Our current issue. Given $(X, \mathcal{J})$ and $A \subset X$. Whether a continuous $f: A \longrightarrow Y$ has a continuous extension $\tilde{f}: X \longrightarrow Y$.

known.

① If A is dense then $\tilde{f}$ is unique.

② If X is special (normal, metric, $\mathbb{R}^n$)
A is closed, $Y = [-a, a] \subset \mathbb{R}$

then Yes.

③ To be discussed, required another concept.

Question. How do we define completeness?

Definition. A metric space (X, d) is complete if every Cauchy sequence converges (in X).

Remark. Both Cauchy sequence and so completeness are only defined with metric

Proposition In a complete metric space X
$Y \subset X$ is complete $\Longleftrightarrow$ Y is closed.

"$\Rightarrow$" Rewrite Y is closed, i.e., $\overline{Y} \subset Y$

Let $x \in \overline{Y}$ Wish: $x \in Y$

G_I How to use sequence!

$\exists$ (y_n) in Y, $y_n \longrightarrow x$ in X

(y_n) is Cauchy as in X

True in Y

$y_n \longrightarrow y = x$
uniqueness

"$\Leftarrow$" Let $(y_n)_{n=1}^{\infty}$ be Cauchy sequence in Y

Wish. $\exists$ $y \in Y$ s.t. $y_n \longrightarrow y$.

Definition. A mapping $f: (X, d_X) \longrightarrow (Y, d_Y)$ is *uniformly continuous* if

$$\forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ such that}$$
$$\forall\ x_1, x_2 \in X \text{ with } d_X(x_1, x_2) < \delta,$$
$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Existence Theorem. Given (X, d_X), (Y, d_Y) where Y is complete and $\bar{A} = X$.
If $f: A \longrightarrow Y$ is uniformly continuous then $\exists$ *unique uniformly continuous* extension $\tilde{f}: X \longrightarrow Y$, i.e., $\tilde{f}|_A \equiv f$.

Idea of proof.
Let $x \in X = \bar{A}$ wish to define $\tilde{f}(x)$.
$\uparrow$ metric

$\uparrow$ converges $\qquad\qquad$ // define $\uparrow$ converges (Y complete)

$\exists\ (a_n^x)_{n=1}^{\infty}$ in A $\xrightarrow[\text{defined on } A]{f}$ $(f(a_n^x))_{n=1}^{\infty}$

converge $\Downarrow$ Cauchy $\qquad\qquad$?? Cauchy ??

$d_X(a_m^x, a_n^x) < \delta \xRightarrow{\text{uniform continuous}} d_Y(f(a_m^x), f(a_n^x)) < \varepsilon$

Question. The above argument has choice!

More rigorous treatment.

Recall Uniform continuity of

$$\tilde{f} : (X, d_X) \longrightarrow (Y, d_Y)$$

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that

if $d_X(x_1, x_2) < \delta$ then $d_Y(\tilde{f}(x_1), \tilde{f}(x_2)) < \varepsilon$ Aim