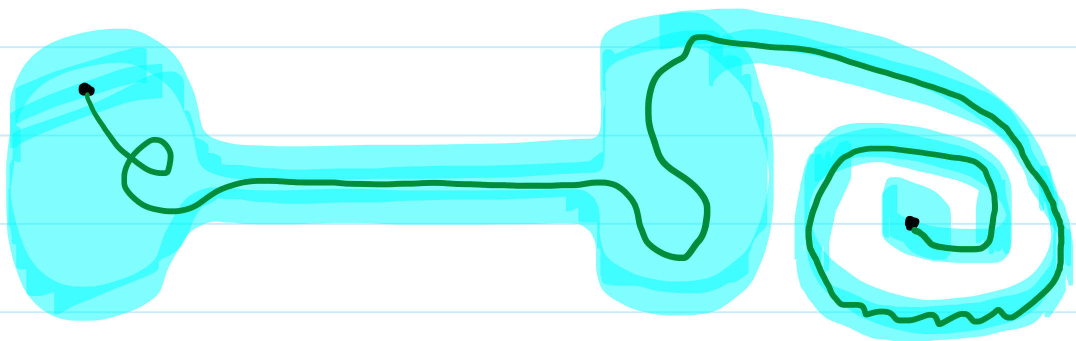
Lect20-20180326

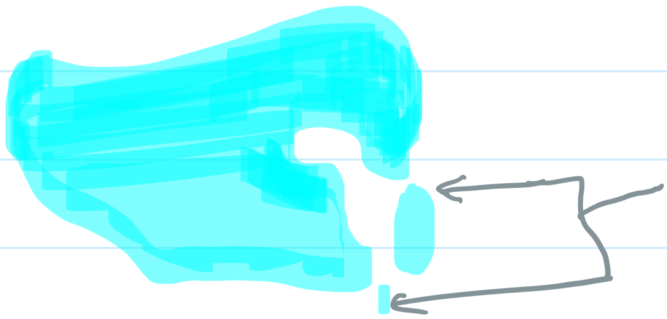
Thursday, 22 March 2018 5:33 PM

Three other connectedness

Path connected — stronger, more visual

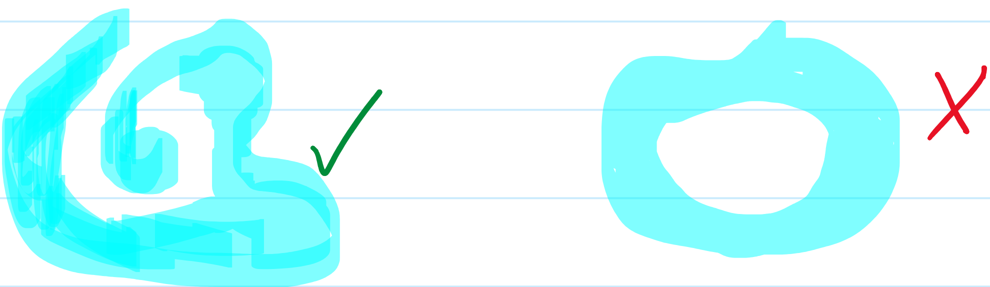


Local connected — 本土意識



connected themselves

Simply connected — different from above



Definition. A space X is path connected if $\forall$ pair $x, y \in X$, $\exists$ continuous mapping $\gamma : [0,1] \longrightarrow X$, $\gamma(0) = x$, $\gamma(1) = y$

Theorem. A path connected space is connected.

Proof. By contradiction, assume
$\exists\ \phi \neq U \subsetneq X,\ U, X\setminus U \in \mathcal{J}$

Then pick $x \in U$ and $y \in X\setminus U$ and

⋮

Then we can found a nontrivial both open and closed subset in $[0,1]$.

Contradiction

Exercise. Careful argument of above.

Definition. Let $\sim$ be an equivalence relation on X that $x \sim y$ if $\exists$ continuous
$\gamma : [0,1] \longrightarrow X,\ \gamma(0) = x,\ \gamma(1) = y.$
For $x_0 \in X$, the equivalence class $[x_0]$ is the path component of x_0.

Obviously, X is path connected $\Longleftrightarrow$ X has exactly one path component.

Lect20-p3

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Question. While path connected is connected,
Is path component = connected component?

Example. Again, $X = Y \cup G$ formed by
$\sin\frac{1}{x}$, $x > 0$

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path component $\neq$ connected component
connected $\not\Rightarrow$ path connected

Theorem. Let $\phi \neq G \subset \mathbb{R}^n$ be an open set.
If G is connected then it is path connected.

Proof. Fix a point $x_0 \in G$ and consider
its path component P in G.

First, P is open.
Wish. $\forall x \in P$, $\exists$ $\underbrace{\text{ball } B,\ x \in B}$ $\subset P$

This is easy

Since G is open, $\exists$ ball B, $x \in B \subset G$

Only need to show $B \subset P$

Arbitrary $y \in B \Rightarrow y \in P$

B is a ball and $x, y \in B$,
$\exists$ straight line from x to y

Also, $\exists$ continuous path from x_0 to x,

$\therefore \exists$ continuous path from x_0 to y

and so $y \in P$.

By similar argument (exercise), $X \setminus P \in \mathcal{J}$.

Thus, by connectedness of G,

$$P = \phi \text{ or } P = G$$

But $x_0 \in P$, $\therefore P = G$ is path connected

Exercise. Show that every pair $x, y \in G$ can be joined by a finite sequence of straight lines parallel to the coordinate axes.

Definition. A space X is *locally connected* if $\forall\, x \in X$ $\exists$ a local base $\mathcal{U}_x$ at x consisting of connected neighborhoods.

Clearly, locally connected $\not\Rightarrow$ connected

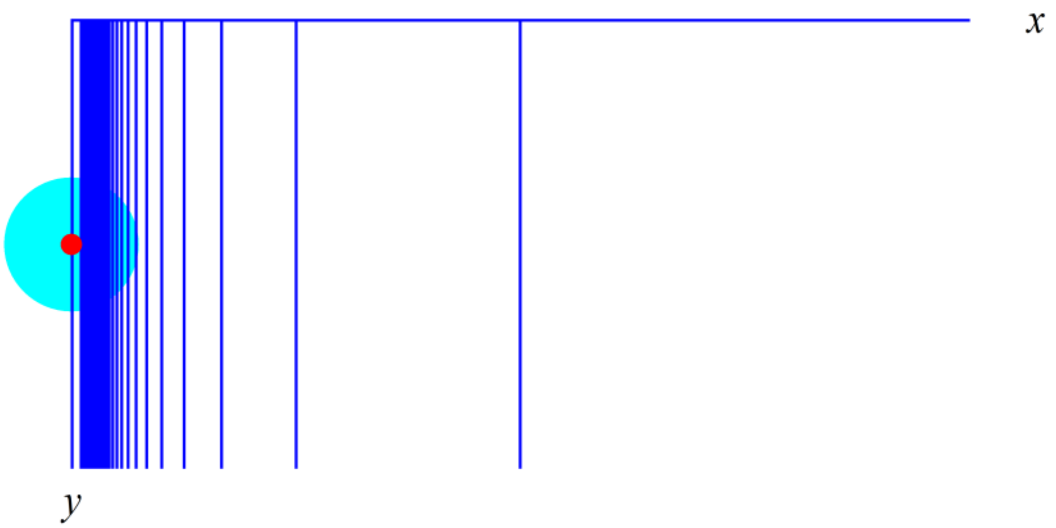
Just $(-\infty, 0) \cup (0, \infty)$ is an example.

Example. Let $F = X \cup Y \cup V \subset \mathbb{R}^2$ where

$$X = \{(x,0) : 0 \leq x \in \mathbb{R}\} \subset x\text{-axis}$$

$$Y = \{(0,y) : y \leq 0\} \subset y\text{-axis}$$

$$V = \bigcup_{n=1}^{\infty} \{(\tfrac{1}{n}, y) : y \leq 0\}$$



F is clearly path connected, and so it is connected

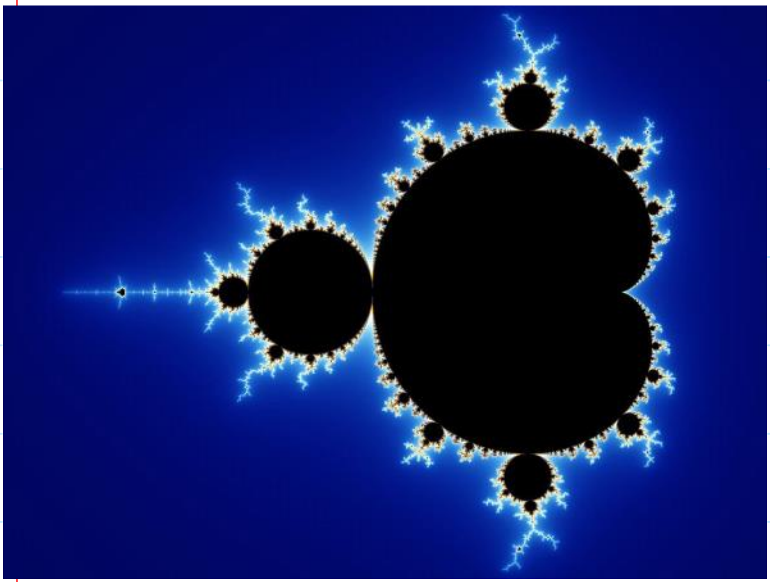
On the other hand, at any $(0,y)$, $y \neq 0$, take $0 < \varepsilon = \frac{|y|}{2}$ and $U = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$.

Then $U \cap F$ is a nbhd of $(0,y)$ in F

But, every sub-nbhd of $(0,y)$ in it is disconnected.

Thus, F is not locally connected.

Example. This is called Mandelbrot Set $M \subset \mathbb{C}$.



$\zeta \in M$ if the complex sequence $\zeta, \zeta^2+\zeta, (\zeta^2+\zeta)^2+\zeta, \ldots$ is bounded.

- It is connected (in fact, simply connected)
- Still, don't know if it is locally connected

Yoccoz, Fields Medal 1994

Proposition. Let X be locally connected. Then each connected component is both $\underbrace{\text{open}}_{\text{Not always true}}$ and $\underbrace{\text{closed}}_{\text{known before}}$.

Infinitely many components cluster to $\underbrace{\text{one component}}_{\text{not open}}$

A is connected
$\Downarrow$
$\bar{A}$ is also.

Component is maximal

Proof. Let $G \subset X$ be a connected component and $x \in G$ be arbitrary.

Take any nbhd of x, e.g., $x \in X \in \mathcal{J}$

By local connectedness,

$\exists$ connected U, $x \in \mathring{U} \subset U$

Since U is connected and G is maximal, we have $U \subset G$. $\therefore x \in \mathring{U} \subset G$.

Hence, G must be open.

Example. Given $\mathbb{R}$ and S^1, both standard.

How do we know $\mathbb{R} \not\cong S^1$?

Reason. $\mathbb{R}$ is non-compact, S^1 is compact.

That is, we used

$\boxed{\text{Continuous image of compact space is compact}}$

Example. What about $[0,1] \not\cong S^1$?

Argument. $[0,1] \setminus \{\frac{1}{2}\}$ is disconnected but $S^1 \setminus \{pt\}$ is connected.

$\Downarrow$ why?

$[0,1] \not\cong S^1$

We need something as below:

$\boxed{\text{Continuous image of connected space is connected}}$

AND

If $f: X \longrightarrow Y$ is a homeomorphism, $f(x_0) = y_0$, then $f|_{X \setminus \{x_0\}} : X \setminus \{x_0\} \longrightarrow Y \setminus \{y_0\}$ is a homeomorphism

Exercise

Exercise. Use similar technique to show that $S^2 \not\cong S^1 \times S^1$, both standard.

What's Underneath. For topological spaces

* Define $k(X) = \begin{cases} +1 & \text{if } X \text{ is compact} \\ -1 & \text{if } X \text{ is noncompact} \end{cases}$

$$k(X) \neq k(Y) \Longrightarrow X \ \cancel{\underline{\underline{\text{homeo}}}}\ Y$$

* Define $c(X)$ = number of connected components

$$c(X) \neq c(Y) \Longrightarrow X \ \cancel{\underline{\underline{\text{homeo}}}}\ Y$$

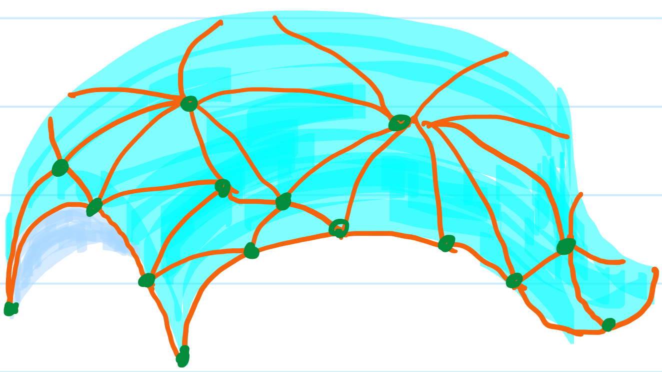
Further, define

$$s(X) = \sup\{c(X \setminus \{x\}) : x \in X\}$$

e.g., $s(S^1) = 1$, $s([0,1]) = 2$

$$s(X) \neq s(Y) \Longrightarrow X \ \cancel{\underline{\underline{\text{homeo}}}}\ Y$$

Example. Let Σ be a surface by intuition



For any $\boxed{\text{triangulation}}$ — some adjacent conditions

$$\chi(X) = V - E + F$$

Euler Characteristic

Note: independent of triangulation

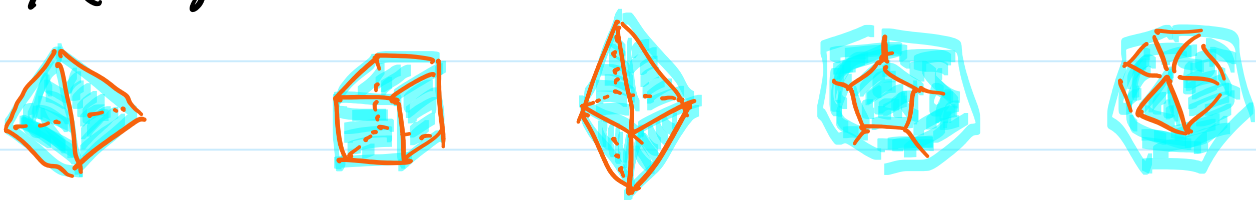
For higher dimensional object,

$$\chi(X) = (-1)^n b_n + \cdots + (-1)^2 b_2 + (-1) b_1 + b_0$$

$$\chi(X) \neq \chi(Y) \Rightarrow X \ \underset{\text{homeo}}{\not\cong} \ Y$$

Remarks.

- $\chi(\mathbb{R}^n) = \chi(D^n) = 1$, $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$
- $\chi(S^n) = (-1)^n + 1$



Tetrahedron	Cube	Octahedron	Dodeca...	Icosa...
$4-6+4$	$6-12+8$	$8-12+6$	$12-30+20$	$20-30+12$

- $\chi(S^1 \times S^1) = 0$ $\quad \chi(S^1 \times \cdots \times S^1) = 0$
- $\chi(RP^2) = 0 = \chi(\text{Klein})$

Theorem. $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$

Calculation is possible