Lect17-20180314

Monday, 12 March 2018 2:20 PM

Definition. $(X, \mathcal{T})$ is *locally compact* if
$\forall\, x \in X\ \exists$ compact nbhd K of x,
i.e., $x \in \mathring{K}$ and K is compact.

Atypical, but adopted historically.

Definition $(X, \mathcal{T})$ *has compact local bases*
if $\forall\, x \in X\ \exists$ local base $\mathcal{K}_x$ at x
consisting of compact nbhds of x.

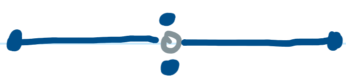
Obviously, X has compact local bases

$$\Downarrow \quad \Uparrow ???$$

X is locally compact

Proposition. If $(X, \mathcal{T})$ is Hausdorff and
locally compact, then it has
compact local bases.

Exercise. Find an example for
Locall compact $\not\Longrightarrow$ having compact Local bases

Sorry.  is not an example !

Exercise. Is cofinite topology a candidate
for the above "$\not\Rightarrow$" example ?

Lect17-p2

Mar 16, Friday, 2018 9:18 AM

Key idea: locally compact and Hausdorff

$\Longrightarrow$ having compact local bases

Let $x \in X$ and $U \in \mathcal{T}$ with $x \in U$

Wish to get: $x \in$ compact nbhd $\subset U$

By definition of locally compact

$\exists$ compact $K \subset X$, $x \in \mathring{K} \subset K$

What is good about K?

$(K, \mathcal{T}|_K)$ is compact Hausdorff

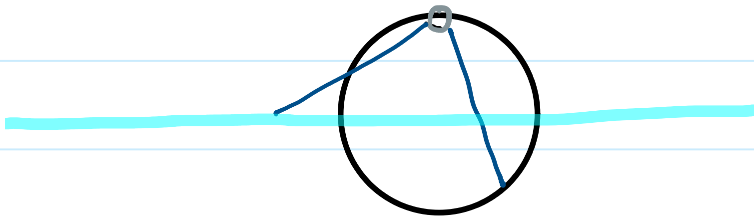
$\therefore$ $\boxed{\text{compact } T_3 \text{, compact } T_4}$

We have $x \in U \cap \mathring{K} \subset U \cap K \in \mathcal{T}|_K$

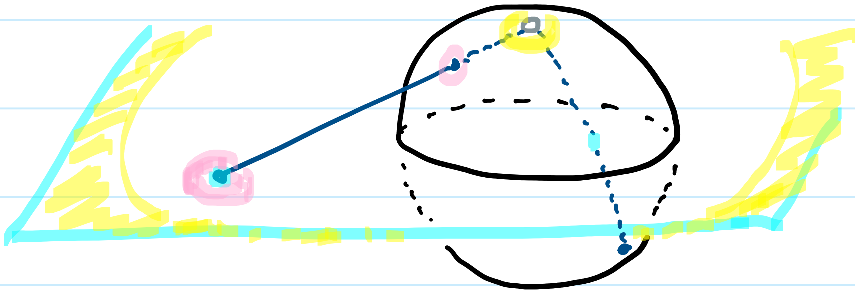
By $\boxed{\text{last time}}$, $\exists$ $U_1 \in \mathcal{T}|_K$, $x \in U_1 \subset \underbrace{\overline{U_1}}_{\text{compact nbhd}} \subset U \cap \mathring{K}$

Stereographic Projection

$\mathbb{R} \longleftrightarrow S^1 \setminus \{*\}$



$\mathbb{R}^2 \longleftrightarrow S^2 \setminus \{*\}$



$\mathbb{R}^n \longleftrightarrow S^n \setminus \{*\}$

$$x = (x_1, \cdots, x_n) \quad \left(\frac{2x_1}{\|x\|^2+1}, \cdots, \frac{2x_n}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1} \right)$$

One-point Compactification

Let $(X, \mathcal{T})$ be locally compact Hausdorff

equivalently, $\exists$ compact local bases

Then $\exists$ compact Hausdorff $(X^*, \mathcal{T}^*)$ such that

(i) $X^* \setminus X$ is singleton

(ii) $\mathcal{T} = \mathcal{T}^*|_X$

(iii) If X is noncompact then $\overline{X} = X^*$

If X is compact then $X^* \setminus X$ is isolated.

Proof. Pick any $\infty \notin X$, let $X^* = X \cup \{\infty\}$.

Define $\mathcal{J}^* = \mathcal{J} \cup \underbrace{\{(X \setminus K) \cup \{\infty\} : K \subset X \text{ is compact}\}}_{\text{Compare with } \mathbb{R}^n \cup \{\infty\} \leftrightarrow S^n}$

Why is $\mathcal{J}^*$ a topology?

Crucial points:

* $\bigcup_\alpha \left[(X \setminus K_\alpha) \cup \{\infty\} \right] = \Big(X \setminus \underbrace{\bigcap_\alpha K_\alpha}_{\text{why compact?}}\Big) \cup \{\infty\}$

* $\underset{\mathcal{J}}{\underset{\cap}{G}} \cup (X \setminus K) \cup \{\infty\} = \Big(X \setminus \underbrace{(K \cap (X \setminus G))}_{\text{Again compact}}\Big) \cup \{\infty\}$

* $\bigcap_{j=1}^{n} \left[(X \setminus K_j) \cup \{\infty\} \right] = \Big(X \setminus \underbrace{\bigcup_{j=1}^{n} K_j}_{\text{Trivial compact}}\Big) \cup \{\infty\}$

* $\underset{\mathcal{J}}{\underset{\cap}{G}} \cap \left[(X \setminus K) \cup \{\infty\} \right] = G \cap \underbrace{(X \setminus K)}_{\text{Again, in } \mathcal{J}}$

Why is $(X^*, \mathcal{J}^*)$ Hausdorff ?

Only need to consider $x \in X$ and $\infty \notin X$

Wish to get: $x \in U \in \mathcal{J}$, $\infty \in (X \setminus K) \cup \{\infty\}$

and $U \cap (X \setminus K) = \phi$

$U \subset K$

That is, need $x \in U \subset \mathring{K} \subset K$

compact nbhd of x

Yes, because X is locally compact.

How to check compactness of $(X^*, \mathcal{J}^*)$?

Take any open cover of X^*

$\|$ WLOG, Why?

$\mathcal{G} \cup \{(X \setminus K) \cup \{\infty\}\}$ $\mathcal{G} \subset \mathcal{J}$

What must $\mathcal{G}$ satisfy to cover X^* ?

$$(\cup\mathcal{G}) \cup (X \setminus K) = X$$

$\therefore \cup\mathcal{G} \supset K$ and

finite subcover obviously exists for X^*

To ask $\infty \overset{?}{\in} \overline{X}$, examine what ?

$$(X\setminus K) \cup \{\infty\} \cap X = X \setminus K \quad \forall \text{ compact } K$$

X noncompact

$X \setminus K \neq \emptyset \ \forall K \Rightarrow \infty \in \overline{X}$

X compact, take $K = X$

$\{\infty\} \in \mathcal{J}^*$, isolated