MATH1030 Further examples on minimal spanning sets.

1. Consider each of the collection of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$ below. Write $V = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots\})$.

   Find a basis for V from amongst the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots$, and express the remaining vectors as linear combinations of the vectors in the basis found.

   (a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$

   (b) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 6 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -7 \\ 3 \\ -3 \\ -14 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$

   (c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ -3 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 3 \\ 8 \\ 0 \\ -5 \end{bmatrix}.$

   (d) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 10 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 7 \\ 8 \\ 11 \\ 13 \end{bmatrix}.$

   (e) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 3 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -4 \\ 9 \\ 5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 2 \\ -3 \\ 8 \\ 7 \end{bmatrix}, \mathbf{u}_5 = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 2 \end{bmatrix}, \mathbf{u}_6 = \begin{bmatrix} 5 \\ -8 \\ 19 \\ 17 \end{bmatrix}.$

Answer.

1. (There are many correct answers for each part.)

   (a) $\mathbf{u}_1, \mathbf{u}_2$ constitute a basis for V.
   $\mathbf{u}_3 = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_4 = -4\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = 4\mathbf{u}_1 - \mathbf{u}_2$.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 & -4 & 4 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for V.
   $\mathbf{u}_3 = 2\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_4$.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (c) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for V.
   $\mathbf{u}_3 = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 + 3\mathbf{u}_2 + 2\mathbf{u}_4$.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (d) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ constitute a basis for V.
   $\mathbf{u}_4 = 3\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3$, $\mathbf{u}_5 = 2\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_4$.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   (e) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ constitute a basis for V.
   $\mathbf{u}_3 = 2\mathbf{u}_1 + 3\mathbf{u}_2$, $\mathbf{u}_5 = \mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_4$, $\mathbf{u}_6 = \mathbf{u}_2 + 2\mathbf{u}_4$.
   Reason:

$$[\ \mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3 \mid \mathbf{u}_4 \mid \mathbf{u}_5 \mid \mathbf{u}_6\ ] \longrightarrow \cdots\cdots \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$