MATH1030 Adjoint of a matrix.

1. Recall the notion of *submatrices of a square matrix* from the handout *Determinants*.

   Let A be an $(n \times n)$-square matrix.

   For each k, ℓ, the (k, ℓ)-th submatrix of A is defined to be the $((n-1) \times (n-1))$-matrix resultant from simultaneously deleted the k-th row and ℓ-th column of A. It is denoted by $A(k|\ell)$.

2. **Definition. (Adjoint of a square matrix.)**

   Let A be an $(n \times n)$-square matrix.

   The $(n \times n)$-square matrix whose (i, j)-th entry is $(-1)^{i+j} \det(A(j|i))$ is called the adjoint (matrix) of the matrix A. This matrix is denoted by $\mathsf{Ad}(A)$.

   So $\mathsf{Ad}(A)$ is explicitly given by

$$\mathsf{Ad}(A) = \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) & (-1)^{1+3}\det(A(3|1)) & \cdots & (-1)^{1+n}\det(A(n|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{2+2}\det(A(2|2)) & (-1)^{2+3}\det(A(3|2)) & \cdots & (-1)^{2+n}\det(A(n|2)) \\ (-1)^{3+1}\det(A(1|3)) & (-1)^{3+2}\det(A(2|3)) & (-1)^{3+3}\det(A(3|3)) & \cdots & (-1)^{3+n}\det(A(n|3)) \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^{n+1}\det(A(1|n)) & (-1)^{n+2}\det(A(2|n)) & (-1)^{n+3}\det(A(3|n)) & \cdots & (-1)^{n+n}\det(A(n|n)) \end{bmatrix}$$

   Remark. In the context of this definition, the (i, j)-th entry of $\mathsf{Ad}(A)$ is usually referred to as the (j, i)-th cofactor of the matrix A.

3. **Illustrations.**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

   Then
$$\mathsf{Ad}(A) = \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{1+1}\det(A(2|2)) \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

   Then
$$\begin{aligned} \mathsf{Ad}(A) &= \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{1+2}\det(A(2|1)) & (-1)^{1+3}\det(A(3|1)) \\ (-1)^{2+1}\det(A(1|2)) & (-1)^{2+2}\det(A(2|2)) & (-1)^{2+3}\det(A(3|2)) \\ (-1)^{3+1}\det(A(1|3)) & (-1)^{3+2}\det(A(2|3)) & (-1)^{3+3}\det(A(3|3)) \end{bmatrix} \\ &= \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{32}a_{13} & a_{12}a_{23} - a_{22}a_{13} \\ -a_{21}a_{33} + a_{31}a_{23} & a_{11}a_{33} - a_{31}a_{13} & -a_{11}a_{23} + a_{21}a_{13} \\ a_{21}a_{32} - a_{31}a_{22} & -a_{11}a_{32} + a_{31}a_{12} & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} \end{aligned}$$

4. **Lemma (1).**

   Suppose A is an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by $a_{i,j}$.

   Then the statements below hold:

   (a) *For each $p = 1, 2, \cdots, n$, $\displaystyle\sum_{k=1}^{n} a_{pk} \cdot (-1)^{p+k} \det(A(p|k)) = \det(A)$.*

   (b) *For each $q = 1, 2, \cdots, n$, $\displaystyle\sum_{k=1}^{n} a_{kq} \cdot (-1)^{k+q} \det(A(k|q)) = \det(A)$.*

   (c) *For each $p, q = 1, 2, \cdots, n$, if $p \neq q$ then $\displaystyle\sum_{k=1}^{n} a_{pk} \cdot (-1)^{q+k} \det(A(q|k)) = 0$.*

   (d) *For each $p, q = 1, 2, \cdots, n$, if $p \neq q$ then $\displaystyle\sum_{k=1}^{n} a_{kq} \cdot (-1)^{k+p} \det(A(k|p)) = 0$.*

   Remark. The interpretation of the respective statements are:

   (a) The (p, p)-th entry of the matrix $A\mathsf{Ad}(A)$ is $\det(A)$.

(b) The (q,q)-th entry of the matrix $\mathsf{Ad}(A)A$ is $\det(A)$.

(c) The (p,q)-th entry of the matrix $A\mathsf{Ad}(A)$ is 0 whenever $p \neq q$.

(d) The (p,q)-th entry of the matrix $\mathsf{Ad}(A)A$ is 0 whenever $p \neq q$.

5. **Proof of Lemma (1).**

Suppose A is an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by $a_{i,j}$.

(a) For each $p = 1, 2, \cdots, n$, the expression $\sum_{k=1}^{n} a_{pk} \cdot (-1)^{p+k} \det(A(p|k))$ is the expansion of $\det(A)$ along the p-th row of A. Hence its value is $\det(A)$.

(b) For each $q = 1, 2, \cdots, n$, the expression $\sum_{k=1}^{n} a_{kq} \cdot (-1)^{k+q} \det(A(k|q))$ is the expansion of $\det(A)$ along the q-th column of A. Hence its value is $\det(A)$.

(c) Let $p, q = 1, 2, \cdots, n$. Suppose $p \neq q$.

Then $\sum_{k=1}^{n} a_{pk} \cdot (-1)^{q+k} \det(A(q|k))$ is the expansion, along its q-th row, the determinant of the matrix A' which is obtained from A when the q-th row of A is replaced with the p-th row of A.

Since A' has two identical rows, namely its p-th row and its q-th row, $\det(A') = 0$.

Hence by definition,

$$\sum_{k=1}^{n} a_{pk} \cdot (-1)^{q+k} \det(A(q|k)) = \det(A') = 0.$$

(d) Let $p, q = 1, 2, \cdots, n$. Suppose $p \neq q$.

Then $\sum_{k=1}^{n} a_{kq} \cdot (-1)^{k+p} \det(A(k|p))$ is the expansion, along its p-th column, the determinant of the matrix A'' which is obtained from A when the p-th column of A is replaced with the q-th column of A.

Since A'' has two identical columns, namely its p-th column and its q-th column, $\det(A'') = 0$.

Hence by definition,

$$\sum_{k=1}^{n} a_{kq} \cdot (-1)^{k+p} \det(A(k|p)) = \det(A'') = 0.$$

6. Lemma (1) can be re-formulated as the result below:

Theorem (ι).

Suppose A be an $(n \times n)$-square matrix. Then $A\mathsf{Ad}(A) = \det(A)I_n$ and $\mathsf{Ad}(A)A = \det(A)I_n$.

7. **Theorem (2).**

Suppose A be an $(n \times n)$-square matrix. Then the statements below hold:

(a) *Suppose A is non-singular. Then $\mathsf{Ad}(A)$ is non-singular.*

(b) *Suppose A is singular. Then $\mathsf{Ad}(A)$ is singular.*

(c) *Suppose A is non-singular. Then the matrix inverse of A is given by $A^{-1} = \dfrac{1}{\det(A)}\mathsf{Ad}(A)$.*

(d) $\det(\mathsf{Ad}(A)) = (\det(A))^{n-1}$ *(whether A is non-singular or not).*

(e) *Suppose A is non-singular. Then $\mathsf{Ad}(\mathsf{Ad}(A)) = (\det(A))^{n-2}A$.*

Proof of Theorem (2).

(a) Suppose A is non-singular. Then $\det(A) \neq 0$.

We have $(\dfrac{1}{\det(A)}A)\mathsf{Ad}(A) = I_n$ and $\mathsf{Ad}(A)(\dfrac{1}{\det(A)}A) = I_n$.

Then, by definition, $\mathsf{Ad}(A)$ is non-singular.

(b) Suppose A is singular. Then $\det(A) = 0$.

We have $A\mathsf{Ad}(A) = \mathcal{O}_{n\times n}$ and $\mathsf{Ad}(A)A = \mathcal{O}_{n\times n}$.

- (Case 1.) Suppose A is the zero matrix. Then $\mathsf{Ad}(A)$ is also the zero matrix. Therefore $\mathsf{Ad}(A)$ is singular.

- (Case 2.) Suppose A is not the zero matrix. Then there is some column of A which is not the zero vector in $\mathbb{R}^n$. Denote it by $\mathbf{v}$.
  Since $\mathsf{Ad}(A)A = \mathcal{O}_{n\times n}$, we have $\mathsf{Ad}(A)\mathbf{v} = \mathbf{0}_n$.
  Then $\mathcal{N}(\mathsf{Ad}(A))$ contains some non-zero vector in $\mathbb{R}^n$, namely $\mathbf{v}$.
  Hence $\mathsf{Ad}(A)$ is singular.

Hence in any case, $\mathsf{Ad}(A)$ is singular.

(c) Suppose A is non-singular. Then $\det(A) \neq 0$.

We have $(\frac{1}{\det(A)}\mathsf{Ad}(A))A = I_n$ and $A(\frac{1}{\det(A)}\mathsf{Ad}(A)) = I_n$.

Then by definition, the matrix inverse of A is given by $A^{-1} = \frac{1}{\det(A)}\mathsf{Ad}(A)$.

(d)
- (Case 1.) Suppose A is non-singular. Then $\det(A) \neq 0$.
  Since $A\mathsf{Ad}(A) = \det(A)I_n$, we have

$$\det(A)\det(\mathsf{Ad}(A)) = \det(A\mathsf{Ad}(A)) = \det(\det(A)I_n) = (\det(A))^n.$$

  Therefore $\det(\mathsf{Ad}(A)) = (\det(A))^{n-1}$.
- (Case 2.) Suppose A is singular. Then $\mathsf{Ad}(A)$ is singular. Then $\det(\mathsf{Ad}(A)) = 0 = (\det(A))^{n-1}$.

In any case, we have $\det(\mathsf{Ad}(A)) = (\det(A))^{n-1}$.

(e) Suppose A is non-singular. Then $\det(A) \neq 0$.
We have

$$\begin{aligned}\det(A)\mathsf{Ad}(\mathsf{Ad}(A)) &= \det(A)I_n\mathsf{Ad}(\mathsf{Ad}(A)) \\ &= A\mathsf{Ad}(A)\mathsf{Ad}(\mathsf{Ad}(A)) = A(\det(\mathsf{Ad}(A))I_n) = \det(\mathsf{Ad}(A))A = (\det(A))^{n-1}A.\end{aligned}$$

Since $\det(A) \neq 0$, we have $\mathsf{Ad}(\mathsf{Ad}(A)) = (\det(A))^{n-2}A$.

8. A consequence of Item (c) in Theorem (2) is the result known as Cramer's Rule, which is one of the earliest discovered result in linear algebra.

Theorem (κ). (Cramer's Rule.)

Suppose A be an $(n \times n)$-square matrix. Suppose A is non-singular.

Then, for any $\mathbf{b} \in \mathbb{R}^n$, the unique solution of the system $\mathcal{LS}(A, \mathbf{b})$ is given by '$\mathbf{x} = \frac{1}{\det(A)}\mathsf{Ad}(A)\mathbf{b}$'.