1. **Definition. (Diagonal matrix.)**

Let D be a $(n \times n)$-square matrix, whose (i, j)-th entry is denoted by d_{ij}.

The matrix D is said to be a diagonal matrix if and only if

$$d_{ij} = 0 \text{ whenever } i \neq j.$$

Remark on notation.

When

$$d_{11} = \alpha_1, \quad d_{22} = \alpha_2, \quad \cdots, \quad d_{nn} = \alpha_n,$$

we may write

$$D = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

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$$D = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \alpha_n \end{bmatrix}$$

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2. **Definition. (Diagonalizability and diagonalization.)**

Let A be an $(n \times n)$-square matrix.

(a) *Suppose U is a non-singular $(n \times n)$-square matrix.*

Then we say that $U^{-1}AU$ is a diagonalization of A if and only if

$U^{-1}AU$ is a diagonal matrix.

(b) *A is said to be diagonalizable if and only if*

there is some non-singular $(n \times n)$-square matrix T such that $T^{-1}AT$ is a diagonalization of A.

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$$U^{-1}AU = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \alpha_n \end{bmatrix}$$

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Remark.

A diagonalizable matrix may have various diagonalization.

- Possible different choices of non-singular matrix U in the expression '$U^{-1}AU$'.
- Possible different choices of diagonal matrix which is given by the 'presentation of diagonalization' '$U^{-1}AU$'.

3. Recall the definition for the notions of *eigenvalue* and *eigenvector* from the handout *Eigenvalues and eigenvectors*:

   Let A be an $(n \times n)$-square matrix (with real entries).
   Let λ be a (real) number. Let $\mathbf{v}$ be a non-zero vector with n (real) entries.
   We say $\mathbf{v}$ is an eigenvector of A with eigenvalue λ (or equivalently, λ is an eigenvalue of A with a corresponding eigenvector $\mathbf{v}$) if and only if $A\mathbf{v} = \lambda\mathbf{v}$.

4. **Theorem (C).**
   Let A is an $(n \times n)$-square matrix.
   Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n \,\right]$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$. (So U is non-singular.)

   Then the statements below are logically equivalent:

   (a) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.*

   (b) *$U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.*

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(a) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.*

(b) *$U^{-1}AU$ is a diagonal matrix, given by $U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.*

Alternative and equivalent way of presenting this equality:
$AU = U\,\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

$$A[u_1|u_2|\cdots|u_n] = [u_1|u_2|\cdots|u_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

5. **Corollary to Theorem (C).**

Let A is an $(n \times n)$-square matrix.
Suppose A has n pairwise distinct eigenvalues.

Then A is diagonalizable.

Proof of Corollary to Theorem (C).
Each of the n eigenvalues of A will correspond to an eigenvector.

Since the eigenvalues are pairwise distinct, the n corresponding eigenvectors will be linearly independent.
These n vectors will then constitute a basis for $\mathbb{R}^n$.

6. **Examples of diagonalizable matrices and their diagonalizations.**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with respective eigenvalues $1, -2$.

Since $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with distinct eigenvalues, they are linearly independent.

Then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathbb{R}^2$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\right]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -5 \end{bmatrix}$.

By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

as expected from theory.

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Then $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for $\mathbb{R}^2$.

Define $U = [\,\mathbf{u}_1 \,|\, \mathbf{u}_2\,]$. So $U = \begin{bmatrix} 5 & 2 \\ -2 & -1 \end{bmatrix}$.

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By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix},$$

as expected from theory.

A different diagonization with a different diagonal matrix:

Define $V = [u_2 | u_1]$. So $V = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$.

V is non-singular, and

$V^{-1} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$.

$V^{-1}AV = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $1, 2, 3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

as expected from theory.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

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By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

as expected from theory.

There are other diagonalizations with distinct diagonal matrices. For example, with $V = [u_2 | u_3 | u_1]$, we have

$$V^{-1}AV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With $W = [u_3 | u_1 | u_2]$, we have

$$W^{-1}AW = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $4, 1, 1$.

We can check that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent. (Fill in the detail.)

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$.

By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as expected from theory.

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $4, 1, 1$.

We can check that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linear independent. (Fill in the detail.)

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for $\mathbb{R}^3$.

Define $U = \left[\, \mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \mathbf{u}_3 \,\right]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$.

By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as expected from theory.

The non-singular matrix used for producing a diagonalization for A can be different but still give the same diagonal matrix.

For example, with

$V = [\mathbf{u}_1 \,|\, \mathbf{u}_2 + \mathbf{u}_3 \,|\, \mathbf{u}_2 - \mathbf{u}_3]$,

we have

$$V^{-1}AV = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

What makes V 'works' is the fact that all linear combinations of $\mathbf{u}_2, \mathbf{u}_3$ which are non-zero are eigenvectors of A with eigenvalue 1.

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with respective eigenvalues $1, -1, 3, -3$.

Since $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with pairwise distinct eigenvalues, they are linear independent.

Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ constitute a basis for $\mathbb{R}^4$.

Define $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right]$.

U is nonsingular, and $U^{-1} = \begin{bmatrix} 5/8 & -1/4 & 0 & -1/8 \\ 1/4 & 1/8 & -1/8 & 0 \\ 0 & 1/8 & 5/24 & 1/12 \\ 1/8 & 0 & -1/12 & 1/24 \end{bmatrix}$.

By direct verification, we see that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

as expected from theory.

7. **Non-examples on diagonalizability.**

(a) Let b be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

u is an eigenvector of A with eigenvalue b, and every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}$.

Then there is no basis for $\mathbb{R}^3$ which is constituted by eigenvectors of A.

Therefore A is not diagonalizable.

(b) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

A has no eigenvalues, and hence no eigenvectors.

Then there is no basis for $\mathbb{R}^4$ which is constituted by eigenvectors of A.

Therefore A is not diagonalizable.

8. **Proof of Theorem (C).**

Let A is an $(n \times n)$-square matrix.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^n$, and $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitute a basis for $\mathbb{R}^n$.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

  [Reminder: We want to verify that a diagonalization for A is given by $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.]

  Then for each $j = 1, 2, \cdots, n$, we have $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$.

  Therefore

$$\begin{aligned} AU &= \left[\, A\mathbf{u}_1 \,\middle|\, A\mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{u}_n \,\right] = \left[\, \lambda_1\mathbf{u}_1 \,\middle|\, \lambda_2\mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \lambda_n\mathbf{u}_n \,\right] \\ &= \left[\, \lambda_1 U\mathbf{e}_1^{(n)} \,\middle|\, \lambda_2 U\mathbf{e}_2^{(n)} \,\middle|\, \cdots \,\middle|\, \lambda_n U\mathbf{e}_n^{(n)} \,\right] = \left[\, U(\lambda_1\mathbf{e}_1^{(n)}) \,\middle|\, U(\lambda_2\mathbf{e}_2^{(n)}) \,\middle|\, \cdots \,\middle|\, U(\lambda_n\mathbf{e}_n^{(n)}) \,\right] \\ &= U\left[\, \lambda_1\mathbf{e}_1^{(n)} \,\middle|\, \lambda_2\mathbf{e}_2^{(n)} \,\middle|\, \cdots \,\middle|\, \lambda_n\mathbf{e}_n^{(n)} \,\right] = U\,\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \end{aligned}$$

  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ constitutes a basis for $\mathbb{R}^n$, U is non-singular and invertible. The matrix U^{-1} is well-defined.

  Then $U^{-1}AU = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, which is a diagonal matrix.

- Suppose $U^{-1}AU$ is a diagonal matrix, given by

$$U^{-1}AU = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

[Reminder: We want to verify that for each j,
$\mathbf{u}_j$ is an eigenvector of A with eigenvalue λ_j.]

Then

$$\begin{aligned}
\left[\, A\mathbf{u}_1 \,\middle|\, A\mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{u}_n \,\right] &= AU = U\,\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n) \\
&= U \left[\, \lambda_1 \mathbf{e}_1^{(n)} \,\middle|\, \lambda_2 \mathbf{e}_2^{(n)} \,\middle|\, \cdots \,\middle|\, \lambda_n \mathbf{e}_n^{(n)} \,\right] \\
&= \left[\, U(\lambda_1 \mathbf{e}_1^{(n)}) \,\middle|\, U(\lambda_2 \mathbf{e}_2^{(n)}) \,\middle|\, \cdots \,\middle|\, U(\lambda_n \mathbf{e}_n^{(n)}) \,\right] \\
&= \left[\, \lambda_1 U\mathbf{e}_1^{(n)} \,\middle|\, \lambda_2 U\mathbf{e}_2^{(n)} \,\middle|\, \cdots \,\middle|\, \lambda_n U\mathbf{e}_n^{(n)} \,\right] \\
&= \left[\, \lambda_1 \mathbf{u}_1 \,\middle|\, \lambda_2 \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \lambda_n \mathbf{u}_n \,\right]
\end{aligned}$$

Therefore, for each $j = 1, 2, \cdots, n$, we have

$$A\mathbf{u}_j = \lambda_j \mathbf{u}_j.$$

Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are eigenvectors of A, with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

9. **Lemma (1).**
Suppose A is an $(n \times n)$-square matrix.

Then A is singular if and only if 0 is an eigenvalue of A.

Furthermore, if A is singular then

every non-zero vector in $\mathcal{N}(A)$ is an eigenvector of A with eigenvalue 0.

Remark.
When

$$\dim(\mathcal{N}(A)) \geq 2,$$

we do not expect any two arbitrary non-zero vectors in $\mathcal{N}(A)$ to be scalar multiples of each other.

This result reminds us that we should not expect eigenvectors corresponding to the same eigenvalue of A to be non-zero scalar multiples of each other.

10. **Lemma (2).**

Let A be an $(n \times n)$-square matrix.
Suppose λ is a real number.

Then the statements below hold:

(a) *λ is an eigenvalue of A if and only if $A - \lambda I_n$ is singular.*

(b) *Now suppose λ is an eigenvalue of A indeed.*

Then for any non-zero vector $\mathbf{x}$ in $\mathbb{R}^n$,

$$\mathbf{x} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda \text{ if and only if } \mathbf{x} \in \mathcal{N}(A - \lambda I_n).$$

11. **Definition. (Eigenspace.)**

Let A be an $(n \times n)$-square matrix.
Suppose λ be an eigenvalue of A.

Then $\mathcal{N}(A - \lambda I_n)$ is called the eigenspace of A with eigenvalue λ.
It is denoted by $\mathcal{E}_A(\lambda)$.

The dimension of $\mathcal{E}_A(\lambda)$ is called the geometric multiplicity of the eigenvalue λ of A.

10. **Lemma (2).**

Let A be an $(n \times n)$-square matrix.

Suppose λ is a real number.

Then the statements below hold:

(a) *λ is an eigenvalue of A if and only if $A - \lambda I_n$ is singular.*

(b) *Now suppose λ is an eigenvalue of A indeed.*

Then for any non-zero vector $\mathbf{x}$ in $\mathbb{R}^n$,

$$\mathbf{x} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda \text{ if and only if } \mathbf{x} \in \mathcal{N}(A - \lambda I_n).$$

Why are we interested in such a concept at all?
Recall that in some examples, we encounter many a matrix with linearly independent vectors with the same eigenvalue. For such eigenvectors, every non-zero vector in their span is an eigenvector with the same eigenvalue. They are best handled with the concept.

11. **Definition. (Eigenspace.)**

Let A be an $(n \times n)$-square matrix.

Suppose λ be an eigenvalue of A.

Then $\mathcal{N}(A - \lambda I_n)$ is called the eigenspace of A with eigenvalue λ.

It is denoted by $\mathcal{E}_A(\lambda)$.

The dimension of $\mathcal{E}_A(\lambda)$ is called the geometric multiplicity of the eigenvalue λ of A.

12. **Theorem (D).**

Let A is an $(n \times n)$-square matrix.

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_p$ are all the eigenvalues of A, pairwise distinct.

Then the statements below are logically equivalent:

(a) *A is diagonalizable.*

(b) *The sum of the respective geometric multiplicities of $\lambda_1, \lambda_2, \cdots, \lambda_p$, as eigenvalues of A, equals n.*

Now suppose A is indeed diagaonalizable.

For each $k = 1, 2, \cdots, p$, write $\dim(\mathcal{E}_A(\lambda_k)) = n_k$, and suppose that

$$\mathbf{v}_{k,1}, \mathbf{v}_{k,2}, \cdots, \mathbf{v}_{k,n_k} \text{ constitute a basis for } \mathcal{E}_A(\lambda_k).$$

Then a basis for $\mathbb{R}^n$ is constituted by

$$\begin{array}{llll} \mathbf{v}_{1,1}, \mathbf{v}_{1,2}, \cdots, \mathbf{v}_{1,n_1}, & & & \\ & \mathbf{v}_{2,1}, \mathbf{v}_{2,2}, \cdots, \mathbf{v}_{2,n_2}, & & \\ & & \ddots & \\ & & & \mathbf{v}_{p,1}, \mathbf{v}_{p,2}, \cdots, \mathbf{v}_{p,n_p}. \end{array}$$

Proof. Omitted. (The argument is not difficult at a conceptual level; we can certainly give it within the context of this course. However it will be tedious unless we introduce the notion of *direct sum*.)

13. **Illustration of the content of Theorem (D).**

(a) Let $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2$ are eigenvectors of A with respective eigenvalues $1, -2$.

The only eigenspaces of A are

$$\mathcal{E}_A(1), \mathcal{E}_A(-2).$$

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-2)$ is 1, with a basis given by $\mathbf{u}_2$.

Since

$$\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-2)) = 2 = \dim(\mathbb{R}^2),$$

A is expected to be diagonalizable.

A diagonalization of A is given by

$$U^{-1}AU = \mathrm{diag}(1, -2),$$

in which

$$U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\right].$$

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $1, 2, 3$.

The only eigenspaces of A are

$$\mathcal{E}_A(1), \mathcal{E}_A(2), \mathcal{E}_A(3).$$

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(2)$ is 1, with a basis given by $\mathbf{u}_2$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

Since

$$\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(2)) + \dim(\mathcal{E}_A(3)) = 3 = \dim(\mathbb{R}^3),$$

A is expected to be diagonalizable.

A diagonalization of A is given by

$$U^{-1}AU = \operatorname{diag}(1, 2, 3),$$

in which

$$U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right].$$

(c) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are eigenvectors of A with respective eigenvalues $4, 1, 1$.

The only eigenspaces of A are

$$\mathcal{E}_A(4), \mathcal{E}_A(1).$$

The dimension of $\mathcal{E}_A(4)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(1)$ is 2, with a basis given by $\mathbf{u}_2, \mathbf{u}_3$.

Since

$$\dim(\mathcal{E}_A(4)) + \dim(\mathcal{E}_A(1)) = 3 = \dim(\mathbb{R}^3),$$

A is expected to be diagonalizable.

A diagonalization of A is given by

$$U^{-1}AU = \operatorname{diag}(4, 1, 1),$$

in which

$$U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right].$$

(d) Let $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}$, and $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}$.

It happens that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ are eigenvectors of A with respective eigenvalues $1, -1, 3, -3$.

The only eigenspaces of A are

$$\mathcal{E}_A(1), \mathcal{E}_A(-1), \mathcal{E}_A(3), \mathcal{E}_A(-3).$$

The dimension of $\mathcal{E}_A(1)$ is 1, with a basis given by $\mathbf{u}_1$.

The dimension of $\mathcal{E}_A(-1)$ is 1, with a basis given by $\mathbf{u}_2$.

The dimension of $\mathcal{E}_A(3)$ is 1, with a basis given by $\mathbf{u}_3$.

The dimension of $\mathcal{E}_A(-3)$ is 1, with a basis given by $\mathbf{u}_4$.

Since

$$\dim(\mathcal{E}_A(1)) + \dim(\mathcal{E}_A(-1)) + \dim(\mathcal{E}_A(3)) + \dim(\mathcal{E}_A(-3)) = 4 = \dim(\mathbb{R}^4),$$

A is expected to be diagonalizable.

A diagonalization of A is given by

$$U^{-1}AU = \operatorname{diag}(1, -1, 3, -3),$$

in which

$$U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\middle|\, \mathbf{u}_4 \,\right].$$

(e) Let b be a real number, and $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

b is the only eigenvalue of A, and
every eigenvector of A is a non-zero scalar multiple of $\mathbf{u}$.

The only eigenspace of A is

$$\mathcal{E}_A(b),$$

which is of dimension 1.

Therefore A is not diagonalizable.

(f) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

A has no eigenvalues, and hence no eigenspace.

Therefore A is not diagonalizable.

14. **Theorem (3).**

Let A be an $(n \times n)$-square matrix.

Suppose A is diagonalizable, with a diagonalization $U^{-1}AU = D$, for some non-singular $(n \times n)$-square matrix U and for some $(n \times n)$-diagonal matrix D.

Then the statements below hold:

(a) *For each positive integer p,*

$$A^p \text{ is diagonalizable,}$$

with a diagonalization given by

$$U^{-1}A^pU = D^p.$$

(b) *Suppose A is non-singular.*

Then D is non-singular, and A^{-1} is diagonalizable, with a diagonalization given by

$$U^{-1}A^{-1}U = D^{-1}.$$

Proof of Theorem (3). Exercise.

Remark. This result tells us that when A is diagonalizable, it will be easy to find its positive powers. (Why? Because it is easy to find the positive powers of a diagonal matrix.)

14. **Theorem (3).**

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Then the statements below hold:

(a) *For each positive integer p,*

$$A^p \text{ is diagonalizable,}$$

with a diagonalization given by

$$U^{-1}A^pU = D^p.$$

When $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $D^p = \mathrm{diag}(\lambda_1^p, \ldots, \lambda_n^p)$.

Illustrations:

$U^{-1}A^2U = U^{-1}AUU^{-1}AU = D^2$

$U^{-1}A^3U = U^{-1}A^2UU^{-1}AU = D^3$.

(b) *Suppose A is non-singular.*

Then D is non-singular, and A^{-1} is diagonalizable, with a diagonalization given by

$$U^{-1}A^{-1}U = D^{-1}.$$

This happens exactly when every diagonal entry of D is non-zero.

When $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $D^{-1} = \mathrm{diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$.

Proof of Theorem (3). Exercise.

Remark. This result tells us that when A is diagonalizable, it will be easy to find its positive powers. (Why? Because it is easy to find the positive powers of a diagonal matrix.)

15. **Examples on application of Theorem (3).**

(a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

It happens that A is diagonalizable, with a diagonalization given by

$$U^{-1}AU = \mathrm{diag}(1, 2, 3),$$

in which $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

Then

$$A = U\,\mathrm{diag}(1, 2, 3)U^{-1}.$$

Note that $U = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ and $U^{-1} = \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$.

Then for each positive integer p, we have

$$\begin{aligned} A^p &= U(\mathrm{diag}(1, 2, 3))^p U^{-1} \\ &= U(\mathrm{diag}(1, 2^p, 3^p))U^{-1} \\ &= \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^p & 0 \\ 0 & 0 & 3^p \end{bmatrix} \begin{bmatrix} 1 & -1 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 + 2^p & 1/2 - 2 \cdot 2^p + (3/2) \cdot 3^p \\ 0 & 2^p & -2 \cdot 2^p + 2 \cdot 3^p \\ 0 & 0 & 3^p \end{bmatrix}. \end{aligned}$$

(b) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

It happens that A is diagonalizable, with a diagonalization given by

$$U^{-1}AU = \text{diag}(4, 1, 1),$$

in which $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \mathbf{u}_3 \,\right]$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Then

$$A = U\,\text{diag}(4, 1, 1)U^{-1}.$$

Note that $U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and $U^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$.

Then for each positive integer p, we have

$$\begin{aligned} A^p &= U(\text{diag}(4, 1, 1))^p U^{-1} \\ &= U(\text{diag}(4^p, 1, 1))U^{-1} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4^p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4^p & 4^p - 3 & 4^p + 3 \\ 4^p - 1 & 4^p + 2 & 4^p - 1 \\ 4^p + 1 & 4^p + 1 & 4^p - 2 \end{bmatrix}. \end{aligned}$$