}{2\pi}\left[\hat\theta(1)-\hat\theta(0)\right]=\frac{1}{2\pi}\left[\theta(1)-\theta(0)\right]$$

Lect23-p5

Monday, 16 April 2018 10:15 AM

Combinatorial Winding Number



$w(\gamma)$
$\quad ||$
signed sum$(L \cap \gamma)$

independent of L

Homomorphism $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \longrightarrow (\mathbb{Z}, +)$

For $\alpha, \beta : [0,1] \longrightarrow \mathbb{R}^2 \setminus \{0\}$, loops at x_0

Let L be a ray from 0,

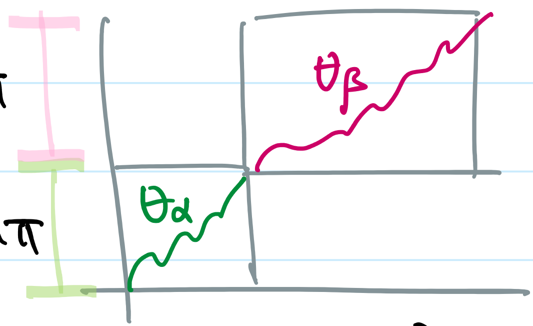
$$L \cap (\alpha * \beta) = (L \cap \alpha) \sqcup (L \cap \beta)$$

$$\therefore \quad w(\alpha * \beta) = w(\alpha) + w(\beta)$$

Alternatively, the continuous choices, θ_α, θ_β will define $\theta_{\alpha*\beta}$ as illustration

$w(\beta) = n$, $2n\pi$

$w(\alpha) = m$, $2m\pi$



$$\therefore \quad w(\alpha * \beta) = m + n$$

Isomormorphism $\pi_1(\mathbb{R}^2\setminus\{0\}, x_0) \longrightarrow (\mathbb{Z}, +)$

Obviously, $c : [0,1] \longrightarrow \{x_0\} \subset \mathbb{R}^2\setminus\{0\}$

$$[c] \longmapsto 0 \in \mathbb{Z}$$

For $\gamma : [0,1] \longrightarrow \mathbb{R}^2\setminus\{0\}$ and

$\bar{\gamma} : [0,1] \longrightarrow \mathbb{R}^2\setminus\{0\}$

$$w(\bar{\gamma}) = -w(\gamma)$$

one-one

For any $n \in \mathbb{Z}$, can construct

$\gamma : [0,1] \longrightarrow \mathbb{R}^2\setminus\{0\}$ by

$$\gamma(s) = (\cos(2n s), \sin(2n s))$$

$\therefore \quad w(\gamma) = n.$

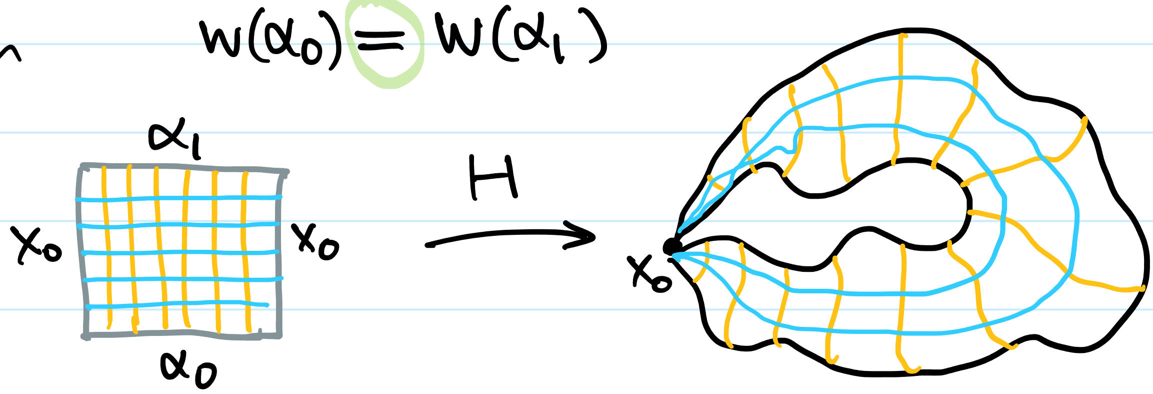
onto

Crucial Argument $\pi_1(\mathbb{R}^2\setminus\{0\}, x_0) = (\mathbb{Z}, +)$

Let $\alpha_0, \alpha_1 : [0,1] \longrightarrow \mathbb{R}^2\setminus\{0\}$ be loops at x_0

and $\alpha_0 \simeq \alpha_1$ rel $\{0,1\}$, i.e., loop homotopic.

Then $\quad w(\alpha_0) = w(\alpha_1)$

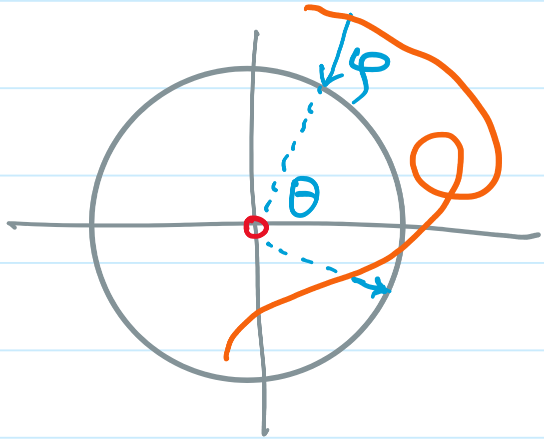


What about

$$\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \overset{?}{=} \pi_1(S^1, x_0)$$
$$\cong (\mathbb{Z}, +)$$

Recall $x_0 \in S^1 \subset \mathbb{R}^2\setminus\{0\} = \mathbb{C}\setminus\{0\}$

$$\begin{array}{ccc} \gamma & \longmapsto & \gamma \\ S^1 & \overset{i}{\longrightarrow} & \mathbb{R}^2\setminus\{0\} \\ & \longleftarrow & \\ \varphi\circ\gamma & \overset{\varphi}{\longleftarrow\!\shortmid} & \gamma \end{array}$$



Theorem. Let $f: X \longrightarrow Y$ be continuous with $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Then $\exists$ homomorphism

$$f_\#: \underset{\cup\!\!\!\shortmid}{\pi_1(X, x_0)} \longrightarrow \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [\text{loop at } y_0] \quad f\circ\gamma$$

- Well-defined : $\alpha_0 \simeq \alpha_1$ rel $\{0,1\}$
  $\Longrightarrow\ ? \simeq ?$ rel $\{0,1\}$
- homomorphic $[\alpha]\cdot[\beta] \longmapsto f_\#[\alpha]\cdot f_\#[\beta]$

$$\begin{array}{ccc} [\alpha]\cdot[\beta] & \longmapsto & f_\#[\alpha]\cdot f_\#[\beta] \\ \| & & \|\,? \\ [\alpha*\beta] & \longmapsto & f_\#[\alpha*\beta] \end{array}$$

Lect23-p8

Apr 17, Tuesday, 2018 9:31 AM

Remark. $id_{\#}: \pi_1(X,x_0) \to \pi_1(X,x_0)$ is exactly the id mapping.

Warning f is 1-1 $\not\Rightarrow$ $f_{\#}$ is 1-1

f is onto $\not\Rightarrow$ $f_{\#}$ is onto

In the above context, we have

$$(S^1, x_0) \underset{\varphi}{\overset{i}{\rightleftarrows}} (\mathbb{R}^2 \setminus \{0\}, x_0)$$

Special: $\gamma(s) \longmapsto i \circ \gamma(s) = \gamma(s)$

$\parallel$

$\varphi \circ i \circ \gamma(s)$

$\varphi \circ i \equiv id_{S^1}$

$$\therefore \quad \pi_1(S^1, x_0) \underset{\varphi_{\#}}{\overset{i_{\#}}{\rightleftarrows}} \pi_1(\mathbb{R}^2 \setminus \{0\}, x_0)$$

Do we have $\varphi_{\#} \circ i_{\#} \equiv id_{\pi_1(S^1, x_0)}$?

Theorem. Let $f: X \to Y$, $g: Y \to Z$ with $x_0 \in X$, $y_0 = f(x_0) \in Y$, $z_0 = g(y_0) \in Z$. Then

$$g_{\#} \circ f_{\#} \equiv (g \circ f)_{\#}: \pi_1(X, x_0) \to \pi_1(Z, z_0)$$

$$g_{\#}(f_{\#}[\gamma]) = g_{\#}([f \circ \gamma]) = [g \circ f \circ \gamma]$$

Now, we have

$$(S^1, x_0) \underset{\varphi}{\overset{i}{\rightleftarrows}} (\mathbb{R}^2\setminus\{0\}, x_0), \quad \varphi\circ i \equiv id_{S^1}$$

$$\therefore\ \pi_1(S^1, x_0) \underset{\varphi_\#}{\overset{i_\#}{\rightleftarrows}} \pi_1(\mathbb{R}^2\setminus\{0\}, x_0)$$

$$\varphi_\# \circ i_\# = (\varphi\circ i)_\# = id_\# = id$$

What can we conclude?

Sorry! We can only have

- $i_\#$ is monomorphic, i.e., injective
- $\varphi_\#$ is epimorphic, i.e., surjective

It could be

$$\pi_1(S^1, x_0) = \begin{cases} 1 \\ \mathbb{Z}/m \end{cases} \underset{\varphi_\#}{\overset{i_\#}{\rightleftarrows}} \pi_1(\mathbb{R}^2\setminus\{0\}, x_0) = \mathbb{Z}$$

What about $(X, x_0) \xrightarrow{f_1} (Y, y_1)$, $(X, x_0) \xrightarrow{f_2} (Y, y_2)$?

We need to compare $\pi_1(Y, y_1)$ and $\pi_1(Y, y_2)$.

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Theorem If X is path connected (or at least x_1, x_2 are joined by a path), then

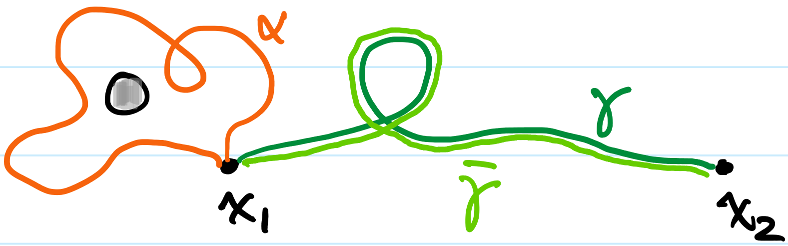
$$\pi_1(X, x_1) \cong \pi_1(X, x_2)$$

isomorphic

Proof. Let $\gamma : [0,1] \longrightarrow X$ be a path joining x_1 to x_2, i.e., $\gamma(0) = x_1$, $\gamma(1) = x_2$

Then $\bar{\gamma}(s) = \gamma(1-s)$ is from x_2 to x_1.

$$[\alpha] \in \pi_1(X, x_1) \longmapsto [\bar{\gamma} * \alpha * \gamma] \in \pi_1(X, x_2)$$



This is a bijection because an inverse is obvious.

Also, it is homomorphic

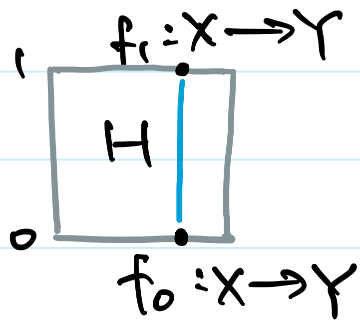
$$\alpha * \beta \longmapsto \bar{\gamma} * \alpha * \beta * \gamma \simeq \bar{\gamma} * \alpha * \gamma * \bar{\gamma} * \beta * \gamma$$

Theorem. Let $f_0 : (X, x_0) \longrightarrow (Y, y_0)$ and $f_1 : (X, x_0) \longrightarrow (Y, y_1)$ be continuous. If $f_0 \simeq f_1$ then the diagram commutes

$$\begin{tikzcd} & \pi_1(Y, y_0) \arrow[d, leftrightarrow, "\text{isomormorphism}"] \\ \pi_1(X, x_0) \arrow[ur, "(f_0)_\#"] \arrow[r, "(f_1)_\#"'] & \pi_1(Y, y_1) \end{tikzcd}$$

Proof. $f_0 \overset{H}{\simeq} f_1 \implies f_0 \circ \alpha \simeq f_1 \circ \alpha$ but not rel $\{0,1\}$

$f_0 \overset{H}{\simeq} f_1 \implies \exists$ path $\gamma : [0,1] \longrightarrow Y$
How? $\gamma(0) = y_0,\ \gamma(1) = y_1.$



$t \overset{\gamma}{\longmapsto} H(x_0, t) : [0,1] \to X$

$\gamma(0) = H(x_0, 0) = f_0(x_0) = y_0$

$\gamma(1) = H(x_0, 1) = f_1(x_0) = y_1$

$$\begin{tikzcd} & {[f_0 \circ \alpha] \text{ at } y_0} \arrow[d] \\ & {[\bar{\gamma} * (f_0 \circ \alpha) * \gamma] \text{ at } y_1} \arrow[d, equal, "\text{Exercise.}"] \\ {[\alpha] \in \pi_1(X, x_0)} \arrow[uur, bend left, "(f_0)_\#"] \arrow[r, "(f_1)_\#"'] & {[f_1 \circ \alpha] \text{ at } y_1} \end{tikzcd}$$

Theorem. Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ be homotopy equivalences on path connected X, Y, i.e., $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.

Then $\pi_1(X, x_0)$, $\pi_1(Y, y_0)$ are isomorphic.

Proof.

$$\begin{tikzcd} \pi_1(X, ?) \arrow[r, bend left, "f_\#"] & \pi_1(Y, ?) \arrow[l, bend left, "g_\#"] \end{tikzcd}$$

$$g_\# \circ f_\# = (g \circ f)_\# = (id_X)_\# = id$$

$\therefore$ $f_\#$ 1-1 and $g_\#$ onto

up to isomorphism

$$f_\# \circ g_\# = (f \circ g)_\# = (id_Y)_\# = id$$

$\therefore$ $f_\#$ onto and $g_\#$ 1-1

Hence, both $f_\#$ and $g_\#$ are isomorphisms.

Lect23-p13

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Definition. A subspace $A \subset X$ is a **retract** of X if $\exists$ continuation $r : X \longrightarrow A$ such that $\forall\, a \in A$, $r(a) = a$; $r|_A \equiv id_A$

The mapping r is called a **retraction.**

Equivalently, $A \xhookrightarrow[\text{inclusion}]{i} X \xrightarrow{r} A$, $r \circ i \equiv id_A$

Example. $S^1 \xhookrightarrow{i} \mathbb{R}^2 \setminus \{0\} \xrightarrow{\varphi} S^1$

Proposition. $r_\#$ is surjective, $i_\#$ is injective.

Definition. A retract $A \subset X$ is called a **deformation retract** if $r \simeq id_X$

Example. S^1 is a deformation retract of $\mathbb{R}^2 \setminus \{0\}$; need to modify φ (Exercise)

Theorem. If $A \subset X$ is a deformation retract then $\pi_1(A, a_0) = \pi_1(X, a_0)$ for $a_0 \in A$

Proof. $r \simeq id_X \implies r_\# = (id_X)_\# = id$

up to isomorphism (base pt)

For $a_0 \in A$ as base point of both A, X, $r_\# \equiv id$.

Lect23-p14

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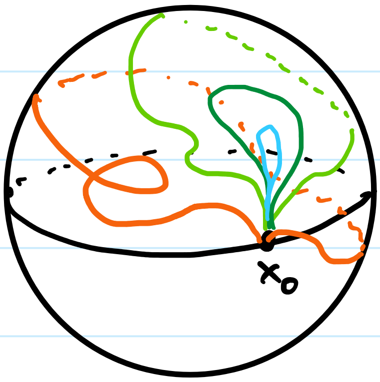
Example. $\pi_1(S^n) = \begin{cases} (\mathbb{Z},+) & \text{if } n=1 \\ 1 & \text{if } n\geq 2 \end{cases}$



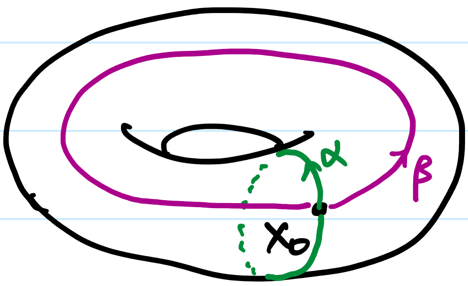
Rigorous proof uses Van Kampen Theorem and induction $S^n, S^{n-1}, S^{n-2}, \ldots, S^1$

Proposition. $\pi_1(X\times Y) = \pi_1(X)\times\pi_1(Y)$

direct product

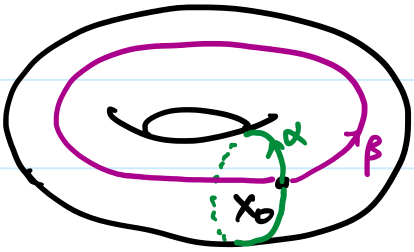
Example. $\pi_1(\text{Torus}) = (\mathbb{Z}\oplus\mathbb{Z}, +)$

$$\begin{array}{ccc} \pi_1(\text{Torus}) & = & (\mathbb{Z}\oplus\mathbb{Z},+) \\ \| & & \| \\ \pi_1(S^1\times S^1) & = & \pi_1(S^1)\times\pi_1(S^1) \end{array}$$



$$\begin{aligned} \pi_1(S^1\times S^1) &\longrightarrow \mathbb{Z}\oplus\mathbb{Z} \\ [\alpha] &\longmapsto (1,0) \\ [\beta] &\longmapsto (0,1) \end{aligned}$$

Similarly, $\pi_1(S^1\times\cdots\times S^1) = (\mathbb{Z}\oplus\cdots\oplus\mathbb{Z},+)$



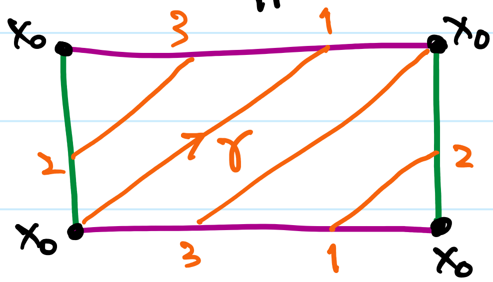
$$\pi_1(S^1\times S^1) \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$$

$$[\alpha] \longmapsto (1,0)$$

$$[\beta] \longmapsto (0,1)$$

$$[\gamma] \longmapsto (3,2)$$

||



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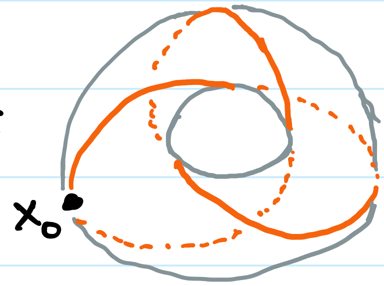
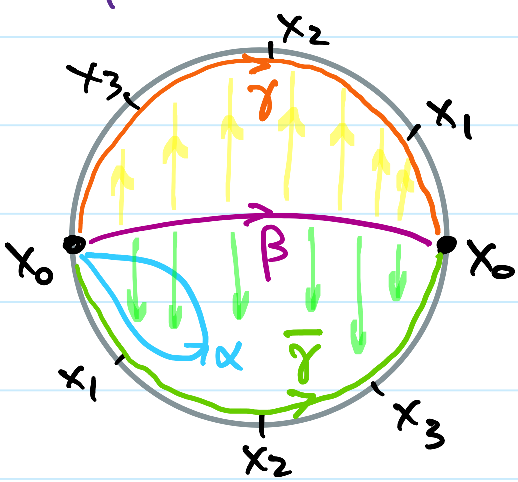


Example. $\pi_1(\mathbb{RP}^2) = (\mathbb{Z}/2, +)$



$$[\alpha] \longmapsto \text{identity}$$

$\beta \simeq \gamma$ and $\beta \simeq \bar{\gamma}$

$$[\beta] = [\beta]^{-1}$$

$$\Updownarrow$$

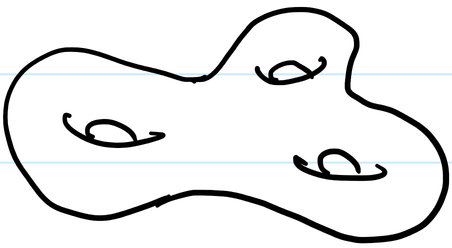
$$[\beta]^2 = \text{identity}$$

After showing abelian

$$\{[\alpha], [\beta]\} \longrightarrow (\{0,1\}, +)$$

Examples. $\pi_1(\text{Surface genus} = g)$

$$\| $$



$$\left\langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g : a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \right\rangle$$