Examples of orthonormal bases.

① The standard basis for $\mathbb{R}^n$, given by

$$e_1, e_2, \dots, e_n,$$

is an orthonormal basis for $\mathbb{R}^n$.

So, in particular:

- $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ constitute an orthonormal basis for $\mathbb{R}^2$.
- $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ constitute an orthonormal basis for $\mathbb{R}^3$.
- $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ constitute an orthonormal basis for $\mathbb{R}^4$.

② Let $u_1, u_2, \dots, u_n$ be vectors in $\mathbb{R}^n$. They constitute an orthonormal basis for $\mathbb{R}^n$ if and only if the $(n\times n)$-square matrix

$$U = [u_1 | u_2 | \cdots | u_n]$$

is an orthogonal matrix.

So, in particular:

(2a) For each $\theta \in \mathbb{R}$,

$$\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}, \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$ constitute an orthonormal basis for $\mathbb{R}^2$.

Such pairs of vectors in $\mathbb{R}^2$ are all the possible orthonormal bases in $\mathbb{R}^2$.

(2b) Now consider the situation in $\mathbb{R}^3$.

Suppose $u_1 = \begin{bmatrix}u_{11}\\u_{21}\\u_{31}\end{bmatrix}$, $u_2 = \begin{bmatrix}u_{12}\\u_{22}\\u_{32}\end{bmatrix}$ are a pair of unit vectors in $\mathbb{R}^3$ which are orthogonal. Define the vector u_3 in $\mathbb{R}^3$ by

$$u_3 = \begin{bmatrix}u_{13}\\u_{23}\\u_{33}\end{bmatrix} = \begin{bmatrix}\det\left(\begin{bmatrix}u_{21} & u_{22}\\ u_{31} & u_{32}\end{bmatrix}\right)\\ \det\left(\begin{bmatrix}u_{31} & u_{32}\\ u_{11} & u_{12}\end{bmatrix}\right)\\ \det\left(\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}\right)\end{bmatrix}.$$

It happens that

$$\langle u_1, u_3\rangle = u_1^t u_3 = \det\left(\begin{bmatrix}u_{11} & u_{11} & u_{12}\\ u_{21} & u_{21} & u_{22}\\ u_{31} & u_{31} & u_{32}\end{bmatrix}\right) = 0$$

Also, $\langle u_2, u_3\rangle = \underbrace{\dots}_{\text{[your work]}} = 0$, and $\langle u_3, u_3\rangle = \underbrace{\dots}_{\text{[your work]}} = 1$.

So u_1, u_2, u_3 constitute an orthonormal basis for $\mathbb{R}^3$.

In school maths, the vector u_3 here is usually denoted by $u_1 \times u_2$, and is called the cross product of u_1, u_2.