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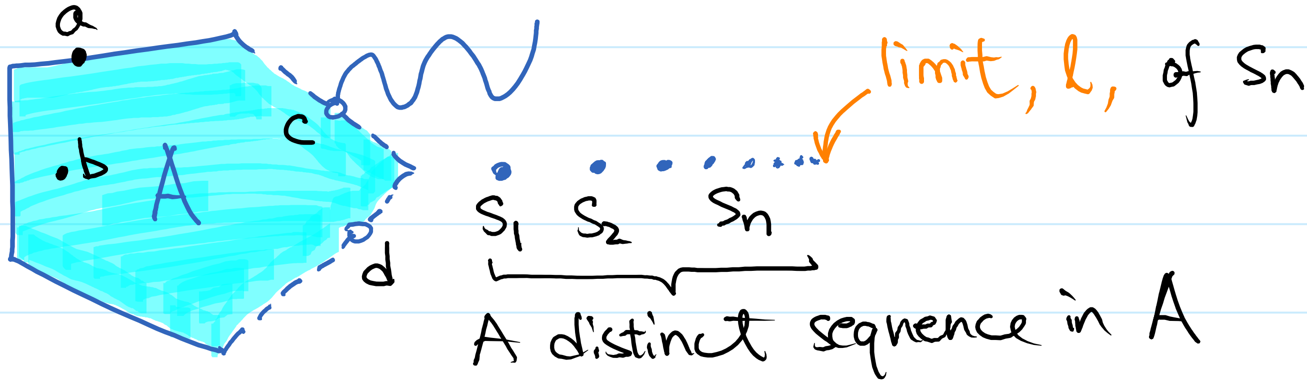
Besides interior points, there are others

Example illustrated in $\mathbb{R}^2$



The points, a, b, c, d, Sqqqq, ℓ;

each is different from others

How??

Note: Sqqqq has small nbhds that do not contain any other points of A

Definition. $x \in A$ is an isolated point of A if $\exists \underbrace{\text{nbhd } N \text{ of } x}_{}$, $N \cap A \setminus \{x\} = \phi$

may use $U \in \mathcal{T}$ and $x \in U$

What is its negation?

Definition. $x \in X$ (may not in A) is a cluster point or an accumulation point of $A \subset X$ (some books call it limit point) if $\forall$ nbhd N of x, $N \cap A \setminus \{x\} \neq \phi$

$A' = \{$all cluster points of $A\}$ is called the derived set of A.

$\bar{A}$ or $Cl(A) = A \cup A'$ is called the closure of A.

Easy observation. $x \in \bar{A} \iff$ $\underbrace{\forall \text{ nbhd } N \text{ of } x}_{\text{again may use } U \in \mathcal{J} \text{ and } x \in U}$, $N \cap A \neq \phi$

Question. The negation.

$x \notin \bar{A} \iff$??

$\exists$ nbhd N of x, $N \cap A = \phi$

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$x \notin \bar{A} \iff \exists$ nbhd N of x,

$N \cap A = \phi$

$\Updownarrow$

$x \in N \subset X \setminus A$

$\therefore\ x \in X \setminus \bar{A} \iff x \in (X \setminus A)^\circ$

Thus, $\boxed{X \setminus \bar{A} = (X \setminus A)^\circ}$

will be used later

Usual Intuition. For $A \subset X$,

$\mathring{A}$: its "skin" removed.

$\bar{A}$: with all its "skin"

Question.* What exactly is "skin"?

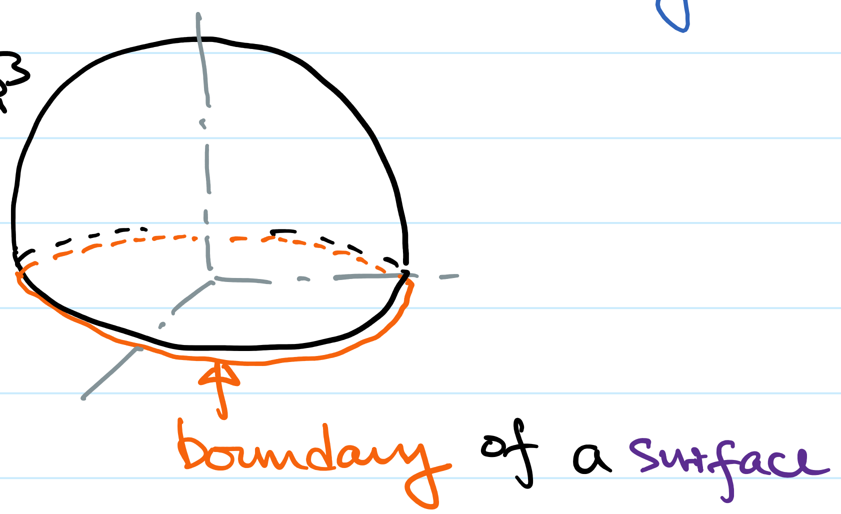
Answer. Check the definition of

Frontier of a set $A \subset X$

(sometime called boundary)

$A =$ Hemisphere $\subset \mathbb{R}^3$

Frontier$(A) = A$



Question

$* \quad G = \mathring{G} \iff G$ is open

$F = \overline{F} \iff$ What is F'?

Now, $\boxed{X \setminus \overline{F} = (X \setminus F)^\circ}$

$\|$ if $F = \overline{F}$

$\iff X \setminus F = (X \setminus F)^\circ$

$\iff X \setminus F$ is open

Definition. A set $F \subset X$ is closed if $X \setminus F$ is open; equivalently

$F = \overline{F} \iff F \supset \overline{F} = F \cup F'$

Propositions.

① F is closed $\iff F \supset F'$

② $\mathring{A}$ is the largest open set contained in A

If $G \in \mathcal{J}$, $G \subset A$, then $G \subset \mathring{A}$

③ $\overline{A}$ is the smallest closed set containing A

If $X \setminus F \in \mathcal{J}$ and $F \supset A$ then $F \supset \overline{A}$

Proof.

① Trivial ② Elementary ③ Obvious

$F = F \cup F'$

* Already know $\mathring{A}$ is open

* Let $G \in \mathcal{J}$, $G \subset A$

Take any $x \in G$

$x \in G \subset A$

$\therefore x \in \mathring{A}$

Consider $G = X \setminus F$

$\mathring{G} = X \setminus \bar{F}$

Let X be a nonempty set.

Possible topology for X ??

Largest $\mathcal{J} = \mathcal{P}(X)$ Discrete

Smallest $\mathcal{J} = \{\phi, X\}$ Indiscrete

Examples.

① Given $\phi \neq A \subsetneq X$ and $A \in \mathcal{J}$

What is the smallest possible $\mathcal{J}$?

$\{\phi, A, X\}$ is it

② If $A, B \in \mathcal{J}$, $\phi \neq \underset{B}{\overset{A}{\neq}} \subsetneq X$ then

What is the smallest possible $\mathcal{J}$?

$\mathcal{J} = \{\phi, A \cap B, A, B, A \cup B, X\}$

Given $S \subset \mathcal{P}(X)$, how to get a minimal topology $\mathcal{T} \supset S$?

Definition. The smallest topology containing S is called the topology generated by S.

For that topology, S is called a subbase.

Brute Force. Try all combinations of arbitrary union and finite intersection.

$\therefore$ It exists $\subset \mathcal{P}(X)$

Theorem. Given any $S \subset \mathcal{P}(X)$, let

$$\mathcal{B} = \{\cap \mathcal{F} : \text{finite } \mathcal{F} \subset S\}$$
$$= \{S_1 \cap S_2 \cap \cdots \cap S_n : S_k \in S\}$$

$$\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\} = \{\bigcup_{\alpha \in I} B_\alpha : B_\alpha \in \mathcal{B}\}$$

Then $\mathcal{T}$ is the smallest topology for X containing S, i.e.,

$\mathcal{T}$ is generated by S, i.e.,

S is a subbase (subbasis) of $\mathcal{T}$.

Example. For $X=\mathbb{R}$, let
$$S=\{(-\infty,b): b\in\mathbb{R}\}\cup\{(a,\infty): a\in\mathbb{R}\}$$
After taking finite intersections, we get
$$\mathcal{B}=S\cup\{(a,b): a<b\in\mathbb{R}\}\cup\{X\}\cup\{\emptyset\}$$
Then arbitrary unions lead to the standard topology $\mathcal{T}_{std}$ for $\mathbb{R}$.

Question. Does it work the other way?

Lower Limit Topology. $\mathcal{T}_{ll}$ is generated by $\{[a,b): a<b\in\mathbb{R}\}$

Question.
$$\mathcal{T}_{std}\overset{?}{\subseteq}\mathcal{T}_{ll} \quad\text{or}\quad \mathcal{T}_{ll}\overset{?}{\subseteq}\mathcal{T}_{std} \quad\text{or else}$$

Yes

Any
$$\underset{\mathcal{T}_{std}}{\underbrace{(a,b)}}=\bigcup_{n=1}^{\infty}\left[a+\tfrac{1}{n},b\right)\in\mathcal{T}_{ll}$$

Exercise. Find a sequence $x_n \in \mathbb{R}$ satisfying

① $x_n \to x$ in $\mathcal{T}_{std}$, i.e.,

$\forall \varepsilon > 0 \ \exists$ integer N such that $\forall n \geq N$

$$x_n \in (x-\varepsilon, x+\varepsilon)$$

② $x_n \not\to x$ in $\mathcal{T}_{ll}$, i.e.,

$\exists \delta > 0 \ \forall$ integer N, $\exists n \geq N$

$$x_n \notin [x, x+\delta)$$

Ah! That's why it's called Lower-limit

Terminology

Let $\mathcal{T}$ be a topology for X.

A subset $\mathcal{B} \subset \mathcal{P}(X)$ is a base (basis) of $\mathcal{T}$ if $\mathcal{T} = \{\cup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$

formed by taking arbitrary unions

Example. $\{(a,b) : a < b \in \mathbb{R}\}$ of $\mathcal{T}_{std}$

$\{(p,q) : p,q \in \mathbb{Q}\}$ of $\mathcal{T}_{std}$

Any $S \subset \mathcal{P}(X) \xrightarrow[\text{Finite } \cap]{} \xrightarrow[\text{Arbitrary } \cup]{}$ Topology

Special cases

$\{(a,b) : a<b \in \mathbb{R}\} \xrightarrow{} \mathcal{J}_{std}$

only arbitrary $\cup$

$\{[a,b) : a<b \in \mathbb{R}\} \xrightarrow{} \mathcal{J}_{ll}$

Theorem. A subset $\mathcal{Q} \subset \mathcal{P}(X)$ is a base for a topology (not yet known) if the conditions are satisfied

(i) $\phi, X \in \mathcal{Q}$

(ii) For each $U, V \in \mathcal{Q}$ and $x \in U \cap V$

$\exists\ W \in \mathcal{Q} \quad x \in W \subset U \cap V$.

Note. Above special cases only have (ii)

Key to proof.

* Define $\mathcal{J}$ by $\mathcal{Q}$
* Verify T1 and T2, standard method involving sets.

T1 is easy.

T2 Let $\bigcup_{\alpha \in I} P_\alpha, \bigcup_{\beta \in J} Q_\beta \in \mathcal{J}$ where $P_\alpha, Q_\beta \in \mathcal{Q}$

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Need $Z = \left(\bigcup_{\alpha\in I} P_\alpha\right) \cap \left(\bigcup_{\beta\in J} Q_\beta\right)$

$= \bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta) \in \mathcal{T}$

(ii) For $P_\alpha \cap Q_\beta$ and any $x \in P_\alpha \cap Q_\beta$

$\exists\ W_x \in \mathcal{Q}$ $x \in W_x \subset P_\alpha \cap Q_\beta$

For every $x \in Z \longmapsto W_x$ as above

$Z \subset \bigcup_x W_x \subset \bigcup_{\alpha,\beta} (P_\alpha \cap Q_\beta) = Z$

Above, we consider if a set $\mathcal{Q} \subset \mathcal{P}(X)$ is qualified to make a topology (not known).

Next concern, given a known topology $\mathcal{T}$ and a set $\mathcal{B} \subset \mathcal{T}$, how to verify that $\mathcal{B}$ is already a base for $\mathcal{T}$.

Theorem. $\mathcal{B}$ is a base for $\mathcal{T}$

i.e., $\mathcal{T} = \{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{B}\}$

$\Leftrightarrow \forall\ G \in \mathcal{T}$ and $x \in G$, $\exists\ U \in \mathcal{B}$, $x \in U \subset G$.

Quick proof

"$\Rightarrow$" Trivial

"$\Leftarrow$" Obvious

Definition. Let $x \in X$ with $\mathcal{J}$. A local base (or nbhd base) at x is a collection $\mathcal{U}_x \subset \mathcal{J}$ such that $\forall$ $\underbrace{G \in \mathcal{J} \text{ with } x \in G}_{\text{nbhd } N \text{ of } x}$

$\exists\ U \in \mathcal{U}_x \quad x \in U \subset G$ or N

Examples.

* A metric space (X, d)
  $\mathcal{U}_x = \{ B(x, \frac{1}{n}) : 1 \leq n \in \mathbb{N} \}$ is a local base. It is countable
  $\mathcal{B} = \bigcup_{x \in X} \mathcal{U}_x = \{ B(x, \frac{1}{n}) : 1 \leq n \in \mathbb{N}, x \in X \}$
  may be uncountable
* Euclidean $\mathbb{R}^n$, $\mathcal{J}_{std}$
  $\mathcal{B} = \{ B(q, \frac{1}{n}) : 1 \leq n \in \mathbb{N}, q \in \mathbb{Q}^n \}$
  is a countable base
* Every base $\mathcal{B}$ for $\mathcal{J}$ defines local bases.
  $\mathcal{U}_x = \{ B \in \mathcal{B} : x \in B \}$
* $(\mathbb{R}, \mathcal{J}_{ll})$ has such local base at $x \in \mathbb{R}$
  $\mathcal{U}_x = \{ [x, \frac{1}{n}) : 1 \leq n \in \mathbb{N} \}$

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Definition. A topological space $(X, \mathcal{J})$ is

(i) 1st countable (C_I) if at every $x \in X$ $\exists$ countable local base

(ii) 2nd countable (C_{II}) if $\exists$ countable base

Facts.

* From above, $C_{II} \Rightarrow C_I$
* But, $C_{II} \not\Leftarrow C_I$

Example. $\{[q, \frac{1}{n}) : 1 \le n \in \mathbb{N}, q \in \mathbb{Q}\}$ and $\{[q, y) : y \in \mathbb{R}, q \in \mathbb{Q}\}$ are not bases for $(\mathbb{R}, \mathcal{J}_{ll})$

Reason.

Consider $r \notin \mathbb{Q}$, any $q \in \mathbb{Q}$, $y \in \mathbb{R}$

What happens if

$$r \in [q, y) \subset [r, r+\varepsilon)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$q \lneq r \qquad\qquad r \lneq q$$

Cannot co-exist