1. **Definition. (Submatrices of a square matrix)**

   Let A be an $(n \times n)$-square matrix.

   For each k, ℓ, the (k, ℓ)-th submatrix of A is defined to be the $((n-1) \times (n-1))$-matrix resultant from simultaneously deleted the k-th row and ℓ-th column of A. It is denoted by $A(k|\ell)$.

2. **Illustration.**

   (a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

   Then $A(1|1) = [a_{22}]$, $A(1|2) = [a_{21}]$, $A(2|1) = [a_{12}]$, $A(2|2) = [a_{11}]$.

   (b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

   Then

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$

$$A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix},$$

$$A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad A(3|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \quad A(3|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

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For each k, ℓ, the (k, ℓ)-th submatrix of A is defined to be the $((n-1) \times (n-1))$-matrix resultant from simultaneously deleted the k-th row and ℓ-th column of A. It is denoted by $A(k|\ell)$.

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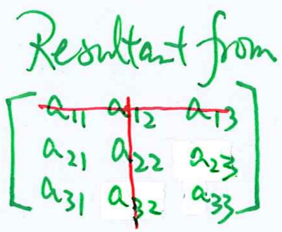
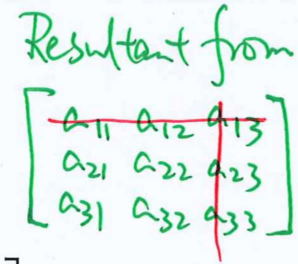
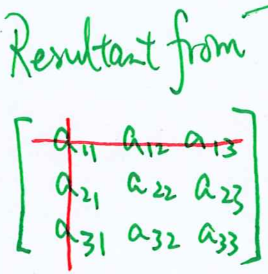
(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.





Then



$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$

$$A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, \quad A(2|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}, \quad A(2|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix},$$

$$A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}, \quad A(3|2) = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}, \quad A(3|3) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

3. **Definition. ('Inductive definition for determinants' through 'expansion' along the first column.)**

Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

(a) *Suppose $n = 1$.*
Then we define the determinant of A, which is denoted by $\det(A)$, to be the number which is the only entry of A.

(b) *Suppose $n > 1$.*
Then we define the determinant of A, which is denoted by $\det(A)$, by

$$\begin{aligned}\det(A) &= (-1)^{1+1}a_{11}\det(A(1|1)) \\ &\quad +(-1)^{2+1}a_{21}\det(A(2|1)) \\ &\quad +(-1)^{3+1}a_{31}\det(A(3|1)) \\ &\quad +\cdots \\ &\quad +(-1)^{n+1}a_{n1}\det(A(n|1)).\end{aligned}$$

Remark. The 'formula'

$$\begin{aligned}\det(A) &= (-1)^{1+1}a_{11}\det(A(1|1)) + (-1)^{2+1}a_{21}\det(A(2|1)) + (-1)^{3+1}a_{31}\det(A(3|1)) \\ &\quad +\cdots+(-1)^{n+1}a_{n1}\det(A(n|1))\end{aligned}$$

is usually referred to as the 'expansion' of a determinant along the first column.

4. **Illustration.**

(a) Suppose $A = [a_{11}]$. Then $\det(A) = a_{11}$.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

We have

$$A(1|1) = [a_{22}], A(2|1) = [a_{12}].$$

Then

$$\det(A) = a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) = a_{11}a_{22} - a_{12}a_{21}.$$

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

We have

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

4. **Illustration.**

(a) Suppose $A = [a_{11}]$. Then $\det(A) = a_{11}$.

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

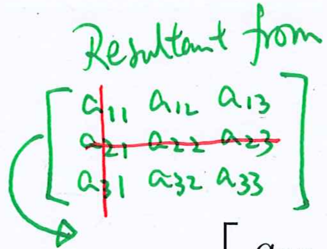
We have

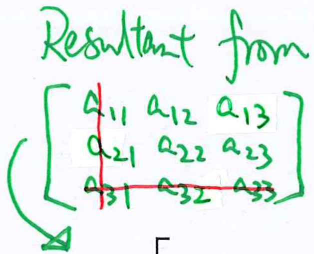
$$A(1|1) = [a_{22}], A(2|1) = [a_{12}].$$

Then

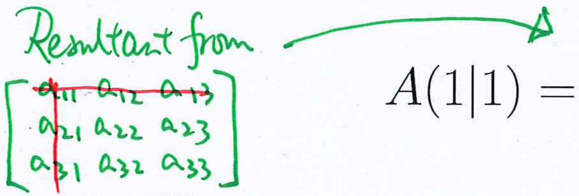
$$\det(A) = a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) = a_{11}a_{22} - a_{12}a_{21}.$$

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.





We have



$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A(2|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}, A(3|1) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(d) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

We have

$$A(1|1)=\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, A(2|1)=\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, A(3|1)=\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, A(4|1)=\begin{bmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) - a_{41}\det(A(4|1)) = \cdots \\ &= \ a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} \\ &\quad +a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\ &\quad +a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\ &\quad +a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42} \\ &\quad -a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44} \\ &\quad -a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} \\ &\quad -a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42} \\ &\quad -a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41} \end{aligned}$$

5. **Examples.**

(a) $\det(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}) = 1 \cdot 9 - 6 \cdot 7 = -33.$

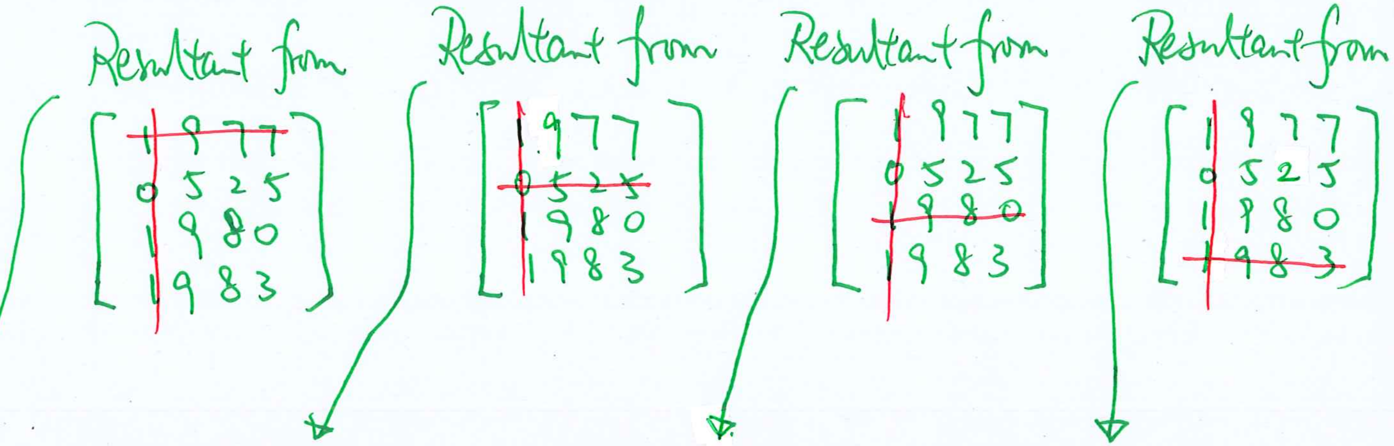
(b)

$$\begin{aligned} \det(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}) &= 1 \cdot \det(\begin{bmatrix} 9 & 8 \\ 1 & 5 \end{bmatrix}) - 6 \cdot \det(\begin{bmatrix} 7 & 0 \\ 1 & 5 \end{bmatrix}) + 0 \cdot \det(\begin{bmatrix} 7 & 0 \\ 9 & 8 \end{bmatrix}) \\ &= 1 \cdot (9 \cdot 5 - 1 \cdot 8) - 6(7 \cdot 5 - 1 \cdot 0) = -173. \end{aligned}$$

(c)

$$
\begin{aligned}
&\det(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}) \\
= \;& 1 \cdot \det(\begin{bmatrix} 5 & 2 & 5 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}) - 0 \cdot \det(\begin{bmatrix} 9 & 7 & 7 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}) + 1 \cdot \det(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}) - 1 \cdot \det(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 0 \end{bmatrix}) \\
= \;& \left(5 \cdot \det(\begin{bmatrix} 8 & 0 \\ 8 & 3 \end{bmatrix}) - 9 \cdot \det(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix})\right) \\
& -0 \\
& + \left(9 \cdot \det(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}) - 5 \cdot \det(\begin{bmatrix} 7 & 7 \\ 8 & 3 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix})\right) \\
& - \left(9 \cdot \det(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}) - 5 \cdot \det(\begin{bmatrix} 7 & 7 \\ 8 & 0 \end{bmatrix}) + 9 \cdot \det(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix})\right) \\
= \;& [5 \cdot 24 - 9 \cdot (-34) + 9 \cdot (-40)] - 0 \\
& + [9 \cdot (-34) - 5 \cdot (-35) + 9 \cdot 21] - [9 \cdot (-40) - 5 \cdot (-56) + 9 \cdot (21)] \\
= \;& 15
\end{aligned}
$$

(c)



$$
\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right)
$$

$$
= 1\cdot\det\left(\begin{bmatrix} 5 & 2 & 5 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}\right) - 0\cdot\det\left(\begin{bmatrix} 9 & 7 & 7 \\ 9 & 8 & 0 \\ 9 & 8 & 3 \end{bmatrix}\right) + 1\cdot\det\left(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}\right) - 1\cdot\det\left(\begin{bmatrix} 9 & 7 & 7 \\ 5 & 2 & 5 \\ 9 & 8 & 0 \end{bmatrix}\right)
$$

$$
= \left(5\cdot\det\left(\begin{bmatrix} 8 & 0 \\ 8 & 3 \end{bmatrix}\right) - 9\cdot\det\left(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}\right) + 9\cdot\det\left(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}\right)\right)
$$

$$
-0
$$

$$
+\left(9\cdot\det\left(\begin{bmatrix} 2 & 5 \\ 8 & 3 \end{bmatrix}\right) - 5\cdot\det\left(\begin{bmatrix} 7 & 7 \\ 8 & 3 \end{bmatrix}\right) + 9\cdot\det\left(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix}\right)\right)
$$

$$
-\left(9\cdot\det\left(\begin{bmatrix} 2 & 5 \\ 8 & 0 \end{bmatrix}\right) - 5\cdot\det\left(\begin{bmatrix} 7 & 7 \\ 8 & 0 \end{bmatrix}\right) + 9\cdot\det\left(\begin{bmatrix} 7 & 7 \\ 2 & 5 \end{bmatrix}\right)\right)
$$

$$
= [5\cdot 24 - 9\cdot(-34) + 9\cdot(-40)] - 0
$$

$$
+[9\cdot(-34) - 5\cdot(-35) + 9\cdot 21] - [9\cdot(-40) - 5\cdot(-56) + 9\cdot(21)]
$$

$$
= 15
$$

6. **Theorem (1). ('Expansion' of a determinant along any arbitrary column.)**
Let A be an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by a_{ij}.
Suppose $n > 1$.
Then, for each $j = 1, 2, \cdots, n$,

$$\begin{aligned}\det(A) &= (-1)^{1+j} a_{1j} \det(A(1|j)) \\ &\quad +(-1)^{2+j} a_{2j} \det(A(2|j)) \\ &\quad +(-1)^{3+j} a_{3j} \det(A(3|j)) \\ &\quad + \cdots \\ &\quad +(-1)^{n+j} a_{nj} \det(A(n|j)).\end{aligned}$$

Proof of Theorem (1). Omitted. (This can be done with mathematical induction.)

7. **Illustration.**

(a) Suppose A is a (3×3)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= \ \ a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)),\\
\det(A) &= -a_{12}\det(A(1|2)) + a_{22}\det(A(2|2)) - a_{32}\det(A(3|2)),\\
\det(A) &= \ \ a_{13}\det(A(1|3)) - a_{23}\det(A(2|3)) + a_{33}\det(A(3|3)).
\end{aligned}$$

(b) Suppose A is a (4×4)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= \ \ a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) - a_{41}\det(A(4|1)),\\
\det(A) &= -a_{12}\det(A(1|2)) + a_{22}\det(A(2|2)) - a_{32}\det(A(3|2)) + a_{42}\det(A(4|2)),\\
\det(A) &= \ \ a_{13}\det(A(1|3)) - a_{23}\det(A(2|3)) + a_{33}\det(A(3|3)) - a_{43}\det(A(4|3)),\\
\det(A) &= -a_{14}\det(A(1|4)) + a_{24}\det(A(2|4)) - a_{34}\det(A(3|4)) + a_{44}\det(A(4|4)).
\end{aligned}$$

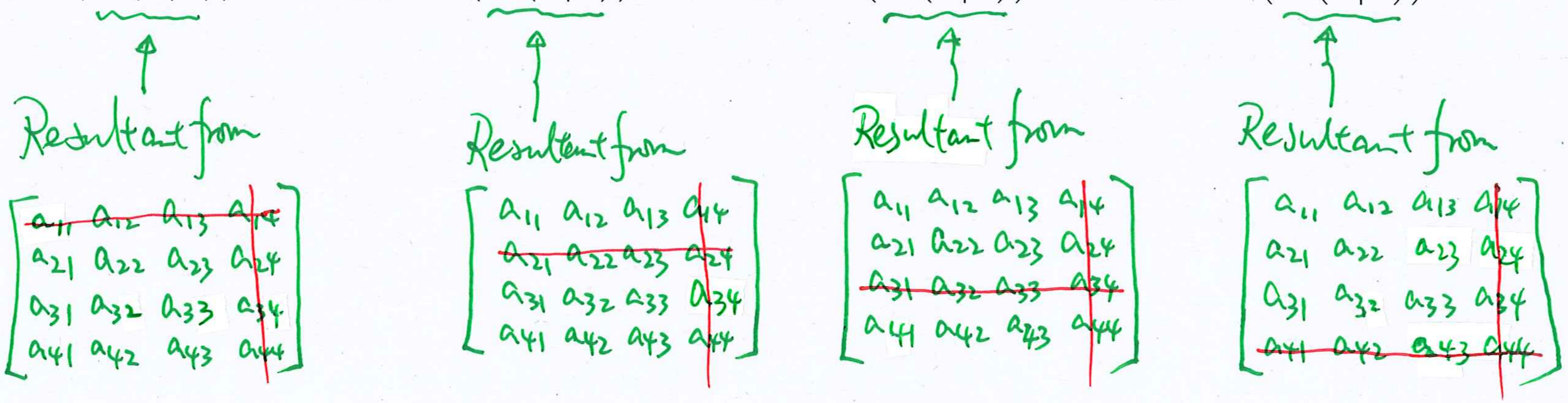
7. **Illustration.**

(a) Suppose A is a (3×3)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)), \\
\det(A) &= -a_{12}\det(A(1|2)) + a_{22}\det(A(2|2)) - a_{32}\det(A(3|2)), \\
\det(A) &= a_{13}\det(A(1|3)) - a_{23}\det(A(2|3)) + a_{33}\det(A(3|3)).
\end{aligned}$$

(b) Suppose A is a (4×4)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{21}\det(A(2|1)) + a_{31}\det(A(3|1)) - a_{41}\det(A(4|1)), \\
\det(A) &= -a_{12}\det(A(1|2)) + a_{22}\det(A(2|2)) - a_{32}\det(A(3|2)) + a_{42}\det(A(4|2)), \\
\det(A) &= a_{13}\det(A(1|3)) - a_{23}\det(A(2|3)) + a_{33}\det(A(3|3)) - a_{43}\det(A(4|3)), \\
\det(A) &= -a_{14}\det(A(1|4)) + a_{24}\det(A(2|4)) - a_{34}\det(A(3|4)) + a_{44}\det(A(4|4)).
\end{aligned}$$



8. **Examples.**

(a)
$$\det\left(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}\right) = -7 \cdot \det\left(\begin{bmatrix} 6 & 8 \\ 0 & 5 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\right) - 1 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 6 & 8 \end{bmatrix}\right) = \cdots = -173.$$

$$\det\left(\begin{bmatrix} 1 & 7 & 0 \\ 6 & 9 & 8 \\ 0 & 1 & 5 \end{bmatrix}\right) = 0 \cdot \det\left(\begin{bmatrix} 6 & 9 \\ 0 & 1 \end{bmatrix}\right) - 8 \cdot \det\left(\begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 7 \\ 6 & 9 \end{bmatrix}\right) = \cdots = -173.$$

(b)

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) &= -9 \cdot \det\left(\begin{bmatrix} 0 & 2 & 5 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}\right) + 5 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 1 & 8 & 0 \\ 1 & 8 & 3 \end{bmatrix}\right) \\ &\quad -9 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 3 \end{bmatrix}\right) + 9 \cdot \det\left(\begin{bmatrix} 1 & 7 & 7 \\ 0 & 2 & 5 \\ 1 & 8 & 0 \end{bmatrix}\right) \\ &= \cdots = 15 \end{aligned}$$

$$\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) = 7\cdot\det\left(\begin{bmatrix} 0 & 5 & 5 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}\right) - 2\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 0 \\ 1 & 9 & 3 \end{bmatrix}\right)$$

$$+8\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 3 \end{bmatrix}\right) - 8\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 5 \\ 1 & 9 & 0 \end{bmatrix}\right)$$

$$= \cdots = 15$$

$$\det\left(\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3 \end{bmatrix}\right) = -7\cdot\det\left(\begin{bmatrix} 0 & 5 & 2 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}\right) + 5\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 1 & 9 & 8 \\ 1 & 9 & 8 \end{bmatrix}\right)$$

$$-0\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}\right) + 3\cdot\det\left(\begin{bmatrix} 1 & 9 & 7 \\ 0 & 5 & 2 \\ 1 & 9 & 8 \end{bmatrix}\right)$$

$$= \cdots = 15$$

9. **Theorem (2). ('Expansion' of a determinant along the first row.)**

Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

Suppose $n > 1$.

Then

$$\begin{aligned} \det(A) &= (-1)^{1+1} a_{11} \det(A(1|1)) \\ &\quad +(-1)^{1+2} a_{12} \det(A(1|2)) \\ &\quad +(-1)^{1+3} a_{13} \det(A(1|3)) \\ &\quad + \cdots \\ &\quad +(-1)^{1+n} a_{1n} \det(A(1|n)). \end{aligned}$$

Proof of Theorem (2). Omitted. (This can be done with mathematical induction.)

10. **Illustration.**

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

We have

$$A(1|1) = [a_{22}], A(2|1) = [a_{12}].$$

Then

$$\det(A) = a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) = a_{11}a_{22} - a_{12}a_{21}.$$

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

We have

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11} \det(A(1|1)) - a_{12} \det(A(1|2)) + a_{13} \det(A(1|3)) \\ &= \cdots \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

10. **Illustration.**

(a) Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

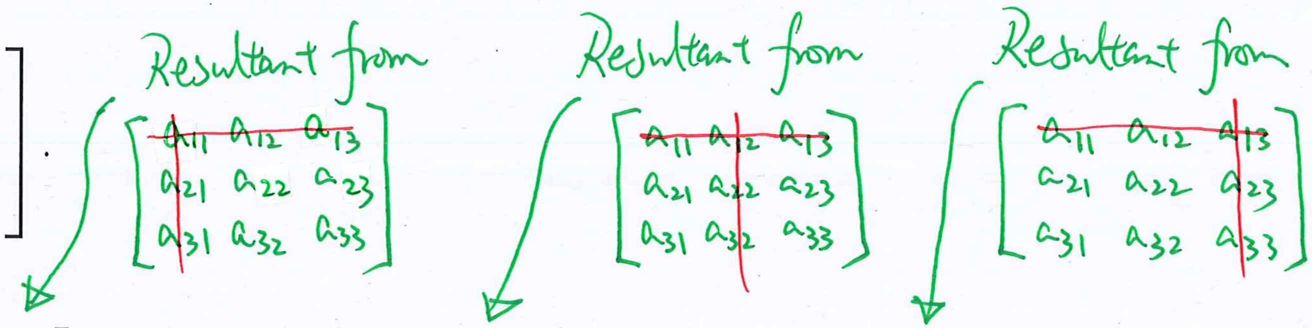
We have

$$A(1|1) = [a_{22}], A(2|1) = [a_{12}].$$

Then

$$\det(A) = a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) = a_{11}a_{22} - a_{12}a_{21}.$$

(b) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.



We have

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, A(1|2) = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, A(1|3) = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) \\ &= \cdots \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

(c) Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$.

We have

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, A(1|2) = \begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}, A(1|3) = \begin{bmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{bmatrix}, A(1|4) = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) - a_{14}\det(A(1|4)) \\ &= \cdots \\ &= \ \ a_{11}a_{22}a_{33}a_{44} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43} \\ &\quad + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} + a_{12}a_{23}a_{31}a_{44} \\ &\quad + a_{13}a_{24}a_{31}a_{42} + a_{13}a_{21}a_{32}a_{44} + a_{13}a_{22}a_{34}a_{41} \\ &\quad + a_{14}a_{23}a_{32}a_{41} + a_{14}a_{22}a_{31}a_{43} + a_{14}a_{21}a_{33}a_{42} \\ &\quad - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{11}a_{23}a_{32}a_{44} \\ &\quad - a_{12}a_{21}a_{33}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} \\ &\quad - a_{13}a_{24}a_{32}a_{41} - a_{13}a_{22}a_{31}a_{44} - a_{13}a_{21}a_{34}a_{42} \\ &\quad - a_{14}a_{23}a_{31}a_{42} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{22}a_{33}a_{41} \end{aligned}$$

11. **Theorem (α).**

Suppose A be a square matrix. Then

$$\det(A^t) = \det(A).$$

Proof of Theorem (α). Omitted. (This can be done with mathematical induction. Apply the definition and Theorem (2).)

12. **Theorem (3). ('Expansion' of a determinant along any arbitrary row.)**

Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

Suppose $n > 1$.

Then, for each $i = 1, 2, \cdots, n$,

$$\begin{aligned} \det(A) \;=\; & (-1)^{i+1} a_{i1} \det(A(i|1)) \\ & +(-1)^{i+2} a_{i2} \det(A(i|2)) \\ & +(-1)^{i+3} a_{i3} \det(A(i|3)) \\ & +\cdots \\ & +(-1)^{i+n} a_{in} \det(A(i|n)). \end{aligned}$$

Proof of Theorem (3). This is a consequence of Theorem (1) and Theorem (α) combined.

13. **Illustration.**

(a) Suppose A is a (3×3)-square matrix, whose (i,j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)), \\
\det(A) &= -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) - a_{23}\det(A(2|3)), \\
\det(A) &= a_{31}\det(A(3|1)) - a_{32}\det(A(3|2)) + a_{33}\det(A(3|3)).
\end{aligned}$$

(b) Suppose A is a (4×4)-square matrix, whose (i,j)-th entry is denoted by a_{ij}. Then

$$\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) - a_{14}\det(A(1|4)), \\
\det(A) &= -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) - a_{23}\det(A(2|3)) + a_{24}\det(A(2|4)), \\
\det(A) &= a_{31}\det(A(3|1)) - a_{32}\det(A(3|2)) + a_{33}\det(A(3|3)) - a_{34}\det(A(3|4)), \\
\det(A) &= -a_{41}\det(A(4|1)) + a_{42}\det(A(4|2)) - a_{43}\det(A(4|3)) + a_{44}\det(A(4|4)).
\end{aligned}$$

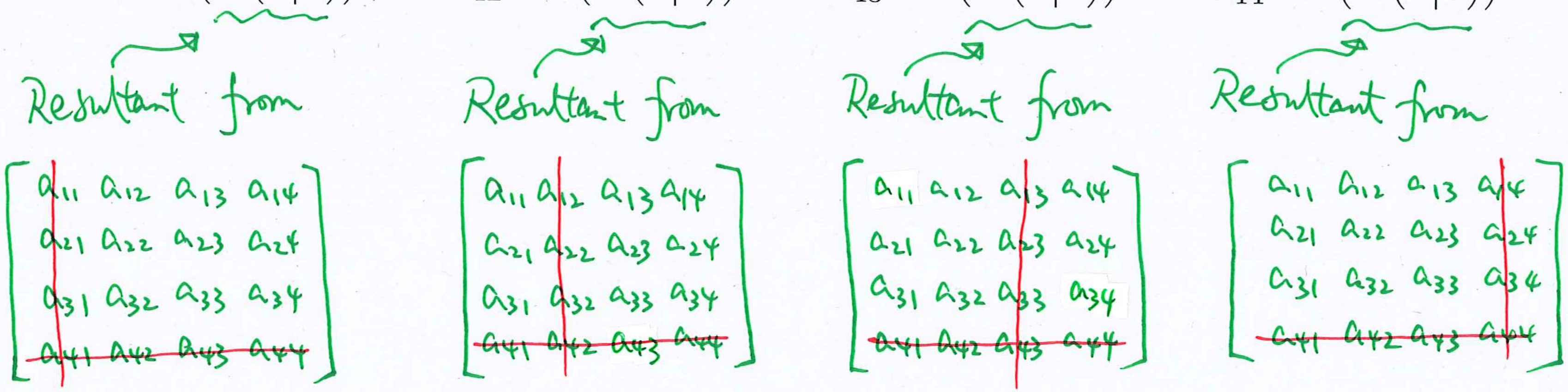
13. **Illustration.**

(a) Suppose A is a (3×3)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$
\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)),\\
\det(A) &= -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) - a_{23}\det(A(2|3)),\\
\det(A) &= a_{31}\det(A(3|1)) - a_{32}\det(A(3|2)) + a_{33}\det(A(3|3)).
\end{aligned}
$$

(b) Suppose A is a (4×4)-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Then

$$
\begin{aligned}
\det(A) &= a_{11}\det(A(1|1)) - a_{12}\det(A(1|2)) + a_{13}\det(A(1|3)) - a_{14}\det(A(1|4)),\\
\det(A) &= -a_{21}\det(A(2|1)) + a_{22}\det(A(2|2)) - a_{23}\det(A(2|3)) + a_{24}\det(A(2|4)),\\
\det(A) &= a_{31}\det(A(3|1)) - a_{32}\det(A(3|2)) + a_{33}\det(A(3|3)) - a_{34}\det(A(3|4)),\\
\det(A) &= -a_{41}\det(A(4|1)) + a_{42}\det(A(4|2)) - a_{43}\det(A(4|3)) + a_{44}\det(A(4|4)).
\end{aligned}
$$



14. **Key theoretical examples.**

(a) Let A be an $(n \times n)$-square matrix, whose (i,j)-th entry is denoted by a_{ij}.

i. Suppose there is some q amongst $1, 2, \cdots, n$ for which the q-th column is all zero. Then

$$\begin{aligned} \det(A) &= (-1)^{1+q} a_{1q} \det(A(1|q)) \\ &\quad +(-1)^{2+q} a_{2q} \det(A(2|q)) \\ &\quad +(-1)^{3+q} a_{3q} \det(A(3|q)) \\ &\quad +\cdots \\ &\quad +(-1)^{n+q} a_{nq} \det(A(n|q)) \\ &= 0. \end{aligned}$$

ii. Suppose there is some p amongst $1, 2, \cdots, n$ for which the p-th row is all zero. Then

$$\begin{aligned} \det(A) &= (-1)^{p+1} a_{p1} \det(A(p|1)) \\ &\quad +(-1)^{p+2} a_{p2} \det(A(p|2)) \\ &\quad +(-1)^{p+3} a_{p3} \det(A(p|3)) \\ &\quad +\cdots \\ &\quad +(-1)^{p+n} a_{pn} \det(A(p|n)) \\ &= 0. \end{aligned}$$

14. **Key theoretical examples.**

(a) Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}.

i. Suppose there is some q amongst $1, 2, \cdots, n$ for which the q-th column is all zero. Then

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Illustration:
$n=4$, $q=2$.

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \\ * & 0 & * & * \end{bmatrix}\right) \\ &= (-1)^{1+2}\cdot 0\cdot\det(\cdots) \\ &\quad +(-1)^{2+2}\cdot 0\cdot\det(\cdots) \\ &\quad +(-1)^{3+2}\cdot 0\cdot\det(\cdots) \\ &\quad +(-1)^{4+2}\cdot 0\cdot\det(\cdots) \\ &= 0. \end{aligned}$$

ii. Suppose there is some p amongst $1, 2, \cdots, n$ for which the p-th row is all zero. Then

$$\begin{aligned} \det(A) &= (-1)^{p+1} a_{p1} \det(A(p|1)) \\ &\quad +(-1)^{p+2} a_{p2} \det(A(p|2)) \\ &\quad +(-1)^{p+3} a_{p3} \det(A(p|3)) \\ &\quad +\cdots \\ &\quad +(-1)^{p+n} a_{pn} \det(A(p|n)) \\ &= 0. \end{aligned}$$

Illustration:
$n=5$, $q=3$.

$$\begin{aligned} \det(A) &= \det\left(\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}\right) \\ &= (-1)^{1+3}\cdot 0\cdot\det(\cdots) + (-1)^{2+3}\cdot 0\cdot\det(\cdots) \\ &\quad +(-1)^{3+3}\cdot 0\cdot\det(\cdots) + (-1)^{4+3}\cdot 0\cdot\det(\cdots) \\ &\quad +(-1)^{5+3}\cdot 0\cdot\det(\cdots) \\ &= 0. \end{aligned}$$

(b) Let D be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by d_{ij}.

Suppose $d_{ij} = 0$ whenever $i \neq j$.

(Such a matrix is called a diagonal matrix.)

We have

$$\det(D) = d_{11} \det(D(1|1)).$$

Note that $D(1|1)$ is itself a diagonal matrix.

Then $\det(D) = d_{11}d_{22}d_{33} \cdots d_{nn}$.

Remark. In particular, $\det(I_n) = 1$.

(b) Let D be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by d_{ij}.

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Note that $D(1|1)$ is itself a diagonal matrix.

Then $\det(D) = d_{11}d_{22}d_{33} \cdots d_{nn}$.

Remark. In particular, $\det(I_n) = 1$.

$$\det(D) = \det\left(\begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & d_{nn} \end{bmatrix}\right)$$

$$= d_{11} \det\left(\begin{bmatrix} d_{22} & 0 & \cdots & 0 \\ 0 & d_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}\right)$$

$$= d_{11}d_{22} \det\left(\begin{bmatrix} d_{33} & 0 & \cdots & 0 \\ 0 & d_{44} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{nn} \end{bmatrix}\right)$$

$$= \cdots$$

$$= d_{11}d_{22} \cdots d_{n-2,n-2} \det\left(\begin{bmatrix} d_{n-1,n-1} & 0 \\ 0 & d_{nn} \end{bmatrix}\right)$$

$$= d_{11}d_{22} \cdots d_{n-1,n-1}d_{nn}$$

'Expansion along the first column' at each equality.

(c) i. Let A be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by a_{ij}. Suppose $a_{ij} = 0$ whenever $i > j$. (Such a matrix is called an upper-triangular matrix.)

A is explicitly given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}$.

By definition, we have

$$\det(A) = a_{11} \det(A(1|1)).$$

Note that

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & \cdots & a_{4n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}.$$

(So $A(1|1)$ is also an upper triangular matrix.)
Then, 'inductively', we have

$$\det(A) = a_{11}a_{22}a_{33} \cdots a_{nn}.$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & \cdots & a_{3n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}.$$

By definition, we have

$$\det(A) = a_{11}\det(A(1|1)).$$

Note that

$$A(1|1) = \begin{bmatrix} a_{22} & a_{23} & a_{24} & \cdots & \cdots & a_{2n} \\ 0 & a_{33} & a_{34} & \cdots & \cdots & a_{3n} \\ 0 & 0 & a_{44} & \cdots & \cdots & a_{4n} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & & a_{nn} \end{bmatrix}.$$

(So $A(1|1)$ is also an upper triangular matrix.)
Then, 'inductively', we have

$$\det(A) = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Illustration:
$n = 5$.

$$\det(A) = \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}\right)$$

$$= a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{55} \end{bmatrix}\right)$$

$$= a_{11}a_{22}\det\left(\begin{bmatrix} a_{33} & a_{34} & a_{35} \\ 0 & a_{44} & a_{45} \\ 0 & 0 & a_{55} \end{bmatrix}\right)$$

$$= a_{11}a_{22}a_{33}\det\left(\begin{bmatrix} a_{44} & a_{45} \\ 0 & a_{55} \end{bmatrix}\right)$$

$$= a_{11}a_{22}a_{33}a_{44}a_{55}$$

'Expansion along the first column' at each equality.

ii. Let B be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by b_{ij}. Suppose $b_{ij} = 0$ whenever $i < j$. (Such a matrix is called a lower-triangular matrix.)

B is explicitly given by
$$B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & \cdots & b_{nn} \end{bmatrix}.$$

Note that B^t is an upper triangular matrix.

Then

$$\det(B) = \det(B^t) = b_{11}b_{22}b_{33}\cdots b_{nn}.$$

ii. Let B be an $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by b_{ij}. Suppose $b_{ij} = 0$ whenever $i < j$. (Such a matrix is called a lower-triangular matrix.)

$$B \text{ is explicitly given by } B = \begin{bmatrix} b_{11} & 0 & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & \cdots & b_{nn} \end{bmatrix}.$$

Note that B^t is an upper triangular matrix.

Then

$$\det(B) = \det(B^t) = b_{11}b_{22}b_{33}\cdots b_{nn}.$$

$$B^t = \begin{bmatrix} b_{11} & b_{21} & b_{31} & \cdots & b_{n1} \\ 0 & b_{22} & b_{32} & \cdots & b_{n2} \\ 0 & 0 & b_{33} & \cdots & b_{n3} \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 \cdots & 0 & b_{nn} \end{bmatrix}$$

(d) i. Denote by ρ the row operation $\alpha R_i + R_k$ on matrices with n rows.

The row operation matrix $M[\rho]$ corresponding to ρ is the $(n \times n)$-square matrix given by

$$M[\rho] = I_n + \alpha E_{k,i}^{n,n}.$$

Note that $M[\rho]$ is:

an upper triangular matrix or a lower triangular matrix.

Then $\det(M[\rho]) = 1$.

ii. Denote by σ the row operation βR_i on matrices with n rows.

The row operation matrix $M[\sigma]$ corresponding to σ is given by

$$M[\sigma] = I_n + (\beta - 1)E_{i,i}^{n,n}.$$

Note that $M[\sigma]$ is an upper triangular matrix, whose diagonal entries are made of:

- $n - 1$ copies of 1 and
- 1 copy of β.

Then $\det(M[\sigma]) = \beta$.

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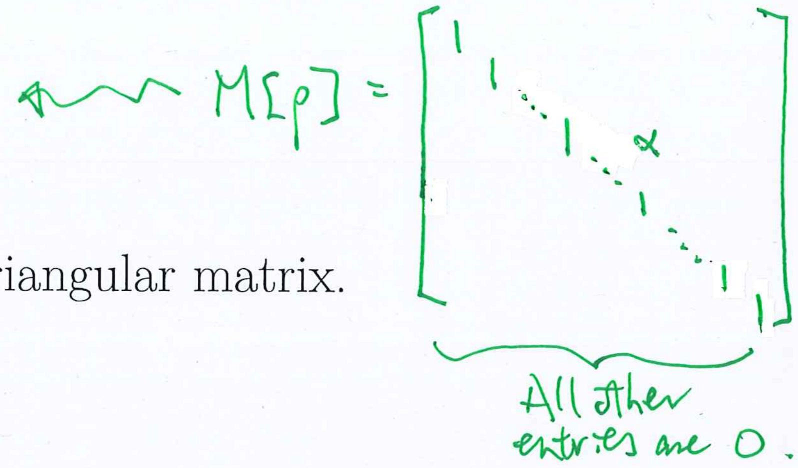

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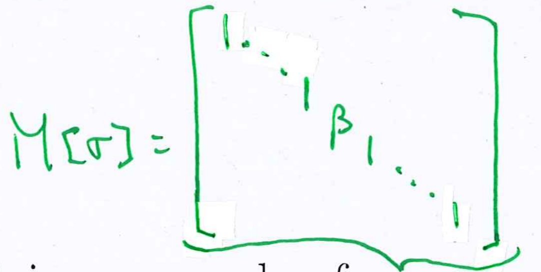

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Then $\det(M[\sigma]) = \beta$.

iii. Denote by τ the row operation $R_i \leftrightarrow R_k$ on matrices with n rows.

The row operation matrix $M[\tau]$ corresponding to τ is given by

$$M[\tau] = I_n - E_{i,i}^{n,n} - E_{k,k}^{n,n} + E_{k,i}^{n,n} + E_{i,k}^{n,n}.$$

$M[\tau]$ is a symmetric matrix with:

- $n - 2$ entries of 1, along the diagonal, and
- 2 entries of 1 off diagonal.

Repeatedly applying the definition, we have

$$\det(M[\tau]) = \det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = -1.$$

(e) Let A be an $(n \times n)$-square matrix. Suppose A is a reduced row-echelon form.

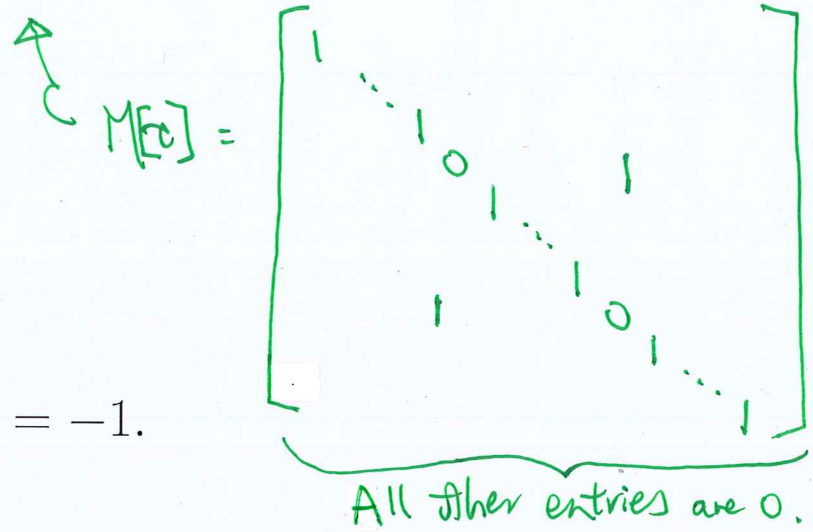
i. Suppose A is non-singular.
Then $A = I_n$.
Therefore $\det(A) = 1$.

ii. Suppose A is singular.
Then A has at least one entire row of 0's.
Therefore $\det(A) = 0$.

iii. Denote by τ the row operation $R_i \leftrightarrow R_k$ on matrices with n rows.

The row operation matrix $M[\tau]$ corresponding to τ is given by

$$M[\tau] = I_n - E_{i,i}^{n,n} - E_{k,k}^{n,n} + E_{k,i}^{n,n} + E_{i,k}^{n,n}.$$



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