Lect15-20180307

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Definition (Heine-Borel). Let $(X, \mathcal{T})$ be a space.
A set $K \subseteq X$ is **compact** if $\mathcal{G} \subset \mathcal{T}$ and $\bigcup \mathcal{G} \supset K$,
$\exists$ finite $\mathcal{F} \subset \mathcal{G}$ such that $\bigcup \mathcal{F} \supset K$.

Equivalent: K is compact wrt the induced topology $\mathcal{T}|_K$

Example. $A \subset \mathbb{R}^n$ is compact $\Longleftrightarrow$ A is closed and bounded.

$\exists$ $R > 0$ such that

$$\underbrace{A \subset \{x \in \mathbb{R}^n : \|x\| \leq R\}}_{\text{itself a compact space}}$$

The above becomes:

$A \subset$ compact set in $\mathbb{R}^n$ is compact $\Longleftrightarrow$ A is closed

Theorem. Let X be compact and $A \subset X$.
A is closed $\Longrightarrow$ A is compact.

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Proof. Let $\mathcal{G} \subset \mathcal{T}$ and $\bigcup \mathcal{G} \supset A$.

Wish. Find a finite $\mathcal{F} \subset \mathcal{G}$, $\bigcup \mathcal{F} \supset A$.

Try to get from an open cover for X.

Since A is closed, i.e., $X \setminus A \in \mathcal{T}$

$\mathcal{G} \cup \{X \setminus A\} \subset \mathcal{T}$ covers X

$\exists$ finite $\mathcal{F} \cup \{X \setminus A\}$ covering X

and $\mathcal{F}$ is a finite subcover for A.

Example. Let $X =$  and

$A =$ 

Then X is compact (why?)

and A is compact (why?)

but A is not closed (really?)

Theorem. Let $f : X \to Y$ be continuous. If $A \subset X$ is compact, then $f(A) \subset Y$ is so.

Consequence. For the above example, consider

$\underbrace{[-1,1] \sqcup [-1,1]}_{\text{both compact}} \xrightarrow{q}$ 

Theorem. Let $f: X \longrightarrow Y$ be continuous. If $A \subset X$ is compact, then $f(A) \subset Y$ is so.

Proof Let $\mathcal{G} \subset \mathcal{J}_Y$ with $\bigcup \mathcal{G} \supset f(A)$.

Wish. Get a finite subcover $\mathcal{F} \subset \mathcal{G}$ from the compactness of A.

Define $\mathcal{E} = \{ f^{-1}(V) : V \in \mathcal{G} \} \subset \mathcal{J}_X$

why?

Then $\bigcup \mathcal{E} \supset A$

Let $a \in A$, $f(a) \in f(A) \subset \bigcup \mathcal{G}$

$\therefore \exists\ V \in \mathcal{G}$ with $f(a) \in V$

$\therefore\ a \in f^{-1}(V) \in \mathcal{E}$

By compactness of A, $\exists$ finite subcover

$\{ f^{-1}(V_1), f^{-1}(V_2), \cdots, f^{-1}(V_n) \} \subset \mathcal{E}$

Then $\mathcal{F} = \{ V_1, \cdots, V_n \} \subset \mathcal{G}$ is a finite subcover for $f(A)$.

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Consequences

(1) X is compact $\Longrightarrow$ $X/\sim \; = q(X)$ is so.

$\overset{?}{\Longleftarrow}$

$\mathbb{R}$ is not $\quad \cancel{\Longleftarrow} \quad S^1 = \mathbb{R}/\mathbb{Z}$ is compact

(2) $P = \prod_{\alpha \in I} X_\alpha$ is compact $\Longrightarrow$ Each $X_\beta = \pi_\beta(P)$ is compact

useless

Is the meaningful "$\Longleftarrow$" true ?

Theorem. If X_α is compact for each $\alpha \in I$ then the product $P = \prod_{\alpha \in I} X_\alpha$ is compact.

Finite I : Proof will be given.

Infinite I : Tychonoff Theorem.

Theorem. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be compact. Then the product space $X \times Y$ is compact.

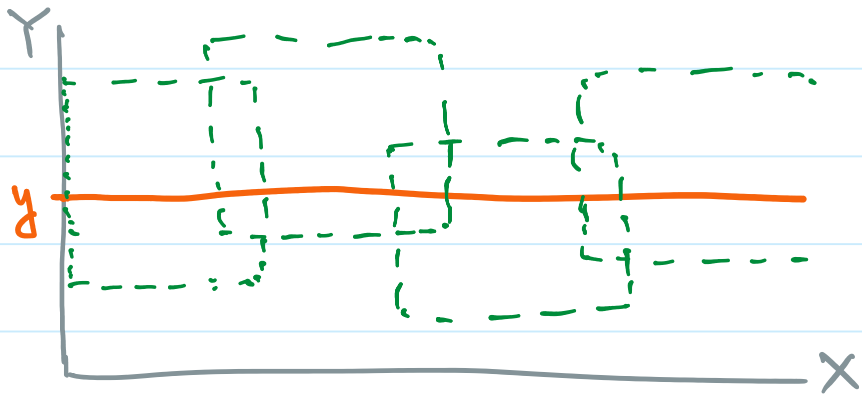
Proof. Let $\mathcal{G} \subset \mathcal{T}_{X \times Y}$ satisfy $\bigcup \mathcal{G} = X \times Y$. Without loss of generality, assume each set in $\mathcal{G}$ is of the form $U \times V$, $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$

Exercise

The general situation is a union of such sets.

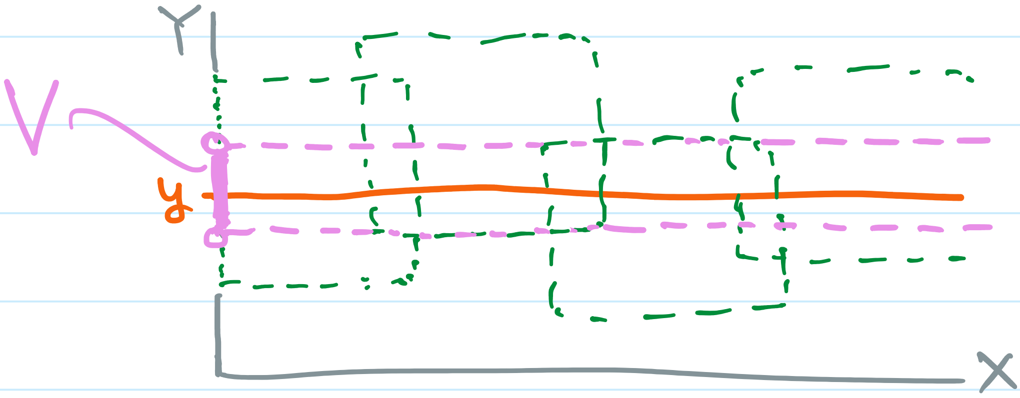
For any fixed $y \in Y$, $X \times \{y\}$ is compact and $\bigcup \mathcal{G} \supset X \times Y \supset X \times \{y\}$. There is a finite subcover $\mathcal{F}^y = \{U_k^y \times V_k^y : k = 1, \cdots, n_y\}$

$$\bigcup_{k=1}^{n_y} U_k^y \times V_k^y \supset X \times \{y\}$$



What can we conclude?

Is this picture true ?



That is, $\exists\, V^y \in \mathfrak{I}_Y$ such that

$$\bigcup_{k=1}^{n_y} U_k^y \times V_k^y \supset X \times V^y \supset X \times \{y\}$$

Almost, $V^y = \bigcap_{k=1}^{n_y} V_k^y \in \mathfrak{I}_Y$

Then, how to proceed ?

Above is true for arbitrary $y \in Y$,

$\{V^y : y \in Y\}$ is an open cover for Y

There is a finite subcover

$\{V^{y_1}, V^{y_2}, \ldots, V^{y_m}\}$ for Y.

$$\mathcal{F} = \mathcal{F}^{y_1} \cup \mathcal{F}^{y_2} \cup \cdots \cup \mathcal{F}^{y_m}$$

$$= \{U_k^{y_l} \times V_k^{y_l} : k = 1, \ldots, n_{y_l} ;\ l = 1, \ldots, m\}$$

Open cover: $\mathcal{G} \subset \mathcal{T}$ with $\underbrace{\bigcup\mathcal{G} \supset K}$

Its negation ??

$$\sim(\bigcup\mathcal{G} \supset K) \iff K \setminus \bigcup\mathcal{G} \neq \emptyset$$

$$K \setminus \bigcup\mathcal{G} = \bigcap \underbrace{\{K \setminus G : G \in \mathcal{G}\}}_{\text{Family of closed sets}}$$

Compactness: If $\mathcal{G} \subset \mathcal{T}$ with $\bigcup\mathcal{G} \supset K$ then $\exists$ finite $\mathcal{F} \subset \mathcal{G}$ with $\bigcup\mathcal{F} \supset K$.

What is the contrapositive?

K is compact $\iff$ For every $\mathcal{C}$ of closed sets in K, if every finite $\mathcal{H} \subset \mathcal{C}$ has $\bigcap\mathcal{H} \neq \emptyset$ then $\bigcap\mathcal{C} \neq \emptyset$

Temporary Notation. Let $\mathcal{A} \subset \mathcal{P}(K)$

Denote $\overline{\mathcal{A}} = \{\overline{A} \subset K : A \in \mathcal{A}\}$

$$\boxed{\begin{array}{l} K \text{ is compact} \iff \text{For every } \mathcal{A} \subset \mathcal{P}(K), \\ \text{if every finite } \mathcal{H} \subset \mathcal{A} \text{ has } \bigcap\overline{\mathcal{H}} \neq \emptyset \\ \text{then } \bigcap\overline{\mathcal{A}} \neq \emptyset. \end{array}}$$

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Try to use this version to prove

X, Y compact $\Longrightarrow$ $X \times Y$ compact

Let $\mathcal{A} \subset \mathcal{P}(X \times Y)$ which satisfies

$\underbrace{\text{every finite } \mathcal{H} \subset \mathcal{A} \text{ has } \cap \mathcal{H} \neq \emptyset}$

FCIP : Finite closure intersection property

Project every sets to X and Y and get

$\mathcal{A}_X = \{ \pi_X(A) : A \in \mathcal{A} \}$
$\mathcal{A}_Y = \{ \pi_Y(A) : A \in \mathcal{A} \}$ $\Big\}$ Verify has FCIP

$\therefore$ Both $\underset{\ni x}{\cap \overline{\mathcal{A}_X}} \neq \emptyset$, $\underset{\ni y}{\cap \overline{\mathcal{A}_Y}} \neq \emptyset \overset{??}{\Longrightarrow} (x,y) \in \cap \overline{\mathcal{A}}$

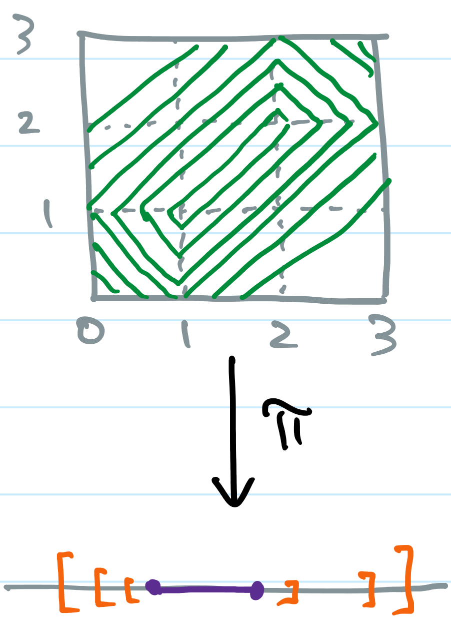
Example. $X = Y = [0,3]$ are compact

$\mathcal{A} \subset \mathcal{P}(X \times Y)$ consists of sets below

It has FCIP.

In fact, $F_n \supset F_{n+1}$

$\cap \mathcal{H} = F_{max}$



$\mathcal{A}_X$ and $\mathcal{A}_Y$ contains intervals $\supset [1,2]$

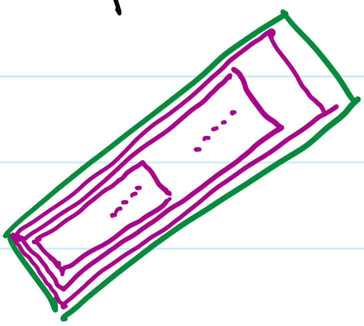
$\therefore \cap \overline{\mathcal{A}_X} = \cap \overline{\mathcal{A}_Y} = [1,2]$

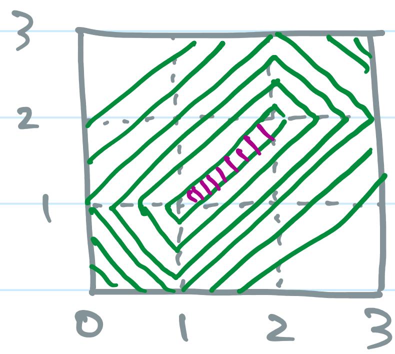
Now, we know that $1 \in \cap \overline{\mathcal{A}_X}$, $2 \in \cap \overline{\mathcal{A}_Y}$

Do we have $(1,2) \in \cap \overline{\mathcal{A}}$

Crucial Idea. $\forall\, \mathcal{A}$ with FCIP, $\exists\, \mathcal{M} \supset \mathcal{A}$ also has FCIP, but better.

For example, each set $\in \mathcal{A}$, add these sets and form $\mathcal{M}$





Then $\cap \overline{\mathcal{M}_X} = \{1\} = \cap \overline{\mathcal{M}_Y}$

and $(1,1) \in \cap \overline{\mathcal{M}} \subset \cap \overline{\mathcal{A}}$

Essential Argument of Tychonoff.

For $\mathcal{A} \subset \mathcal{P}(\prod_{\alpha \in I} X_\alpha)$ having FCIP,

use Zorn's Lemma to get a maximal

$\mathcal{A} \subset \mathcal{M} \subset \mathcal{P}(\prod_{\alpha \in I} X_\alpha)$ having FCIP.

Then $\mathcal{M}_\alpha = \{\pi_\alpha(M) : M \in \mathcal{M}\}$ also FCIP

By compactness of X_α, $\exists\, x_\alpha \in \cap \overline{\mathcal{M}_\alpha}$

Maximality of $\mathcal{M} \Rightarrow (x_\alpha) \in \cap \overline{\mathcal{M}} \subset \cap \overline{\mathcal{A}}$