

## 1. **Definition. (Eigenvalues and eigenvectors.)**

Let  $A$  be an  $(n \times n)$ -square matrix (with real entries).

Let  $\lambda$  be a (real) number. Let  $\mathbf{v}$  be a non-zero vector with  $n$  (real) entries.

We say  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$  (or equivalently,  $\lambda$  is an eigenvalue of  $A$  with a corresponding eigenvector  $\mathbf{v}$ ) if and only if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

**Remark.** In this course, we restrict all discussion to real numbers.

However, in other mathematics courses and in many practical situations, allowing complex numbers into play is not only desirable but also natural and necessary.

This definition is formulated in such a way that it can be adapted to the ‘world of complex numbers’ immediately: simply change the word ‘real’ to ‘complex’.

### **Further remarks on terminologies.**

(a) We may write

*‘the number  $\lambda$  is an eigenvalue of  $A$ ’*

(without mentioning any specific corresponding eigenvector) if there is some non-zero vector  $\mathbf{u}$  so that  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .

(b) We may write

*‘the non-zero vector  $\mathbf{v}$  is an eigenvector of  $A$ ’*

(without mentioning its corresponding eigenvalue) if the equality  $A\mathbf{v} = \mu\mathbf{v}$  holds for some number  $\mu$ .

# 1. Definition. (Eigenvalues and eigenvectors.)

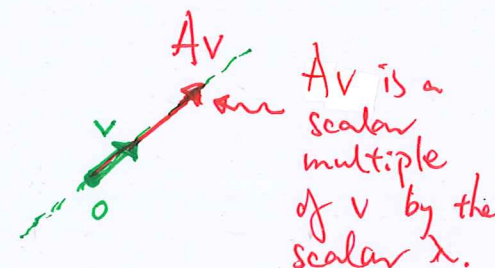
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$$\underline{A\mathbf{v} = \lambda\mathbf{v}.}$$

This is to ensure the definition to be 'meaningful', as we are aware that  $A\mathbf{0} = \mu\mathbf{0}$  for every real number  $\mu$ .



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## 2. Lemma (1).

Let  $A$  be an  $(n \times n)$ -square matrix. The statements below hold:

(a) Let  $\mathbf{v}$  be a non-zero vector in  $\mathbb{R}^n$ . Let  $\lambda, \mu$  be real numbers.

Suppose  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$  and also with eigenvalue  $\mu$ .

Then  $\lambda = \mu$ .

(b) Let  $\mathbf{v}$  be a non-zero vector in  $\mathbb{R}^n$ . Let  $\lambda$  be a real number.

Suppose  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .

Then, for any non-zero real number  $\beta$ ,

the vector  $\beta\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .

(c) Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  be non-zero vectors in  $\mathbb{R}^n$ . Let  $\lambda$  be a real number.

Suppose each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .

Then, for any real numbers  $\alpha_1, \alpha_2, \dots, \alpha_k$ , if  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k \neq \mathbf{0}$ , then

the vector  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ .

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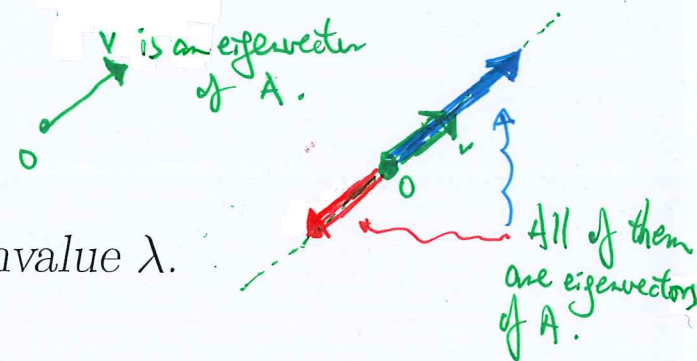
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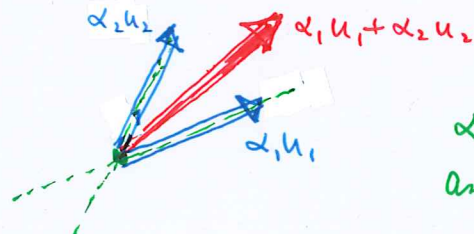
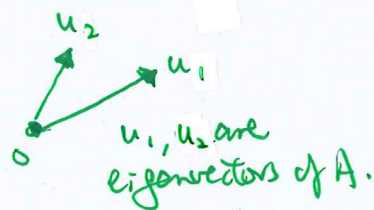


(c) Let  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  be non-zero vectors in  $\mathbb{R}^n$ . Let  $\lambda$  be a real number.

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$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2$  is an eigenvector of A. Et cetera.

## Remark.

In plain words, what this result says is about the square matrix  $A$ :

- (a) Every eigenvector of  $A$  corresponds to a unique eigenvalue.
- (b) Every non-zero scalar multiple of an eigenvector of  $A$  is also an eigenvector of  $A$ , and they correspond to the same eigenvalue.

(Be careful: we have not ruled out the possibility that non-zero vectors which are not scalar multiple of each other can be eigenvectors of  $A$  with the same eigenvalue.)

- (c) When it is not the zero vector, a linear combination of eigenvectors of  $A$  with the same eigenvalue is an eigenvector of  $A$  with that eigenvalue.

### 3. Examples.

(a) Let  $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$ , and  $\mathbf{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ .

None of  $\mathbf{u}, \mathbf{v}$  is the zero vector in  $\mathbb{R}^2$ .

i. We have  $A\mathbf{u} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 1 \cdot \mathbf{u}$ . Then  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue 1.

Every non-zero scalar multiple of  $\mathbf{u}$  is also an eigenvector of  $A$  with eigenvalue 1.

Detail:

- Suppose  $\alpha \in \mathbb{R}$  and  $\alpha \neq 0$ . Then

$$A(\alpha\mathbf{u}) = \alpha A\mathbf{u} = \alpha \cdot 1\mathbf{u} = 1 \cdot \alpha\mathbf{u}.$$

ii. We have  $A\mathbf{v} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = -2\mathbf{v}$ . Then  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $-2$ .

Every non-zero scalar multiple of  $\mathbf{v}$  is also an eigenvector of  $A$  with eigenvalue  $-2$ .

Detail:

- Suppose  $\alpha \in \mathbb{R}$  and  $\alpha \neq 0$ . Then

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha(-2\mathbf{v}) = -2\alpha\mathbf{v}.$$

(b) Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ , and  $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ . None of  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  is the zero vector in  $\mathbb{R}^3$ .

i. We have  $A\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{u}$ . Then  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue 1.

Every non-zero scalar multiple of  $\mathbf{u}$  is also an eigenvector of  $A$  with eigenvalue 1. Detail:

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ii. We have  $A\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2\mathbf{v}$ . Then  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue 2.

Every non-zero scalar multiple of  $\mathbf{v}$  is also an eigenvector of  $A$  with eigenvalue 2. Detail:

- Suppose  $\alpha \in \mathbb{R}$  and  $\alpha \neq 0$ . Then

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha \cdot 2\mathbf{v} = 2 \cdot \alpha\mathbf{v}.$$

iii. We have  $A\mathbf{w} = \begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} = 3\mathbf{w}$ . Then  $\mathbf{w}$  is an eigenvector of  $A$  with eigenvalue 3.

Every non-zero scalar multiple of  $\mathbf{w}$  is also an eigenvector of  $A$  with eigenvalue 3. Detail:

- Suppose  $\alpha \in \mathbb{R}$  and  $\alpha \neq 0$ . Then

$$A(\alpha\mathbf{w}) = \alpha A\mathbf{w} = \alpha \cdot 3\mathbf{w} = 3 \cdot \alpha\mathbf{w}.$$

(c) Let  $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ , and  $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ .

None of  $\mathbf{u}$ ,  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  is the zero vector in  $\mathbb{R}^3$ .

i. We have  $A\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4\mathbf{u}$ . Then  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue 4.

Every non-zero scalar multiple of  $\mathbf{u}$  is also an eigenvector of  $A$  with eigenvalue 4.

ii. We have  $A\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \cdot \mathbf{v}_1$ . Then  $\mathbf{v}_1$  is an eigenvector of  $A$  with eigenvalue 1.

We have  $A\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{v}_2$ . Then  $\mathbf{v}_2$  is an eigenvector of  $A$  with eigenvalue 1.

Every linear combination of  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  which is not the zero vector in  $\mathbb{R}^3$  is also an eigenvector of  $A$  with eigenvalue 1. Detail:

- Suppose  $\alpha_1, \alpha_2 \in \mathbb{R}$ , and  $\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ . Suppose  $\mathbf{w} \neq \mathbf{0}_3$ .

Then

$$A\mathbf{w} = A(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1A\mathbf{v}_1 + \alpha_2A\mathbf{v}_2 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = 1 \cdot \mathbf{w}.$$

**Remark.** It is possible for a square matrix to have eigenvectors which are not scalar multiples of each other but which correspond to the same eigenvalue.

More generally, it is possible for a square matrix to have several linearly independent eigenvectors with the same eigenvalue.



$$(d) \text{ Let } A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{bmatrix}, \text{ and } \mathbf{t} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ -1 \\ -5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ -5 \\ -3 \\ 15 \end{bmatrix}.$$

Note that none of  $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$  is the zero vector in  $\mathbb{R}^4$ .

$$\text{i. We have } A\mathbf{t} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 1 \cdot \mathbf{t}. \text{ Then } \mathbf{t} \text{ is an eigenvector of } A \text{ with eigenvalue } 1.$$

$$\text{ii. We have } A\mathbf{u} = \begin{bmatrix} -1 \\ -5 \\ 1 \\ 5 \end{bmatrix} = -1 \cdot \mathbf{u}. \text{ Then } \mathbf{u} \text{ is an eigenvector of } A \text{ with eigenvalue } -1.$$

$$\text{iii. We have } A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 9 \end{bmatrix} = 3\mathbf{v}. \text{ Then } \mathbf{v} \text{ is an eigenvector of } A \text{ with eigenvalue } 3.$$

$$\text{iv. We have } A\mathbf{w} = \begin{bmatrix} -3 \\ 15 \\ 9 \\ -45 \end{bmatrix} = -3\mathbf{w}. \text{ Then } \mathbf{w} \text{ is an eigenvector of } A \text{ with eigenvalue } -3.$$

#### 4. Further examples and non-examples.

(a) Let  $A = \mathcal{O}_{n \times n}$ .

For any  $\mathbf{v} \in \mathbb{R}^n$ ,  $A\mathbf{v} = \mathbf{0}_n = 0 \cdot \mathbf{v}$ .

It follows that every non-zero vector in  $\mathbb{R}^n$  is an eigenvector of  $A$  with eigenvalue 0.

(b) Let  $A = I_n$ .

For any  $\mathbf{v} \in \mathbb{R}^n$ ,  $A\mathbf{v} = \mathbf{v} = 1 \cdot \mathbf{v}$ .

It follows that every non-zero vector in  $\mathbb{R}^n$  is an eigenvector of  $A$  with eigenvalue 1.

(c) Let  $b_1, b_2, b_3 \in \mathbb{R}$ , and  $A = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$ .

For each  $j = 1, 2, 3$ , we have  $A\mathbf{e}_j^{(3)} = b_j\mathbf{e}_j^{(3)}$ .

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(c) Let  $b_1, b_2, b_3 \in \mathbb{R}$ , and  $A = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$ .

For each  $j = 1, 2, 3$ , we have  $A\mathbf{e}_j^{(3)} = b_j\mathbf{e}_j^{(3)}$ .

It follows that  $\mathbf{e}_j^{(3)}$  is an eigenvector of  $A$  with eigenvalue  $b_j$ .

*Digestion:*

*if  $c$  is not equal to any one of  $b_1, b_2, b_3$  then  $c$  is not an eigenvalue of  $A$ .*

*Justification. Note that  $c \neq b_1, c \neq b_2, c \neq b_3$ .*

*Suppose such a  $c$  were an eigenvalue of  $A$ , with a corresponding eigenvector  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ .*

$$\text{Then } \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix} = c\mathbf{x} = A\mathbf{x} = \begin{bmatrix} b_1x_1 \\ b_2x_2 \\ b_3x_3 \end{bmatrix}.$$

*Since  $\mathbf{x}$  is an eigenvector of  $A$ ,  $\mathbf{x} \neq \mathbf{0}$ .  
Then  $x_j \neq 0$  for some  $j$ . For this  $j$ ,  $cx_j = b_jx_j$ .  
Then  $c = b_j$ . (No!!!)*

(d) Let  $b$  be a real number, and  $A = \begin{bmatrix} b & 1 & 0 \\ 0 & b & 1 \\ 0 & 0 & b \end{bmatrix}$ , and  $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ .

We have  $A\mathbf{u} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = b\mathbf{u}$ . Then  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue  $b$ .

We verify that  $b$  is the only eigenvalue of  $A$  and every eigenvector of  $A$  is a scalar multiple of  $\mathbf{u}$ :

- Suppose  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ . Denote the  $j$ -th entry of  $\mathbf{v}$  by  $v_j$  for each  $j = 1, 2, 3$ .

$$\text{Then } \lambda\mathbf{v} = A\mathbf{v} = \begin{bmatrix} bv_1 + v_2 \\ bv_2 + v_3 \\ bv_3 \end{bmatrix} = b\mathbf{v} + \begin{bmatrix} v_2 \\ v_3 \\ 0 \end{bmatrix}. \text{ Therefore } \begin{bmatrix} v_2 \\ v_3 \\ 0 \end{bmatrix} = (\lambda - b)\mathbf{v} = (\lambda - b) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Comparing the entries, we have

$$(\lambda - b)v_3 = 0, \quad v_2 = (\lambda - b)v_1, \quad v_3 = (\lambda - b)v_2.$$

Since  $(\lambda - b)v_3 = 0$ , we have  $\lambda = b$  or  $v_3 = 0$ . We claim that  $\lambda = b$ :

\* Suppose it were true that  $\lambda \neq b$ . Then  $\lambda - b \neq 0$  and  $v_3 = 0$ . So  $v_2 = \frac{v_3}{\lambda - b} = 0$  and  $v_1 = \frac{v_2}{\lambda - b} = 0$ .

Now  $\mathbf{v} = \mathbf{0}$ . But this is impossible because  $\mathbf{v}$  is an eigenvector of  $A$ .

Therefore  $\lambda = b$ . Then  $v_2 = v_3 = 0$ . Hence  $\mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = v_1\mathbf{u}$ .

**Remark.** It is possible for an  $(n \times n)$ -square matrix to have ‘relatively few’ eigenvectors, in the sense that there is no chance to form a basis for  $\mathbb{R}^n$  out of eigenvectors.

(e) Let  $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ .

We verify that  $A$  has no eigenvalue (and no eigenvector):

- Suppose  $\mathbf{u}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ . Denote the  $j$ -th entry of  $\mathbf{u}$  by  $u_j$ .

$$\text{We have } \lambda \mathbf{u} = A\mathbf{u} = \begin{bmatrix} u_1 - u_4 \\ u_1 + u_2 \\ u_2 + u_3 \\ u_3 + u_4 \end{bmatrix} = \mathbf{u} + \begin{bmatrix} -u_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}. \text{ Then } (\lambda - 1) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = (\lambda - 1)\mathbf{u} = \begin{bmatrix} -u_4 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Comparing the entries, we have

$$(\lambda - 1)u_1 = -u_4, \quad (\lambda - 1)u_2 = u_1, \quad (\lambda - 1)u_3 = u_2, \quad (\lambda - 1)u_4 = u_3$$

Then  $u_1 = -(\lambda - 1)^4 u_1$ . Therefore  $[(\lambda - 1)^4 + 1]u_1 = 0$ .

We have  $(\lambda - 1)^4 + 1 = 0$  or  $u_1 = 0$ .

Since  $\lambda$  is a real number,  $(\lambda - 1)^4 + 1 > 0$ . Then  $u_1 = 0$ .

Therefore  $u_2 = u_3 = u_4 = 0$  also.

Hence  $\mathbf{u} = \mathbf{0}$ . But this is impossible, because  $\mathbf{u}$  is an eigenvector of  $A$ .

**Remark.** It is possible for a square matrix to have no eigenvalue (and hence to have no eigenvector).

**5. Lemma (2).**

*Let  $A$  be an  $(n \times n)$ -square matrix.*

*Suppose  $\mathbf{u}, \mathbf{v}$  are eigenvectors of  $A$  with distinct eigenvalues.*

*Then  $\mathbf{u}, \mathbf{v}$  are linearly independent.*

**6. Lemma (3).**

*Let  $A$  be an  $(n \times n)$ -square matrix.*

*Suppose  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent eigenvectors of  $A$ .*

*Further suppose  $\mathbf{v}$  is an eigenvector of  $A$  whose eigenvalue is distinct from the eigenvalue of each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ .*

*Then  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{v}$  are linearly independent.*

5. **Lemma (2).**

Let  $A$  be an  $(n \times n)$ -square matrix.

Suppose  $\mathbf{u}, \mathbf{v}$  are eigenvectors of  $A$  with distinct eigenvalues.

Then  $\mathbf{u}, \mathbf{v}$  are linearly independent.

The previous examples seem to suggest that a square matrix cannot have 'too many' eigenvalues.

So do Lemma (2) and Lemma (3).

6. **Lemma (3).**

Let  $A$  be an  $(n \times n)$ -square matrix.

Suppose  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent eigenvectors of  $A$ .

Further suppose  $\mathbf{v}$  is an eigenvector of  $A$  whose eigenvalue is distinct from the eigenvalue of each of  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ .

Then  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{v}$  are linearly independent.

## 7. Question.

What are the immediate consequences of Lemma (2) and Lemma (3) combined?

### Answer.

Suppose  $A$  is an  $(n \times n)$ -square matrix, and  $\lambda_1, \lambda_2, \lambda_3, \dots$  are pairwise distinct eigenvalues of  $A$ .

(a) For each  $j$ , suppose  $\mathbf{v}_j$  is an eigenvector of  $A$  with eigenvalue  $\lambda_j$ .

By Lemma (2),  $\mathbf{v}_1, \mathbf{v}_2$  are linearly independent.

Then by Lemma (3),  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly independent.

Again by Lemma (3),  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  are linearly independent.

Repeatedly by Lemma (3),  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent for each  $k$ .

It follows that any finitely many vectors from amongst  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$  are linearly independent.

(b) Now, for each  $j$ , we indeed pick some eigenvector of  $A$ , say,  $\mathbf{v}_j$ , with eigenvalue  $\lambda_j$ .

As argued above, the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$  are linearly independent.

Further recall that any  $n + 1$  vectors in  $\mathbb{R}^n$  are linearly dependent.

Then the list  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$  has to terminate somewhere: it is  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ , for some  $p \leq n$ , in disguise.

This in turn implies that the  $p$  pairwise distinct numbers  $\lambda_1, \lambda_2, \dots, \lambda_p$  are all the eigenvalues that  $A$  has.

We summarize the above discussion into Theorem (A) and Theorem (B) below.



## 8. **Theorem (A).**

Suppose  $A$  is a square matrix, and  $\lambda_1, \lambda_2, \lambda_3, \dots$  are pairwise distinct eigenvalues.

Further suppose

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$$

are eigenvectors of  $A$  with respective eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \dots .$$

Then  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$  are linearly independent.

## **Theorem (B).**

Suppose  $A$  is an  $(n \times n)$ -square matrix.

Then  $A$  has at most  $n$  pairwise distinct eigenvalues.

## **Remark.**

What we will be most interested is the scenario in which  $A$  is an  $(n \times n)$ -square matrix and there is some basis for  $\mathbb{R}^n$  which are eigenvectors of  $A$ . (Their eigenvalues are not necessarily pairwise distinct.)

## 9. Proof of Lemma (2).

Let  $A$  be an  $(n \times n)$ -square matrix.

Suppose  $\mathbf{u}, \mathbf{v}$  are eigenvectors of  $A$  with distinct eigenvalues, say,  $\mu, \lambda$  respectively.

[Reminder: we want to deduce that  $\mathbf{u}, \mathbf{v}$  are linearly independent.

This amounts to verifying: ‘For any  $\alpha, \beta \in \mathbb{R}$ , if  $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$  then  $\alpha = \beta = 0$ .’]

Pick any  $\alpha, \beta \in \mathbb{R}$ . Suppose  $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ .

[Reminder: We want to deduce  $\alpha = \beta = 0$ .

Ask: How to make use of assumption? Answer: Multiply this equality by  $A$  from the left.]

By assumption,  $A\mathbf{u} = \mu\mathbf{u}$  and  $A\mathbf{v} = \lambda\mathbf{v}$ . Then

$$\alpha\mu\mathbf{u} + \beta\lambda\mathbf{v} = \alpha A\mathbf{u} + \beta A\mathbf{v} = A(\alpha\mathbf{u} + \beta\mathbf{v}) = A\mathbf{0} = \mathbf{0}.$$

We also have

$$\alpha\lambda\mathbf{u} + \beta\lambda\mathbf{v} = \lambda(\alpha\mathbf{u} + \beta\mathbf{v}) = \mathbf{0}.$$

Then

$$\alpha(\mu - \lambda)\mathbf{u} = \mathbf{0}.$$

By assumption,  $\mathbf{u}$  is not the zero vector. Also by assumption,  $\mu - \lambda \neq 0$ . Then  $\alpha = 0$ .

Now we have  $\beta\mathbf{v} = \mathbf{0}$ . By assumption,  $\mathbf{v}$  is not the zero vector. Then  $\beta = 0$ .

## 10. Proof of Lemma (3).

Let  $A$  be an  $(n \times n)$ -square matrix. Suppose  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent eigenvectors of  $A$ , say, with eigenvalues  $\mu_1, \mu_2, \dots, \mu_k$  respectively.

Further suppose  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ , and  $\lambda \neq \mu_1, \lambda \neq \mu_2, \dots, \lambda \neq \mu_k$ .

[Reminder: we want to deduce that  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{v}$  are linearly independent.

This amounts to verifying: ‘For any  $\alpha_1, \alpha_2, \dots, \alpha_k, \beta \in \mathbb{R}$ , if  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k + \beta\mathbf{v} = \mathbf{0}$  then  $\alpha_1 = \alpha_2 = \dots = \alpha_k = \beta = 0$ .’]

Pick any  $\alpha_1, \alpha_2, \dots, \alpha_k, \beta \in \mathbb{R}$ . Suppose  $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k + \beta\mathbf{v} = \mathbf{0}$ .

[Reminder: We want to deduce  $\alpha_1 = \alpha_2 = \dots = \alpha_k = \beta = 0$ .

Ask: How to make use of assumption? Answer: Multiply this equality by  $A$  from the left.]

By assumption,  $A\mathbf{u}_j = \mu_j\mathbf{u}_j$  for each  $j = 1, 2, \dots, k$ , and  $A\mathbf{v} = \lambda\mathbf{v}$ . Then

$$\begin{aligned}\alpha_1\mu_1\mathbf{u}_1 + \alpha_2\mu_2\mathbf{u}_2 + \dots + \alpha_k\mu_k\mathbf{u}_k + \beta\lambda\mathbf{v} &= \alpha_1A\mathbf{u}_1 + \alpha_2A\mathbf{u}_2 + \dots + \alpha_kA\mathbf{u}_k + \beta A\mathbf{v} \\ &= A(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k + \beta\mathbf{v}) = A\mathbf{0} = \mathbf{0}\end{aligned}$$

We also have

$$\alpha_1\lambda\mathbf{u}_1 + \alpha_2\lambda\mathbf{u}_2 + \dots + \alpha_k\lambda\mathbf{u}_k + \beta\lambda\mathbf{v} = \lambda(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \dots + \alpha_k\mathbf{u}_k + \beta\mathbf{v}) = \mathbf{0}.$$

Then  $\alpha_1(\mu_1 - \lambda)\mathbf{u}_1 + \alpha_2(\mu_2 - \lambda)\mathbf{u}_2 + \dots + \alpha_k(\mu_k - \lambda)\mathbf{u}_k = \mathbf{0}$ .

By assumption,  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$  are linearly independent.

Then for each  $j$ , we have  $\alpha_j(\mu_j - \lambda) = 0$ . By assumption,  $\lambda \neq \mu_j$ . Then  $\alpha_j = 0$ .

Now we have  $\beta\mathbf{v} = \mathbf{0}$ . By assumption  $\mathbf{v}$  is not the zero vector. Then  $\beta = 0$ . The result follows.