Lect18-20180319

Wednesday, 14 March 2018 12:34 PM

Notions of compactness

Heine-Borel (HB)

Every open cover has a finite subcover

Bolzano-Weierstrass (BW)

Every infinite set has a cluster point

Sequentially compact (SC)

Every sequence has a convergent subseq.

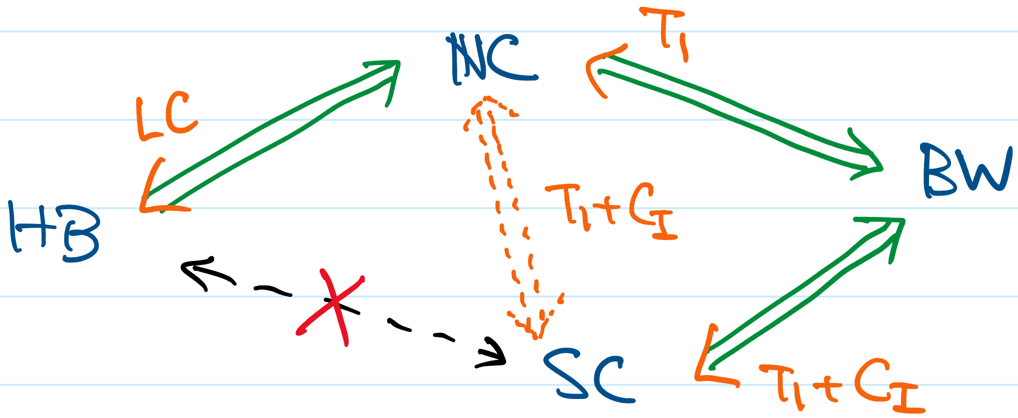
Countably compact (NC)

Every countable cover has a finite subcover.

Lindelöf (LC)

Every open cover has a countable subcover.

Equivalence Diagram (easy, need condition)



"HB $\Rightarrow$ NC" Trivially obvious.

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"SC $\Rightarrow$ BW" Idea

Infinite set ---> infinite distinct sequence

convergent subsequence ---> limit = cluster point

"$\sim$BW $\Rightarrow$ $\sim$NC" Idea and proof.

Wish: construct a bad countable cover

Infinite set A with $A' = \emptyset$

---> subset $B \longleftrightarrow \mathbb{N}$ with $B' = \emptyset$

Give me one example in $\mathbb{R}^2$,

countable and no cluster point!

Obviously, $\mathbb{Z}^2 = \{(m,n) \in \mathbb{R}^2 : m,n \in \mathbb{Z}\}$

Make a bad open cover of $\mathbb{R}^2$

no finite subcover

Easily, $\{\mathbb{R}^2 \setminus \mathbb{Z}^2\} \cup \{B((m,n), \frac{1}{2}) : (m,n) \in \mathbb{Z}^2\}$

Now, we have a countable $B=\{b_n : n\in\mathbb{N}\}$
with $B'=\phi$.

Analogous to $\mathbb{Z}^2$, can we use $X\setminus B$?

$\overline{B}=B\cup B'=B$, $\therefore$ B is closed, $X\setminus B\in\mathcal{T}$

Again, analogous to $\mathbb{Z}^2$, is B discrete?

Take any $b_n\in B$, $b_n\notin B'$, that is

$\exists$ nbhd of b_n, say $U_n\in\mathcal{T}$, such that

$U_n\cap B\setminus\{b_n\}=\phi$

$\longrightarrow U_n\cap B=\{b_n\}$

$\therefore \mathcal{G}=\{X\setminus B\}\cup\{U_n : n\in\mathbb{N}\}$ is a countable cover of X without finite subcover.

"BW $\xrightarrow{T_1+C_I}$ SC"

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

The set $\{x_n : n\in\mathbb{N}\}=A$

finite : $\exists$ constant subsequence

Infinite : By BW, $\exists$ cluster point x.

Wish: Use $C_I + T_1$ to construct subsequence.

At $x \in X$, $\exists$ countable local base
$\{U_n : n \in \mathbb{N}\}$
Since $x \in A'$, $\underset{\substack{\cup\!\!\!\!\!\! \\ x_{n_1}}}{U_1 \cap A \setminus \{x\} \neq \phi}$

x_{n_1} What next?

Use U_2, but for x_{n_2}, need $n_2 > n_1$
$(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence requires
$n_1 < n_2 < n_3 < \cdots$ (MATH 2050)

We have $A = \{x_n : n \in \mathbb{N}\}$, may assume $x_m \neq x_n$
and $x \in A'$ has a local base $\{U_n : n \in \mathbb{N}\}$

Already get $x_{n_1} \in U_1 \cap A \setminus \{x\}$, n_1 is minimal

By T_1, $x \in U_1 \cap U_2 \setminus \{x_1, x_2, \cdots, x_{n_1}\} \in \mathcal{T}$

By $x \in A'$, $\exists\, x_{n_2} \in U_1 \cap U_2 \setminus \{x_1, x_2, \cdots, x_{n_1}\}$

$n_2 > n_1$

Inductively, after $x_{n_1}, x_{n_2}, \cdots, x_{n_k}$,

$$x \in \bigcap_{j=1}^{k+1} U_j \setminus \{x_l : l = 1, \cdots, k\} \in \mathcal{T}$$

and $\exists$ $x_{n_{k+1}}$

The subsequence $(x_{n_k})_{k \in \mathbb{N}} \xrightarrow{\text{clearly}} x$.

"$\sim$ INC $\xrightarrow{T_1}$ $\sim$BW"

Let $\{G_n : n \in \mathbb{N}\}$ be a countable open cover for X, i.e., $\bigcup_{n\in\mathbb{N}} G_n = X$, such that it has no finite subcover.

Wish: Get an infinite set A with $A' = \phi$.

Start with any $x_1 \in X = \bigcup_{n=1}^{\infty} G_n$

$\therefore \exists\, n_1 \in \mathbb{N},\ x_1 \in G_{n_1}$

assume minimal, i.e., $x_1 \notin G_l \ \forall\, l = 1, \cdots, n_1 - 1$

Note that $X \neq G_1 \cup G_2 \cup \cdots \cup G_{n_1}$, no finite subcover

$\therefore \exists\, x_2 \in X \setminus \bigcup_{k=1}^{n_1} G_k$

Similarly, $\exists\, n_2 > n_1,\ x_2 \in G_{n_2}$,

$x_2 \notin G_l \ \forall\, l < n_2$. $x_2 \neq x_1$

Inductively, we have distinct $x_k \in G_{n_k}$

$x_k \notin G_l \quad \forall\, l < n_k$ $n_1 < n_2 < \cdots < n_k$

Let $A = \{x_k : k \in \mathbb{N}\}$, which is infinite

By BW, $\exists\, x \in A' \subset X = \bigcup_{n=1}^{\infty} G_n$

Thus, $\exists\, m \in \mathbb{N},\ x \in G_m$

where is it among n_k?

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We have the situation

$$\underbrace{n_1 < n_2 < \cdots < n_N}_{} \le m < \underbrace{n_{N+1} < n_{N+2} < \cdots}_{}$$

$$\begin{array}{cccccc} x_1 & x_2 & \cdots & x_N & x & x_k,\ k>m \\ \in & \in & & \in & \in & \cancel{\in} \\ G_{n_1} & G_{n_2} & \cdots & G_{n_N} & G_m & G_m \end{array}$$

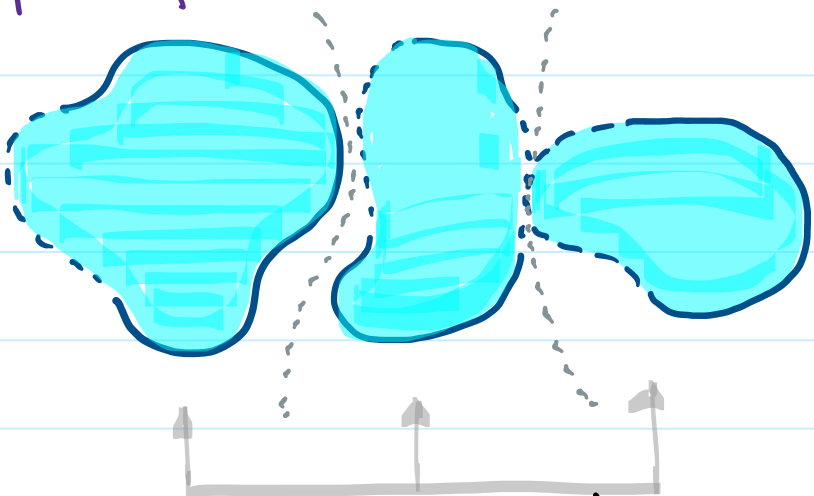
Let $V = G_m \setminus \{x_1, x_2, \cdots, x_{N-1}, x_N\}$

or $G_m \setminus \{x_1, x_2, \cdots, x_{N-1}\}$ if $x = x_N$

Then $x \in V \in \mathcal{J}$ T_1

But $V \cap A \setminus \{x\} = \phi$ contradicts $x \in A'$.

Concept of disconnected



May be separated by open sets

Definition. A space $(X, \mathcal{T}_X)$ is disconnected if $\exists\ \phi \neq U, V \in \mathcal{T}_X$ such that wrt X

$$X = U \cup V \quad \text{and} \quad U \cap V = \phi$$

Example.

* $X = (0,2) = \underbrace{(0,r]}_{\text{Not open in } (0,2)} \cup (r,2)$, $0 < r < 2$

* $X = [0,1) \cup (1,2)$ is disconnected. $[0,1)$ and $(1,2)$: both open in X.

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Note : $\underbrace{X = U \cup V \quad \text{and} \quad U \cap V = \emptyset}$

$\boxed{U, V \in \mathcal{T}_X, \ \boxed{\begin{array}{l} U = X \setminus V \\ V = X \setminus U \end{array}}} \Rightarrow$ U, V are open & closed

Proposition. X is disconnected $\Longleftrightarrow$ $\exists\ \phi \neq U \subsetneq X$, U is both open & closed.

$\Updownarrow$

$\exists\, U \subset X, \quad U \neq \phi \ \& \ U \neq X \ \& \ U \in \mathcal{T} \ \& \ X \setminus U \in \mathcal{T}$

What is its negation?

$\forall\, U \subset X, \quad U = \phi$ or $U = X$ or $U \notin \mathcal{T}$ or $X \setminus U \notin \mathcal{T}$

Is there a different writing?

Exercise. Compare the truth tables of

$P \vee Q \quad , \quad \sim P \to Q$

Fact: There are ${}^4C_1 + {}^4C_2 + {}^4C_3$ different ways

Definition. $(X, \mathcal{T})$ is connected if

$\forall\, S \subset X$ with both S and $X \setminus S \in \mathcal{T}$,

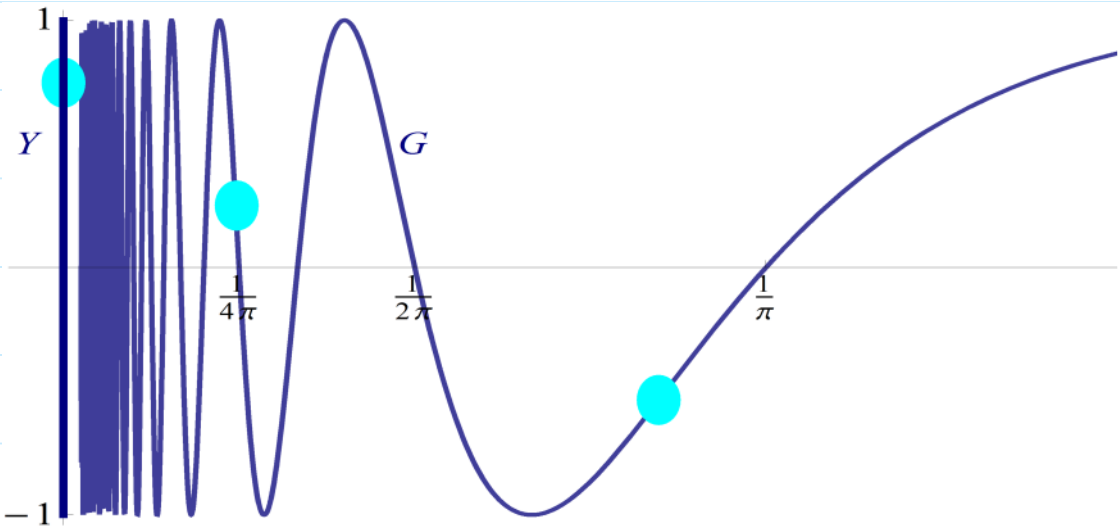
$S = \phi$ or $S = X$.

In other words, X is connected $\Longleftrightarrow$ only ϕ and X are both open & closed.

Famous Example. $X = Y \cup G \subset \mathbb{R}^2$ where

$Y = \{(x,y) \in \mathbb{R}^2 : x = 0\}$, i.e., y-axis

$G = \{(x,y) \in \mathbb{R}^2 : y = \sin\frac{1}{x}, x > 0\}$



Is it connected or disconnected?

What will be your strategy?

1. Let $S \subset X = Y \cup G$ be both open and closed.
   Then what??

2. Neither Y nor G is both open and closed
   $\therefore S \neq Y$ and $S \neq G$

3. Can S separate Y or G? Why not?

Conclusion: $S = \phi$ or $S = X$

(a) Y is not open

Take $(0,\frac{1}{2})\in Y$. Every open nbhd of $(0,\frac{1}{2})$ contains $(-\varepsilon,\varepsilon)\times(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon)$, $\varepsilon>0$.

$$\underbrace{\left(\frac{1}{2n\pi+\frac{\pi}{6}},\ \frac{1}{2}\right)}_{\text{in } X\setminus Y},\quad n>\frac{1}{2}\left(\frac{1}{\pi\varepsilon}-\frac{1}{6}\right)$$

$$\therefore\ (-\varepsilon,\varepsilon)\times\left(\tfrac{1}{2}-\varepsilon,\tfrac{1}{2}+\varepsilon\right)\cap Y \not\subset Y$$

(b) $G=X\setminus Y$ is not closed

(c) Y is connected (homeomorphic to $\mathbb{R}$)

(d) G is connected Why?

Both (c) and (d) will be seen later.

(e) Let $S\subset X=Y\cup G$ be both open & closed

$(S\cap Y)$ both open & closed in Y | ϕ or Y

$(S\cap G)$ both open & closed in G | ϕ or G

$\therefore\ (S\subset G$ or $S\supset Y)$ and $(S\subset Y$ or $S\supset G)$

$\therefore\ S=\phi$ or $\underbrace{S=G \text{ or } S=Y}_{\text{excluded}}$ or $S=X$

Proposition. Let $f: X \longrightarrow Y$ be continuous.
If X is connected then $f(X)$ is connected.

Remark. In the example, G is the image of $(x, \sin\frac{1}{x}) : (0,\infty) \longrightarrow \mathbb{R}^2$

Proof. Let $S \subset f(X)$ be both open & closed,

$$S = V \cap f(X), \quad V \in \mathcal{T}_Y, \quad Y \setminus V \in \mathcal{T}_Y$$

Then $f^{-1}(V)$ and $X \setminus f^{-1}(V) \in \mathcal{T}_X$

$$\therefore f^{-1}(V) = \phi \quad \text{or} \quad f^{-1}(V) = X$$

$$S = V \cap f(X) = \phi \qquad\qquad V = f(X) \Rightarrow S = V \cap f(X) = f(X)$$

Proposition. X is connected $\Longleftrightarrow$
$\forall\, \phi \neq A, B \subset X$ with $A \cap B = \phi$ and $A \cup B = X$,
$\bar{A} \cap B \neq \phi$ or $A \cap \bar{B} \neq \phi$

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Examples.

* $X = (0,2)$ which is connected

$$= \underbrace{(0,r]}_{A} \cup \underbrace{(r,2)}_{B} , \quad 0<r<2$$

B, $\bar{B} = [r,2)$ in X

$$A \cap \bar{B} = \{r\} \neq \phi$$

* $X = \underbrace{[0,1)}_{A=\bar{A}} \cup \underbrace{(1,2)}_{B=\bar{B}}$, disconnected

$A = \bar{A}$ $\quad$ $B = \bar{B}$ in X

* $X = Y \cup G$, connected

$\bar{Y} = Y \quad \therefore \bar{Y} \cap G = \phi$

$\bar{G} \supset \{0\} \times [-1,1]$, $\therefore Y \cap \bar{G} \neq \phi$

"$\Rightarrow$" By contrapositive, assume $\exists\ A, B \subset X$

$$\underbrace{X = A \cup B,\ A \cap B = \phi,\ A \cap \bar{B} = \phi,\ \bar{A} \cap B = \phi}$$

$A = X \setminus B$ and $\bar{A} = X \setminus B$

$\therefore A$ is closed

Similarly, B is closed

"$\Leftarrow$" Also by contrapositive, assume $X = U \cup V$,

$U \cap V = \phi$, U, V both open and closed.

Then $\bar{U} \cap V = U \cap V = \phi = U \cap V = U \cap \bar{V}$.