1. **Definition. (Subspaces of subspace of $\mathbb{R}^n$.)**

Let V, W be subspaces of $\mathbb{R}^n$.

We say V is a subspace of W if and only if the statement (†) holds:

(†) *For any $\mathbf{x} \in \mathbb{R}^n$, if $\mathbf{x} \in V$ then $\mathbf{x} \in W$.*

Remark.

In plain words, the statement (†) reads: '*every vector of V belongs to W*'.

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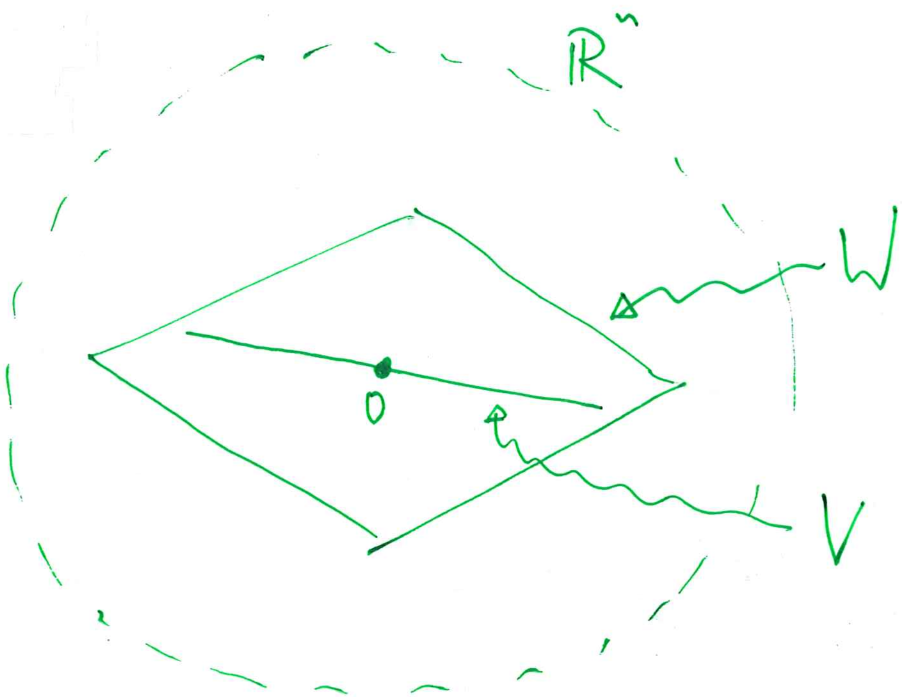
Visualization of the definition for the notion 'subspace of a subspace of $\mathbb{R}^n$'.



- W is a subspace of $\mathbb{R}^n$.
- V is a subspace of $\mathbb{R}^n$.
- Every vector which belongs to V also belongs to W. (This is what (†) is saying.)

2. **Illustrations.**

(a) Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 & 3 & 5 \end{bmatrix}$.

With a direct application of the definition, we can show that:

- $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(B)$,
- $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(C)$,
- $\mathcal{N}(A)$ is a subspace of $\mathcal{N}(D)$,
- $\mathcal{N}(B)$ is a subspace of $\mathcal{N}(D)$, and
- $\mathcal{N}(C)$ is a subspace of $\mathcal{N}(D)$.

After some harder work, we can also show that:

- $\mathcal{N}(B)$ is not a subspace of $\mathcal{N}(C)$, and
- $\mathcal{N}(C)$ is not a subspace of $\mathcal{N}(B)$.

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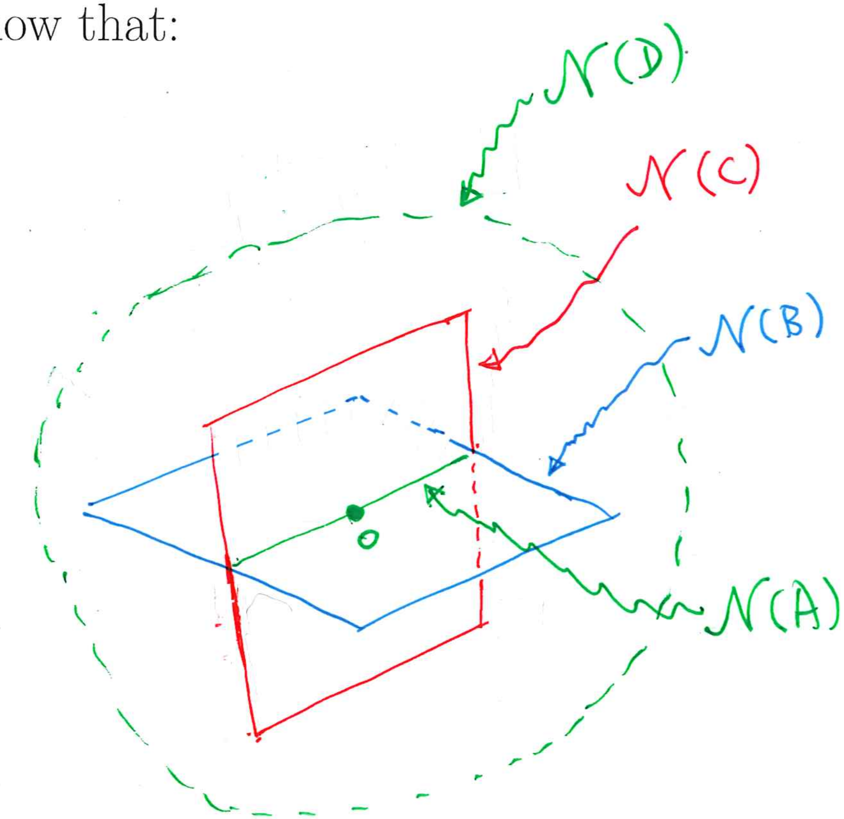

(b) Let $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and

$T = \mathsf{Span}\left(\{\mathbf{x}_1, \mathbf{x}_2\}\right)$, $U = \mathsf{Span}\left(\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}\right)$, $V = \mathsf{Span}\left(\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}\right)$, $W = \mathsf{Span}\left(\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}\right)$.

With a direct application of the definition, we can show that:

- T is a subspace of U,
- T is a subspace of V,
- T is a subspace of W,
- U is a subspace of W, and
- V is a subspace of W.

After some harder work, we can also show that:

- U is not a subspace of V, and
- V is not a subspace of U.

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With a direct application of the definition, we can show that:

- T is a subspace of U,
- T is a subspace of V,
- T is a subspace of W,
- U is a subspace of W, and
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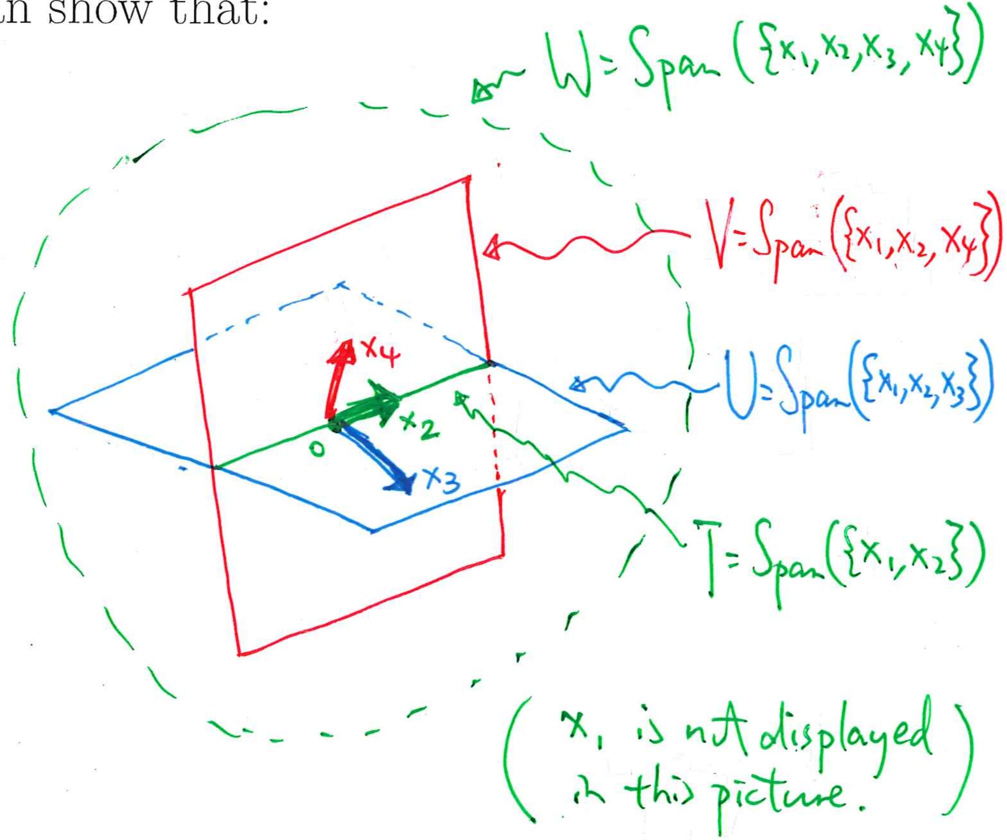
After some harder work, we can also show that:

- U is not a subspace of V, and
- V is not a subspace of U.



3. Again recall the Replacement Theorem (Theorem (F)) in the handout *More on minimal spanning set*:

   Let W be a subspace of $\mathbb{R}^n$.
   Let $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ be vectors in W. Suppose none of these vectors is the zero vector.

   Suppose $\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ are linearly independent.
   Further suppose $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$ constitute a basis for W.

   Then:

   - *$q \geq p$, and*
   - *there is a basis for W which is constituted by*
     * *$\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_p$ together with*
     * *some $q - p$ vectors from amongst $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_q$.*

This result, combined with the notion of *subspaces of a subspace of* $\mathbb{R}^n$, leads to Theorem (I), which is a useful tool for comparing various subspaces of $\mathbb{R}^n$.

4. **Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$.

Suppose V is a subspace of W.

Then $\dim(V) \leq \dim(W)$.

Moreover, equality holds if and only if $V = W$.

Remark.

From the handout *Dimension*, we have learnt that

- every subspace of $\mathbb{R}^n$ is of dimension at most n, and
- $\mathbb{R}^n$ is the one and only one n-dimensional subspace of $\mathbb{R}^n$.

What we have learnt earlier can be seen as a manifestation of Theorem (I) in the special case in which $W = \mathbb{R}^n$.

5. **Proof of Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$.

Suppose V is a subspace of W.

Write $\dim(V) = k$. Pick some basis for V, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

Then by assumption, each of them belongs to W.

(a) By definition, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are linearly independent.
So they are k linearly independent vectors in W.
Hence $\dim(V) = k \leq \dim(W)$ by Theorem (H).

(b) Suppose $V = W$. Then $\dim(V) = \dim(W)$.

(c) Suppose $\dim(V) = \dim(W)$. Then $\dim(W) = k$.
Pick some basis for W, say, $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$.

Recall that $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are k linearly independent vectors in W.

Then by the Replacement Theorem, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$, together with possibly some vectors from amongst $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_k$, constitute a basis for W.

Since $\dim(W) = k$, the k vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ already constitute a basis for W.

It follows that $W = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}) = V$.

5. **Proof of Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$.

Suppose V is a subspace of W.

Write $\dim(V) = k$. Pick some basis for V, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

So by definition, $V = \text{Span}(\{v_1, v_2, \dots, v_k\})$ and $v_1, v_2, \dots, v_k$ are linearly independent.

Then by assumption, each of them belongs to W.

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6. **Corollary (1) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(W) = p$.

Further suppose V is a subspace of W.
Also suppose that there are p vectors in V which are linearly independent.

Then $V = W$.

7. **Proof of Corollary (1) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(W) = p$.
Further suppose V is a subspace of W.
Also suppose that there are p vectors in V which are linearly independent.

- Since V is a subspace of W, we have $\dim(V) \leq \dim(W)$.
- Since there are p vectors in V which are linearly independent, we have

$$\dim(V) \geq p = \dim(W),$$

Then $\dim(V) = \dim(W)$.

Now, by Theorem (I), since V is a subspace of W, we have $V = W$.

6. **Corollary (1) to Theorem (I).**

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Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(W) = p$.
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- Since V is a subspace of W, we have $\dim(V) \leq \dim(W)$.
- Since there are p vectors in V which are linearly independent, we have

$$\dim(V) \geq p = \dim(W),$$

Then $\dim(V) = \dim(W)$.

Now, by Theorem (I), since V is a subspace of W, we have $V = W$.

8. **Corollary (2) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(V) = p$.

Further suppose V is a subspace of W.

Also suppose that there are p vectors of W so that every vector of W is a linear combination of these p vectors.

Then $V = W$.

9. **Proof of Corollary (2) to Theorem (I).**

Let V, W be subspaces of $\mathbb{R}^n$. Suppose $\dim(V) = p$.

Further suppose V is a subspace of W.

Also suppose that there are p vectors of W, say, $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$, so that every vector of W is a linear combination of these p vectors.

- Since $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ are all vectors in W, every linear combination of $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ is a vector in W.
  Then $W = \mathsf{Span}\,\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p\}$.
  Therefore there is a basis for W from amongst the p vectors $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$.
  Hence $\dim(W) \leq p = \dim(V)$.
- Since V is a subspace of W, we have $\dim(V) \leq \dim(W)$.

Then $\dim(V) = \dim(W)$. Now, by Theorem (I), since V is a subset of W, we have $V = W$.

10. **Theorem (J). (Re-formulation of the notion of basis in terms of dimension.)**

Let W be a subspace of $\mathbb{R}^n$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.

Then the statements below are logically equivalent:

($\sharp$) *$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for W.*

($\natural$) *$\dim(W) = p$, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.*

($\flat$) *$\dim(W) = p$, and every vector of W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.*

11. **Proof of Theorem (J).**

Let W be a subspace of $\mathbb{R}^n$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W.

[The statement ($\sharp$) implies each of the statements ($\natural$), ($\flat$) immediately.

What matters is whether whether each of the statements ($\natural$), ($\flat$) separately implies ($\sharp$).]

- [We ask whether ($\natural$) implies ($\sharp$).]
  Suppose $\dim(W) = p$ and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.
  Define $V = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p\})$. [We want to show that $V = W$.]
  By definition, $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for the p-dimensional subspace V of $\mathbb{R}^n$.
  Now we have $\dim(V) = p = \dim(W)$.
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in W, every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ belongs to W.
  Then V is a subspace of W.
  Now, by Corollary (2) to Theorem (I), we have $V = W$.
  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for W.
- [We ask whether ($\flat$) implies ($\sharp$).]
  Suppose $\dim(W) = p$, and every vector in W is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.
  Since $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are vectors in W, every linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ is a vector in W.
  Then there is a basis for W, with, say, q vectors, from amongst the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$.
  Without loss of generality, assume they are $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_q$.
  These q vectors are linearly independent.
  Since these q vectors constitute a basis for W, we have $\dim(W) = q$.
  Then $p = \dim(W) = q$. Therefore $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ are linearly independent.
  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ constitute a basis for W.

12. **Theorem (J'). (Re-formulation of Theorem (J) in terms of systems of equations.)**

Let W be a subspace of $\mathbb{R}^n$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ be vectors in W, and U is the $(n \times p)$-matrix given by $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_p \,\right]$.

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$ *is a basis for W.*

(b) $\dim(W) = p$, *and the homogeneous system $\mathcal{LS}(U, \mathbf{0})$ has no non-trivial solution.*

(c) $\dim(W) = p$, *and for any $\mathbf{b} \in V$, the system $\mathcal{LS}(U, \mathbf{b})$ is consistent.*