1. **Lemma (1).**

Let H, B *be* $(n \times n)$*-square matrices.*

Suppose H *is a row-operation matrix.*

Then

$$\det(HB) = \det(H)\det(B).$$

Proof of Lemma (1).

Let H, B be $(n \times n)$-square matrices.

Suppose H is a row-operation matrix. There are three possibilities:

- H is the row operation matrix corresponding to the row operation $\alpha R_i + R_k$ for some distinct i, k and for some real number α.
- H is the row operation matrix corresponding to the row operation βR_i for some non-zero real number β.
- H is the row operation matrix corresponding to the row operation $R_i \leftrightarrow R_k$ for some distinct i, k.

(a) Suppose H is the row operation matrix corresponding to the row operation $\alpha R_i + R_k$ for some distinct i, k and for some real number α.
Then $\det(H) = 1$.
HB is obtained by B by adding a scalar multiple of the i-th row to the k-th row.
Then $\det(HB) = \det(B)$.
Therefore $\det(HB) = 1 \cdot \det(B) = \det(H)\det(B)$.

(b) Suppose H is the row operation matrix corresponding to the row operation βR_i for some non-zero real number β.
Then $\det(H) = \beta$.
HB is obtained by B by multiplying every entry of the i-th row by β.
Then $\det(HB) = \beta \det(B)$.
Therefore $\det(HB) = \beta \det(B) = \det(H)\det(B)$.

(c) Suppose H is the row operation matrix corresponding to the row operation $R_i \leftrightarrow R_k$ for some distinct i, k.
Then $\det(H) = -1$.
HB is obtained by B by interchanging the i-th row and the k-th row.
Then $\det(HB) = -\det(B)$.
Therefore $\det(HB) = -\det(B) = \det(H)\det(B)$.

Hence, in any case, $\det(HB) = \det(H)\det(B)$.

2. **Corollary to Lemma (1).**

Let $H_1, H_2, \cdots, H_k$ be $(n \times n)$-matrices.

Suppose $H_1, H_2, \cdots, H_k$ are row operation matrices.

Then

$$\det(H_k H_{k-1} \cdots H_2 H_1) = \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1).$$

Proof of Corollary to Lemma (1).

Let $H_1, H_2, \cdots, H_k$ be $(n \times n)$-matrices.

Suppose $H_1, H_2, \cdots, H_k$ are row operation matrices.

Then

$$\begin{aligned}
\det(H_k H_{k-1} \cdots H_2 H_1) &= \det(H_k) \det(H_{k-1} \cdots H_2 H_1) \\
&= \det(H_k) \det(H_{k-1}) \det(H_{k-2} \cdots H_2 H_1) \\
&= \cdots \\
&= \det(H_k) \det(H_{k-1}) \cdots \det(H_3) \det(H_2 H_1) \\
&= \det(H_k) \det(H_{k-1}) \cdots \det(H_2) \det(H_1).
\end{aligned}$$

2. **Corollary to Lemma (1).**

Let $H_1, H_2, \cdots, H_k$ be $(n \times n)$-matrices.

Suppose $H_1, H_2, \cdots, H_k$ are row operation matrices.

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Proof of Corollary to Lemma (1).

Let $H_1, H_2, \cdots, H_k$ be $(n \times n)$-matrices.

Suppose $H_1, H_2, \cdots, H_k$ are row operation matrices.

Then

$$\begin{aligned}\det(H_k H_{k-1} \cdots H_2 H_1) &= \det(H_k)\det(H_{k-1} \cdots H_2 H_1)\\ &= \det(H_k)\det(H_{k-1})\det(H_{k-2} \cdots H_2 H_1)\\ &= \cdots\\ &= \det(H_k)\det(H_{k-1}) \cdots \det(H_3)\det(H_2 H_1)\\ &= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1).\end{aligned}$$

Repeated applications of Lemma(1).

3. **Theorem (2).**

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular.

Then

$$\det(AB) = \det(A)\det(B).$$

Proof of Theorem (2).

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular.

Then there are some k row-operation matrices, say, $H_1, H_2, \cdots, H_k$, so that

$$A = H_k H_{k-1} \cdots H_2 H_1.$$

Therefore

$$\begin{aligned}
\det(AB) &= \det(H_k H_{k-1} \cdots H_2 H_1 B) \\
&= \det(H_k)\det(H_{k-1} \cdots H_2 H_1 B) \\
&= \det(H_k)\det(H_{k-1})\det(H_{k-2} \cdots H_2 H_1 B) \\
&= \cdots \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1 B) \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1)\det(B) \\
&= \det(H_k H_{k-1} \cdots H_2 H_1)\det(B) = \det(A)\det(B)
\end{aligned}$$

Then $\det(AB) = \det(A)\det(B)$.

3. **Theorem (2).**

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular.

Then

$$\det(AB) = \det(A)\det(B).$$

Proof of Theorem (2).

Let A, B be $(n \times n)$-square matrices. Suppose A is nonsingular.

Then there are some k row-operation matrices, say, $H_1, H_2, \cdots, H_k$, so that

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Therefore

$$\begin{aligned}
\det(AB) &= \det(H_k H_{k-1} \cdots H_2 H_1 B) \\
&= \det(H_k)\det(H_{k-1} \cdots H_2 H_1 B) \\
&= \det(H_k)\det(H_{k-1})\det(H_{k-2} \cdots H_2 H_1 B) \\
&= \cdots \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1 B) \\
&= \det(H_k)\det(H_{k-1}) \cdots \det(H_2)\det(H_1)\det(B) \\
&= \det(H_k H_{k-1} \cdots H_2 H_1)\det(B) = \det(A)\det(B)
\end{aligned}$$

Repeated applications of Lemma (1),

Application of Corollary to Lemma (1).

Then $\det(AB) = \det(A)\det(B)$.

4. **Lemma (3).**

Let C be an $(n \times n)$-square matrix. Suppose C is singular.

Then $\det(C) = 0$.

Proof of Lemma (3).

Let C be an $(n \times n)$-square matrix. Suppose C is singular.

Denote by C' the reduced row-echelon form which is row-equivalent to C.

Since C is singular, C' is also singular. (Why?)

Then, since C' is a singular reduced row-echelon form, C' has at least one entire row of 0's.

Therefore $\det(C') = 0$.

Since C is row-equivalent to C', there is some non-singular $(n \times n)$-square matrix A such that $C = AC'$.

Then, by Theorem (2),

$$\det(C) = \det(AC') = \det(A)\det(C') = 0.$$

4. **Lemma (3).**

Let C be an $(n \times n)$-square matrix. Suppose C is singular. Then $\det(C) = 0$.

Proof of Lemma (3).

Let C be an $(n \times n)$-square matrix. Suppose C is singular.

Alternative argument:
Suppose C is singular.
Then the columns of C are linear dependent.
Therefore some column of C is a linear combination of the rest of the columns of C.
Then $\det(C) = 0$.

Denote by C' the reduced row-echelon form which is row-equivalent to C.

Since C is singular, C' is also singular. (Why?)

Then, since C' is a singular reduced row-echelon form, C' has at least one entire row of 0's.

Therefore $\det(C') = 0$.

Since C is row-equivalent to C', there is some non-singular $(n \times n)$-square matrix A such that $C = AC'$.

Then, by Theorem (2),

$$\det(C) = \det(AC') = \det(A)\det(C') = 0.$$

5. **Theorem (4).**
Let A, B *be* $(n \times n)$*-square matrices. Suppose* A *is singular.*

Then
$$\det(AB) = 0 = \det(A)\det(B).$$

Proof of Theorem (4).
Let A, B be $(n \times n)$-square matrices. Suppose A is singular.

Then by Theorem (3), we have $\det(A) = 0$.
Therefore $\det(A)\det(B) = 0$.

Since A is singular, AB is also singular. (Why?)
Then by Theorem (3), we have $\det(AB) = 0$.

Therefore
$$\det(AB) = 0 = \det(A)\det(B).$$

6. Combining Theorem (2) and Theorem (4), we obtain the result below:

Theorem (ζ).
Suppose A, B are $(n \times n)$-square matrices.

Then

$$\det(AB) = \det(A)\det(B).$$

Remark.
Actually it further follows that

$$\begin{aligned} \det(AB) &= \det(A)\det(B) \\ &= \det(B)\det(A) \\ &= \det(BA). \end{aligned}$$

However, note that AB and BA are not necessarily the same matrix.

7. An immediate consequence of Theorem (ζ) is Theorem (η).

Theorem (η).
Suppose A is an $(n \times n)$-square matrix.

Then the statements below holds:

(a) *For any positive integer p,*

$$\det(A^p) = (\det(A))^p.$$

(b) *Suppose A is invertible.*

Then $\det(A) \neq 0$, *and* $\det(A^{-1}) = (\det(A))^{-1}$.

8. Statement (b) in Theorem (η) tells us that if a square matrix is invertible then its determinant is non-zero.

It is natural to ask whether it is true that if the determinant of a square matrix is non-zero then the matrix concerned is invertible. The answer is provided by Theorem (5).

Theorem (5).
Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$.

Then A is invertible.

7. An immediate consequence of Theorem (ζ) is Theorem (η).

Theorem (η).
Suppose A is an $(n \times n)$-square matrix.

Then the statements below holds:

(a) *For any positive integer p,*

$$\det(A^p) = (\det(A))^p.$$

(b) *Suppose A is invertible.*

Then $\det(A) \neq 0$, *and* $\det(A^{-1}) = (\det(A))^{-1}$.

Why? How?
$\det(A^p) = \det(AA^{p-1})$
$= \det(A)\det(A^{p-1})$
$= (\det(A))^2 \det(A^{p-2})$
$\vdots$
$= (\det(A))^p$

Why? How?
$1 = \det(I_n) = \det(A^{-1}A)$
$= \det(A^{-1})\det(A)$
Then $\det(A) \neq 0$.
Also, $\det(A^{-1}) = 1/\det(A)$.

8. Statement (b) in Theorem (η) tells us that if a square matrix is invertible then its determinant is non-zero.

It is natural to ask whether it is true that if the determinant of a square matrix is non-zero then the matrix concerned is invertible. The answer is provided by Theorem (5).

Theorem (5).
Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$.

Then A is invertible.

Proof of Theorem (5).

Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$.

[We want to deduce that A is non-singular.
How? We try to show that A is row-equivalent to I_n.]

Denote by A' the reduced row-echelon form which is row-equivalent to A.

[Ask: Is it true that $A' = I_n$? To find the answer, we ask whether $\det(A') \neq 0$ or not.]

There exists some non-singular $(n \times n)$-square matrix H such that $A' = HA$.
By Theorem (ζ), we have

$$\det(A') = \det(H)\det(A).$$

Since H is non-singular, we have $\det(H) \neq 0$.

By assumption, $\det(A) \neq 0$. Then $\det(A') \neq 0$.

By assumption A' is a reduced row-echelon form.
Since $\det(A') \neq 0$, there is no row of A' which is a row of 0's.
Then every row of A' contains a leading one.

Therefore $A' = I_n$.

Hence A is row equivalent to I_n. Then A is non-singular.

Proof of Theorem (5).

Let A be an $(n \times n)$-square matrix. Suppose $\det(A) \neq 0$.

[We want to deduce that A is non-singular.
How? We try to show that A is row-equivalent to I_n.]

What is so special about I_n in this context?
I_n is the one and only one $(n\times n)$-reduced row echelon form which has no rows of 0's and has a non-zero determinant.

Denote by A' the reduced row-echelon form which is row-equivalent to A.

[Ask: Is it true that $A' = I_n$? To find the answer, we ask whether $\det(A') \neq 0$ or not.]

There exists some non-singular $(n \times n)$-square matrix H such that $A' = HA$.

By Theorem (ζ), we have

$$\det(A') = \det(H)\det(A).$$

Since H is non-singular, we have $\det(H) \neq 0$.

By assumption, $\det(A) \neq 0$. Then $\det(A') \neq 0$.

By assumption A' is a reduced row-echelon form.
Since $\det(A') \neq 0$, there is no row of A' which is a row of 0's.
Then every row of A' contains a leading one.

Therefore $A' = I_n$.

Hence A is row equivalent to I_n. Then A is non-singular.

9. Combining Theorem (η) and Theorem (5), we obtain the result below:

Theorem (θ).
Suppose A is an $(n \times n)$-square matrix.

Then the statements below are logically equivalent:

(a) *A is non-singular.*

(b) *A is invertible.*

(c) $\det(A) \neq 0$.

10. **Corollary to Theorem (θ).**
Suppose A is an $(n \times n)$-square matrix.

Then the statements below are logically equivalent:

(a) *A is singular.*

(b) *A is not invertible.*

(c) $\det(A) = 0$.

11. We now compile and re-organized all the various re-formulations for the notions of non-singularity and invertibility that we have learnt so far into one single result:

Theorem (ι). (Various re-formulations for the notions of non-singularity and invertibility.)
Let A be an $(n \times n)$-matrix.

(a) *The statements below are logically equivalent:*

i. *A is non-singular.*
ii. *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*
iii. *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*
iv. *A is row-equivalent to I_n.*
v. *A is invertible.*
vi. *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*
vii. *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*
viii. *For any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.*
ix. *For any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at least one solution.*
x. *For any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at most one solution.*

(b) *The statements below are logically equivalent:*

i. A *is non-singular.*

ii. A^t *is non-singular.*

iii. *For any vector* $\mathbf{v}$ *in* $\mathbb{R}^n$, *if* $A^t\mathbf{v} = \mathbf{0}$ *then* $\mathbf{v} = \mathbf{0}$.

iv. *The trivial solution is the only solution of the homogeneous system* $\mathcal{LS}(A^t, \mathbf{0})$.

v. A^t *is row-equivalent to* I_n.

vi. A^t *is invertible.*

vii. *There exists some* $(n \times n)$*-square matrix* J *such that* $JA^t = I_n$.

viii. *There exists some* $(n \times n)$*-square matrix* K *such that* $A^tK = I_n$.

ix. *For any vector* $\mathbf{b}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{b})$ *has one and only one solution, namely,* '$\mathbf{x} = (A^t)^{-1}\mathbf{b}$'.

x. *For any vector* $\mathbf{c}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{c})$ *has at least one solution.*

xi. *For any vector* $\mathbf{d}$ *in* $\mathbb{R}^n$, *the system* $\mathcal{LS}(A^t, \mathbf{d})$ *has at most one solution.*

(c) *Denote the j-th column of A by $\mathbf{u}_j$ for each $j = 1, 2, \cdots, n$.*
The statements below are logically equivalent:

i. *A is non-singular.*
ii. *Every vector in $\mathbb{R}^n$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.*
iii. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.*
iv. *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.*
v. *The dimension of the column space of A is n.*
vi. *The dimension of the null space of A is 0.*
vii. $\det(A) \neq 0$.

(d) *Denote the i-th row of A by $\mathbf{w}_i$ for each $i = 1, 2, \cdots, n$.*
The statements below are logically equivalent:

i. *A is non-singular.*
ii. *A^t is non-singular.*
iii. *Every vector in $\mathbb{R}^n$ is a linear combination of ${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$.*
iv. *${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$. are linearly independent.*
v. *${\mathbf{w}_1}^t, {\mathbf{w}_2}^t, \cdots, {\mathbf{w}_n}^t$. constitute a basis for $\mathbb{R}^n$.*
vi. *The dimension of the row space of A is n.*
vii. *The dimension of the null space of A^t is 0.*
viii. $\det(A^t) \neq 0$.

(e) *Now further suppose A is non-singular, with a sequence of row operations*

$$A = C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots\cdots \underset{\rho_{p-2}}{\longrightarrow} C_{p-1} \underset{\rho_{p-1}}{\longrightarrow} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k.
Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A|I_n]$:

$$\begin{aligned}
[A|I_n] &= [C_1|I_n] \underset{\rho_1}{\longrightarrow} [C_2|H_1] \\
&\underset{\rho_2}{\longrightarrow} [C_3|H_2H_1] \\
&\underset{\rho_3}{\longrightarrow} \\
&\quad\vdots \\
&\quad\vdots \\
&\underset{\rho_{p-2}}{\longrightarrow} [C_{p-1}|H_{p-2}\cdots H_2H_1] \\
&\underset{\rho_{p-1}}{\longrightarrow} [C_p|H_{p-1}\cdots H_2H_1] = [I_n|A^{-1}].
\end{aligned}$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1}\cdots H_2H_1, \qquad\qquad A = H_1{}^{-1}H_2{}^{-1}\cdots H_{p-1}{}^{-1}.$$