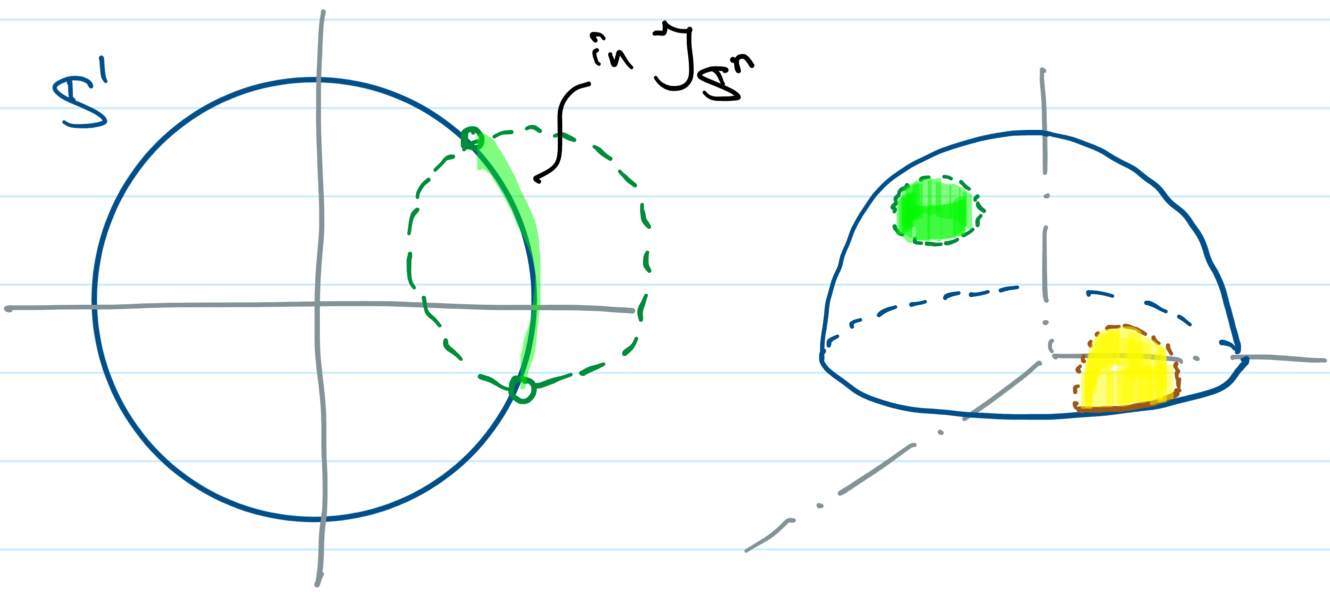
Given a topological space $(X, \mathcal{T})$ and $A \subset X$.
Hope to define a topology for A which is coming from X, "inherited".

Definition. $\mathcal{T}|_A = \{G \cap A : G \in \mathcal{T}\}$ is called the subspace topology or relative topology or induced topology on A from X.

Exercise. Verify (T1) & (T2).

Example. $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \subset \mathbb{R}^n, \mathcal{T}_{std}$



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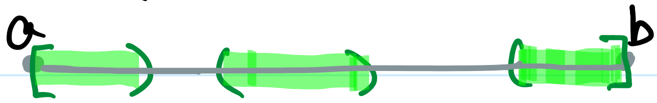
Example. $(X, \mathcal{T}) = (\mathbb{R}, \mathcal{T}_{std})$

* $A = [a,b]$



$$\underbrace{[a, a+\varepsilon),\ (c,d),\ (b-\delta, b]}_{\text{form a base for } \mathcal{T}|_{[a,b]}}$$

* $A = (1,2) \cup [3,4)$
  - Which one is open in A? $(3,4)$ or $[3,4)$
  - Is the set $(1,2)$ closed in A?

Question. What are the induced topologies on $\mathbb{Q}$ from $(\mathbb{R}, \mathcal{T}_{std})$ or $(\mathbb{R}, \mathcal{T}_{ll})$?

From the above, given $P \subsetneq A \subset X$ in $(X, \mathcal{T})$

$$P \in \mathcal{T} \xrightarrow{\text{trivial}} P \in \mathcal{T}|_A$$

$$\xleftarrow{\text{need condition}}$$

Simple condition. If $A \in \mathcal{T}$ then

$P \subsetneq A$ and $P \in \mathcal{T}|_A \Rightarrow P \in \mathcal{T}$.

Converse? If second line then $A \in \mathcal{T}$?

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Hereditary Issues.

Let $Z \subset A \subset X$ with $(X, \mathcal{T})$

Three topologies,

$$\mathcal{T}|_A, \quad \mathcal{T}|_Z, \quad (\mathcal{T}|_A)|_Z$$

Intersection is transitive

$$(☀) \cap (\Delta \cap ★) = (☀ \cap \Delta) \cap ★$$

Open, closed, interior, closure, etc.

$$\mathrm{Int}_A(Z) = \mathrm{Int}_X(Z) \cap A$$

$$\mathrm{Cl}_A(Z) = \mathrm{Cl}_X(Z) \cap A$$

Sequence convergence

Future discussion

Mapping restrictions

$$f : X \longrightarrow Y, \qquad f|_A : A \longrightarrow Y$$

continuous $\Longrightarrow$ continuous

Key idea. Let $V \in \mathcal{T}_Y$

Wish: $(f|_A)^{-1}(V) \in \mathcal{T}|_A$

$$(f|_A)^{-1}(V) = f^{-1}(V) \cap A, \qquad f^{-1}(V) \in \mathcal{T}_X$$

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Mapping Extension

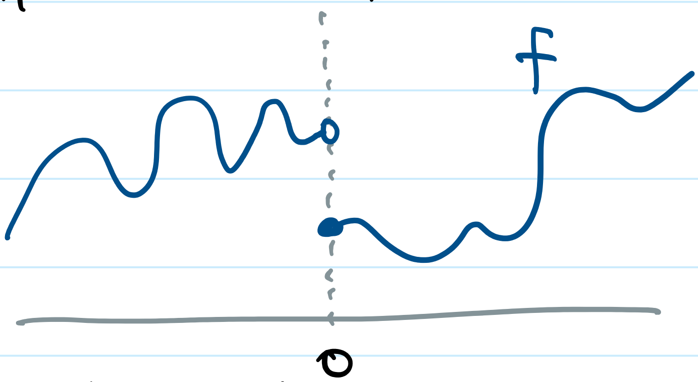
Clearly, $\underset{\text{continuous}}{f|_A : A \longrightarrow Y} \;\not\Longrightarrow\; \underset{\text{continuous}}{f : X \longrightarrow Y}$

What if we have

$f|_{A_\alpha} : A_\alpha \longrightarrow Y$ continuous, $\bigcup_{\alpha \in I} A_\alpha = X$

Bad Example. On $(\mathbb{R}, \mathcal{J}_{std})$,

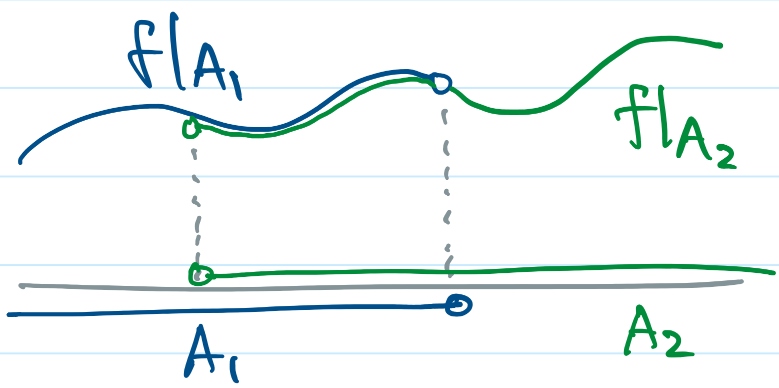
$A_1 = (-\infty, 0)$, $A_2 = [0, \infty)$



Both $f|_{A_1}$, $f|_{A_2}$ are continuous

Any Remedy ?

$f|_{A_1}$, $f|_{A_2}$ continuous $\Longrightarrow$ f continuous



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Proposition. Given $f:(X,\mathcal{J}) \longrightarrow Y$ and $X = \bigcup_{\alpha\in I} G_\alpha$ for $G_\alpha \in \mathcal{J}$.

If each $f|_{G_\alpha} : G_\alpha \longrightarrow Y$ is continuous

then so is $f: X \longrightarrow Y$.

Key idea.

$$f^{-1}(V) = f^{-1}(V) \cap \bigcup_{\alpha\in I} G_\alpha$$

$$= \bigcup_{\alpha\in I} \underbrace{\left[f^{-1}(V) \cap G_\alpha \right]}_{\text{open in } G_\alpha} = \bigcup_{\alpha\in I} \underbrace{\left(f|_{G_\alpha}\right)^{-1}(V)}_{}$$

open in G_α, need care

Equivalent Version. Given $X = \bigcup_{\alpha\in I} G_\alpha$ as above.

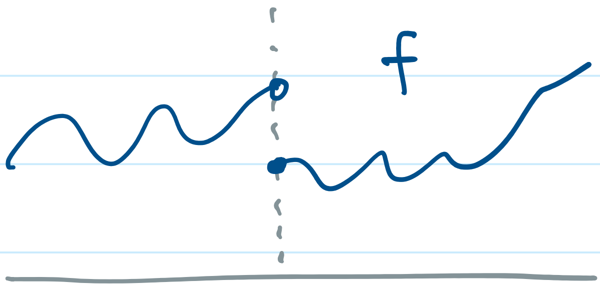
If $f_\alpha : G_\alpha \longrightarrow Y$ is a family of continuous mappings satisfying $f_\alpha \equiv f_\beta$ on $G_\alpha \cap G_\beta$

then $\exists$ continuous $f: X \longrightarrow Y$ extension,

i.e. $f|_{G_\alpha} \equiv f_\alpha$

Trivial to define the suitable f.

Bad becomes Good !!



It is continuous on $(\mathbb{R}, \mathcal{J}_{ll})$.

What about union of closed sets ?

Proposition. Let A, B be closed and $X = A \cup B$. If both $f|_A$, $f|_B$ are continuous then $f : X \to Y$ is so.

Proof Almost the same as before

Simply $f^{-1}(H) = f^{-1}(H) \cap (A \cup B)$
$= (f|_A)^{-1}(H) \cup (f|_B)^{-1}(H)$

Exercise.

* Write the equivalent version.
* Give examples about infinite union of closed sets, some works some doesn't.

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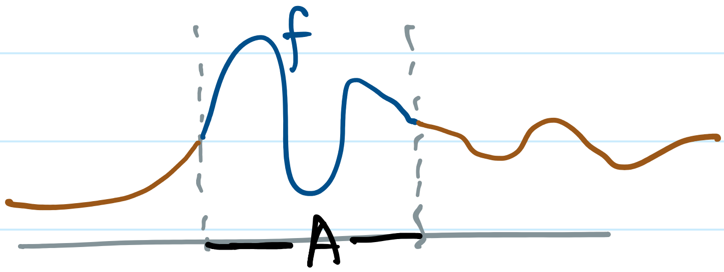
Harder Question (for later

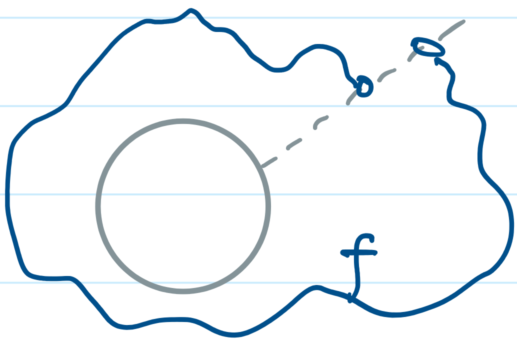
Only given $A \subset X$ and continuous $f: A \longrightarrow Y$

$\underbrace{\text{Do we have}}_{\text{existence}}$ continuous $\tilde{f}: X \longrightarrow Y$, $\tilde{f}|_A = f$?



$X = S^1$

$A = S^1 \setminus \{*\}$



$\tilde{f}$ does not exist!