MATH1030 How to check whether some given vectors constitute a basis for a given subspace of $\mathbb{R}^n$.

1. Recall the definition for the notion of *basis for a subspace of* $\mathbb{R}^n$.

   Let V be a subspace of $\mathbb{R}^n$.
   We declare that if V is the zero subspace of $\mathbb{R}^n$ then the empty set is the basis for V.
   From now on suppose V is not the zero subspace of $\mathbb{R}^n$.
   Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in V.
   The vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are said to constitute a basis for V if and only if both of the statements (BL), (BS) *below hold:*

   (BL) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *are linearly independent.*

   (BS) *Every vector in V is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

   Also recall Theorem (J) from the handout *Inequalities on dimension*:

   Let W be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.
   Then the statements below are logically equivalent:

   ($\sharp$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *constitute a basis for W.*

   ($\natural$) $\dim(W) = p$, *and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.*

   ($\flat$) $\dim(W) = p$, *and every vector of W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

2. **'Algorithms' associated with Theorem (J).**

   When we can only rely on the definition of for the notion of basis, it will be a tedious task to check whether a given collections of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ in $\mathbb{R}^n$ constitutes a basis for a given subspace V of $\mathbb{R}^n$. (Why? In order to verify that every vector in W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$, we have to first produce a basis, say, $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \cdots$ for W. This is already no easy task.)

   Theorem (J) provides a short-cut in many situations. If the dimension of V has already been known, then we will only need to verify the validity of one (instead of both) of the statements (BL), (BS) for the collection of vectors concerned.

3. **'Algorithm' for checking whether a concretely given collection of vectors of $\mathbb{R}^n$ is a basis for the null space of a matrix with n columns.**

   Let A be a matrix with n columns, and $V = \mathcal{N}(A)$. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in $\mathbb{R}^n$.

   We proceed to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for V, as described below:

   - **Step (1).**
     Check whether $A\mathbf{u}_1 = A\mathbf{u}_2 = \cdots = A\mathbf{u}_p = \mathbf{0}$.
     If no, then conclude that some of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ do not belong to V and that these vectors do not constitute a basis for V.
     If yes, then proceed to Step (2).
   - **Step (2).**
     (From now on, we suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in V$.)
     Find the reduced row-echelon form A' which is row equivalent to A.
     By inspecting A', determine whether $\dim(V) = p$ holds. (Recall that $\dim(V)$ is the number of free columns of A'.)
     If no, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ do not constitute a basis for V.
     If yes, then proceed to Step (3).
   - **Step (3).**
     (From now on, we also suppose $p = \dim(V)$.)
     Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ]$.
     Determine the reduced row-echelon form U' which is row-equivalent to U.
     By inspecting U', determine whether $\mathcal{N}(U) = \{\mathbf{0}_p\}$.
     If no, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly dependent and that they do not constitute a basis for V.
     If yes, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent and that they do constitute a basis for V.

4. **Illustrations.**

(a) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix}$, *and* $V = \mathcal{N}(A)$. *Let* $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2$ *constitute a basis for* V.

We have $A\mathbf{u}_1 = \mathbf{0}_3$ and $A\mathbf{u}_2 = \mathbf{0}_3$. Then $\mathbf{u}_1, \mathbf{u}_2 \in V$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 3 & 1 & 5 & -7 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are two free columns in A'. Then $\dim(V) = \dim(\mathcal{N}(A')) = 2$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2\ ]$. We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 1 & 5 \\ -1 & -3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = U'$$

We have $\mathcal{N}(U) = \{\mathbf{0}_2\}$. Then $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent.

Hence $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for V.

(b) *Let* $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix}$, *and* $V = \mathcal{N}(A)$. *Let* $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 8 \\ -2 \\ -1 \\ -2 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *constitute a basis for* V.

We have $A\mathbf{u}_1 = \mathbf{0}_5$, $A\mathbf{u}_2 = \mathbf{0}_5$ and $A\mathbf{u}_3 = \mathbf{0}_5$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V$.

We obtain the reduced row-echelon form A' which is row-equivalent to A by applying a sequence of row operations to A:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A'$$

There are three free columns in A'. Then $\dim(V) = \dim(\mathcal{N}(A')) = 3$.

Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$. We obtain the reduced row-echelon form U' which is row-equivalent to U:

$$U = \begin{bmatrix} 2 & 2 & -1 \\ -5 & 2 & 8 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & -1 & -2 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U'$$

We have $\mathcal{N}(U) = \{\mathbf{0}_3\}$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for V.

5. **'Algorithm' for checking whether a concretely given collection of vectors of $\mathbb{R}^n$ is a basis for the span of another concretely given collection of vectors of $\mathbb{R}^n$.**

Let $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_k$ be vectors in $\mathbb{R}^n$, and $V = \mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_k\})$.

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in $\mathbb{R}^n$.

We proceed to determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for V, as described below:

- **Step (0).**
  Form the $(n \times k)$-matrix $Z = [\ \mathbf{z}_1 \mid \mathbf{z}_2 \mid \cdots \mid \mathbf{z}_k\ ]$ and the $(n \times p)$-matrix $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_p\ ]$.
- **Step (1).**
  Form the $(n \times (k+p))$-matrix $[\ Z \mid U\ ]$.
  Obtain some appropriate $(n \times (k+p))$-matrix $[\ Z' \mid U'\ ]$ which is row-equivalent to $[\ Z \mid U\ ]$, and in which Z' is the reduced row-echelon form row-equivalent to Z.
- **Step (2).**
  By inspecting $[\ Z' \mid U'\ ]$, determine whether each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ belongs to V.
  If no, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ do not constitute a basis for V.
  If yes, then proceed to Step (3).

- **Step (3).**
(From now on, we suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p \in V$.)
By inspecting Z', determine determine whether $\dim(V) = p$ holds.
If no, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ do not constitute a basis for V.
If yes, then proceed to Step (4).
- **Step (4).**
(From now on, we also suppose $\dim(V) = p$.)
Determine the reduced row-echelon form $\hat{U}$ which is row-equivalent to U.
By inspecting $\hat{U}$, determine whether $\mathcal{N}(U) = \{\mathbf{0}_p\}$.
If no, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly dependent and that they do not constitute a basis for $\mathcal{N}(A)$.
If yes, then conclude that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent and that they do constitute a basis for $\mathcal{N}(A)$.

6. **Illustrations.**

(a) *Let* $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{z}_4 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\})$.

Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 7 \\ 4 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 8 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 7 \\ 3 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *constitute a basis for* V.
Define $Z = [\ \mathbf{z}_1 \mid \mathbf{z}_2 \mid \mathbf{z}_3 \mid \mathbf{z}_4\ ]$.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.
We apply row operations on $[\ Z \mid U\ ]$ so as to obtain a matrix $[\ Z' \mid U'\ ]$ in which Z' is the reduced row-echelon form which is row equivalent to Z:

$$[\ Z \mid U\ ] = \left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 3 & 4 & 4 & 3 & 7 & 8 & 7 \\ 2 & 2 & 1 & 1 & 4 & 3 & 3 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\ Z \mid U\ ]$$

Each column in U' is a linear combination of the columns of Z'. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathcal{C}(Z) = V$.

Note that $Z' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $\dim(V) = r(Z) = r(Z') = 3$.

We obtain the reduced row-echelon form $\hat{U}$ which is row-equivalent to U:

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 0 \\ 7 & 8 & 7 \\ 4 & 3 & 3 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have $\mathcal{N}(U) = \{\mathbf{0}_3\}$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.
Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for V.

(b) *Let* $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \\ -1 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \\ -1 \\ 9 \end{bmatrix}, \mathbf{z}_4 = \begin{bmatrix} 1 \\ -1 \\ -5 \\ 2 \\ 0 \end{bmatrix}$, *and* $V = \mathsf{Span}\ (\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\})$.

Let $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -2 \\ 9 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 5 \\ 5 \\ 15 \\ 1 \\ 8 \end{bmatrix}$.

We check whether $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ *constitute a basis for* V.
Define $Z = [\ \mathbf{z}_1 \mid \mathbf{z}_2 \mid \mathbf{z}_3 \mid \mathbf{z}_4\ ]$.
Define $U = [\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3\ ]$.
We apply row operations on $[\ Z \mid U\ ]$ so as to obtain a matrix $[\ Z' \mid U'\ ]$ in which Z' is the reduced row-echelon form which is row equivalent to Z:

$$[\ Z \mid U\ ] = \left[\begin{array}{cccc|ccc} 1 & 1 & 3 & 1 & 0 & 2 & 5 \\ 2 & 1 & 2 & -1 & 1 & 1 & 5 \\ 7 & 3 & 5 & -5 & 4 & 2 & 15 \\ 1 & 1 & -1 & 2 & 0 & -2 & 1 \\ -1 & 0 & 9 & 0 & -1 & 9 & 8 \end{array}\right] \longrightarrow \cdots\cdots \longrightarrow \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -9/4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1/4 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\ Z \mid U\ ]$$

Each column in U' is a linear combination of the columns of Z'. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathcal{C}(Z) = V$.

Note that $Z' = \begin{bmatrix} 1 & 0 & 0 & -9/4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $\dim(V) = r(Z) = r(Z') = 3$.

We obtain the reduced row-echelon form $\hat{U}$ which is row-equivalent to U:

$$\begin{bmatrix} 0 & 2 & 5 \\ 1 & 1 & 5 \\ 4 & 2 & 15 \\ 0 & -2 & 1 \\ -1 & 9 & 8 \end{bmatrix} \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We have $\mathcal{N}(U) = \{\mathbf{0}_3\}$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent.

Hence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for V.