Example

Consider each of the collection of vectors below, and the respective subspace of $\mathbb{R}^n$ (for the appropriate n) for which the vectors constitute a basis. Apply the Gram-Schmidt orthogonalization to obtain an orthogonal basis for the subspace concerned.

(1) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(2) $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(3) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

(4) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

Answer.

(1) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix}$.

Remark. There are many correct answers. For instance, another correct answer is given by

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. (Why?)

This remark applies to other parts in this exercise.

(2) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2/3 \\ 1/2 \end{bmatrix}$, $\begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$.

(Another correct answer is given by

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/2 \\ 2/3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$.)

(3) $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4/3 \\ -5/3 \\ 0 \\ 1/3 \end{bmatrix}$, $\begin{bmatrix} -4/7 \\ -2/7 \\ -1/7 \\ 6/7 \end{bmatrix}$.

(4) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ -1 \\ -1/2 \end{bmatrix}$, $\begin{bmatrix} 1/4 \\ 3/4 \\ -1/4 \\ -1/2 \\ -3/4 \end{bmatrix}$.