1. Recall Theorem (B) from the handout *Linear Combinations*:

   Let $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$ *be vectors in* $\mathbb{R}^m$.
   Every linear combination of (finitely many) linear combinations of $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$ *is a linear combination of* $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$.

   Also recall the definition for the notion of *span*:

   Let $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *be ('finitely many') vectors in* $\mathbb{R}^m$.
   The span of (the set of vectors) $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *is defined to be the set*

   $$\left\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} \text{ is a linear combination of } \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n \right\}$$

   We denote this set by $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

2. In this handout we need to handle 'equality questions' on sets. We introduce the definition for the notion of *set equality* (or recall it from the handout *The use of set notations in linear algebra*):

   Let K, L *be sets (of vectors in* $\mathbb{R}^n$*).*
   We say that K, L *are equal to each other, and write* $K = L$ *if and only if both of* (†), (‡) *are true:*

   (†) *For any* $\mathbf{u}$, *if* $\mathbf{u} \in K$ *then* $\mathbf{u} \in L$.
   (‡) *For any* $\mathbf{v}$, *if* $\mathbf{v} \in L$ *then* $\mathbf{v} \in K$.

1. Recall Theorem (B) from the handout *Linear Combinations*:

   Let $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$ be vectors in $\mathbb{R}^m$.
   Every linear combination of (finitely many) linear combinations of $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$ is a linear combination of $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_n$.

   Also recall the definition for the notion of *span*:

   Let $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ be ('finitely many') vectors in $\mathbb{R}^m$.
   The span of (the set of vectors) $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ is defined to be the set

   $$\{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} \text{ is a linear combination of } \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\}$$

   We denote this set by $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

2. In this handout we need to handle 'equality questions' on sets. We introduce the definition for the notion of *set equality* (or recall it from the handout *The use of set notations in linear algebra*):

   Let K, L be sets (of vectors in $\mathbb{R}^n$).
   We say that K, L are equal to each other, and write $K = L$ if and only if both of (†), (‡) are true:

   (†) *For any $\mathbf{u}$, if $\mathbf{u} \in K$ then $\mathbf{u} \in L$.* (Every vector in K belongs to L also.)
   (‡) *For any $\mathbf{v}$, if $\mathbf{v} \in L$ then $\mathbf{v} \in K$.* (Every vector in L belongs to K also.)

3. **Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.
Suppose each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.
Further suppose each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.
Then

$$\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}).$$

Remark.

The equality

$$\text{`}\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})\text{'}$$

is a set equality.
What such an equality means is that the statements (†), (‡) below hold simultaneously:

(†) For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.
(‡) For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

In plain words:

- (†) reads: every vector in $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$ belongs to $\mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.
- (‡) reads: every vector in $\mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$ belongs to $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

4. **Proof of Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be vectors in $\mathbb{R}^m$.

Suppose each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Further suppose each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

We deduce the statement (†):

(†) 'For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.'

- Pick any $\mathbf{y} \in \mathbb{R}^m$. Suppose $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

  [Reminder: We want to see why $\mathbf{y}$ belongs to $\mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.]

  By definition, $\mathbf{y}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  Each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

  Then, by Theorem (B), $\mathbf{y}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

  Therefore $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

Modifying the above argument for (†), we also deduce the statement (‡):

(‡) 'For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.'

It follows that $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

4. **Proof of Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ be vectors in $\mathbb{R}^m$.

Suppose each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Further suppose each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

We deduce the statement (†):

(†) 'For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.'

- Pick any $\mathbf{y} \in \mathbb{R}^m$. Suppose $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

  Consequence of what is supposed.

  [Reminder: We want to see why $\mathbf{y}$ belongs to $\mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.]

  What does it amount to? 'y is a linear combination of $v_1, v_2, \ldots, v_k$.'

  By definition, $\mathbf{y}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  Each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

  Then, by Theorem (B), $\mathbf{y}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

  Therefore $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

How? Interchange the roles of '$u_1, \ldots, u_n$' and '$v_1, \ldots, v_k$' in the passage above.

Modifying the above argument for (†), we also deduce the statement (‡):

(‡) 'For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$ then $\mathbf{y} \in \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.'

It follows that $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

5. The converse of Theorem (1) is an immediate consequence of Lemma (2).

Lemma (2).

Suppose $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *are vectors in* $\mathbb{R}^m$.

Then each of $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *belongs to* $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

Theorem (3). (Converse of Theorem (1).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

Suppose

$$\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}).$$

Then each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Also each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

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Then each of $\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n$ *belongs to* $\mathsf{Span}\,(\{\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n\})$.

Observe: z_1 is a linear combination of $z_1, z_2, \ldots, z_n$, with $z_1 = 1 \cdot z_1 + 0 \cdot z_2 + \ldots + 0 \cdot z_n$. z_2 is a linear combination of $z_1, z_2, \ldots, z_n$. z_3 is a linear combination of $z_1, z_2, \ldots, z_n$. Et cetera.

Theorem (3). (Converse of Theorem (1).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

Suppose

$$\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}).$$

Then each of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Also each of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

6. We may combine Theorem (1) and Theorem (3) to obtain Theorem (K):

Theorem (K).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

The statements below are logically equivalent:

(a) *Each of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, *and each of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

(b) $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\ (\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

7. **Corollary to Theorem (K).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ *be vectors in* $\mathbb{R}^m$.

The statements below are logically equivalent:

(a) *Each of* $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p\}) = \mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

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Theorem (K).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

The statements below are logically equivalent:

(a) *Each of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$, *and each of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

(b) $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

7. **Corollary to Theorem (K).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ *be vectors in* $\mathbb{R}^m$.

The statements below are logically equivalent:

(a) *Each of* $\mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{t}_1, \mathbf{t}_2, \cdots, \mathbf{t}_p\}) = \mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\})$.

If (a) holds then: certainly each of $t_1, t_2, \ldots, t_p$ is a linear combination of $u_1, u_2, \ldots, u_n$ and trivially each of $u_1, u_2, \ldots, u_n$ is a linear combination of $u_1, u_2, \ldots, u_n$. Moreover, each of $u_1, u_2, \ldots, u_n$ is trivially a linear combination of $u_1, u_2, \ldots, u_n, t_1, t_2, \ldots, t_p$. So (b) follows.

If (b) holds then: each of $u_1, u_2, \ldots, u_n, t_1, t_2, \ldots, t_p$ is a linear combination of $u_1, u_2, \ldots, u_n$. In particular, each of $t_1, t_2, \ldots, t_p$ is a linear combination of $u_1, u_2, \ldots, u_n$.

8. **Illustrations of Theorem (K).**

(a) $$\mathsf{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix},\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}2\\6\\10\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right)=\mathsf{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}\right\}\right).$$

Reason: Each of $\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}2\\6\\10\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}$.

Below is the detail:

$$\begin{bmatrix}3\\7\\11\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}+\begin{bmatrix}2\\4\\6\end{bmatrix},\qquad\begin{bmatrix}2\\6\\10\end{bmatrix}=2\begin{bmatrix}1\\3\\5\end{bmatrix},\qquad\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\4\\6\end{bmatrix}-\begin{bmatrix}1\\3\\5\end{bmatrix}.$$

(b) $\mathsf{Span}\left(\left\{\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}\right\}\right)=\mathsf{Span}\left(\left\{\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}\right)$.

Reason: Each of $\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}$.

Below is the detail:

$$\begin{bmatrix}3\\7\\11\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}+\begin{bmatrix}2\\4\\6\end{bmatrix},\qquad \begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\4\\6\end{bmatrix}-\begin{bmatrix}1\\3\\5\end{bmatrix}.$$

Also, each of $\begin{bmatrix}1\\3\\5\end{bmatrix},\begin{bmatrix}2\\4\\6\end{bmatrix}$ is a linear combination of $\begin{bmatrix}3\\7\\11\end{bmatrix},\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Below is the detail:

$$\begin{bmatrix}1\\3\\5\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3\\7\\11\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix},\qquad \begin{bmatrix}2\\4\\6\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3\\7\\11\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}.$$

9. We now state a pair of results (Lemma (4), Lemma (5)) describing whether square-matrix multiplication from the left to matrices 'preserves' equality relations between column spaces (though not necessarily the individual column spaces).

Lemma (4).

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.

Suppose

- *A is a $(m \times m)$-square matrix, and*
- *$\mathcal{C}(H) = \mathcal{C}(G)$.*

Then $\mathcal{C}(AH) = \mathcal{C}(AG)$.

Lemma (5). (A 'partial converse' of Lemma (5).)

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.

Suppose

- *A is a non-singular $(m \times m)$-square matrix, and*
- *$\mathcal{C}(AH) = \mathcal{C}(AG)$.*

Then $\mathcal{C}(H) = \mathcal{C}(G)$.

10. We combine Lemma (4) and Lemma (5) to obtain Theorem (L) below:

Theorem (L).

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.
Suppose A is a non-singular $(m \times m)$-square matrix.

Then the statements below are logically equivalent:

(a) $\mathcal{C}(H) = \mathcal{C}(G)$.

(b) $\mathcal{C}(AH) = \mathcal{C}(AG)$.

Remark.

In plain words, this result is saying that

the equality between column spaces of matrices (though not necessarily the individual matrices themselves) are preserved upon

the multiplication by the same non-singular matrix from the left to the matrices.

When we think in terms of row operations, this result is saying that

the equality between column spaces of matrices (though not necessarily the individual matrices themselves) are preserved upon

the application of the same sequence of row operations to the matrices.

Interpretation of the content of Theorem (L) in terms of row operations

equality between column spaces of two matrices with the same number of rows:

$$\mathcal{C}(H) = \mathcal{C}(G)$$

Same sequence of row operations applied to individual matrices which give the respective column spaces:

$$\mathcal{C}(H) \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} \mathcal{C}(H') \qquad \mathcal{C}(G) \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} \mathcal{C}(G')$$

column spaces of the resultant matrices:

$$\mathcal{C}(H') \qquad \mathcal{C}(G')$$

Ask: How are these two collections of vectors related?

Answer.

$$\mathcal{C}(H') = \mathcal{C}(G')$$

11. **Proof of Lemma (4).**

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.
Suppose A is a $(m \times m)$-square matrix, and $\mathcal{C}(H) = \mathcal{C}(G)$.

[We are going to verify the set equality $\mathcal{C}(AH) = \mathcal{C}(AG)$.

This amount to deducing (with the assumption stated earlier) that both (†), (‡) are true:

(†) For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathcal{C}(AG)$ then $\mathbf{y} \in \mathcal{C}(AH)$.

(‡) For any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathcal{C}(AH)$ then $\mathbf{y} \in \mathcal{C}(AG)$.

We will the arguments in two separate passages, one for each of (†), (‡).]

- [Here we verify (†).]
  Suppose $\mathbf{y} \in \mathcal{C}(AH)$.
  Then, by the definition of $\mathcal{C}(AH)$, there exist some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = AH\mathbf{x}$.

  [Reminder: We want to deduce $\mathbf{y} \in \mathcal{C}(AG)$.
  So we ask whether we can conceive some appropriate $\mathbf{w} \in \mathbb{R}^k$ which satisfies $\mathbf{y} = (AG)\mathbf{w}$.
  How to conceive such a $\mathbf{w}$?
  Compare the equality '$\mathbf{y} = AH\mathbf{x}$' which we have already known to be true, with the desired equality '$\mathbf{y} = AG\mathbf{w}$', which we hope to be true.
  This suggests we ask if there is indeed some $\mathbf{w} \in \mathbb{R}^k$ which satisfies $H\mathbf{x} = G\mathbf{w}$.
  It turns out that the answer is *yes*.]

  By the definition of $\mathcal{C}(H)$, $H\mathbf{x} \in \mathcal{C}(H)$.
  Then by assumption $H\mathbf{x} \in \mathcal{C}(G)$.
  Then, by the definition of $\mathcal{C}(G)$, there exists some $\mathbf{w} \in \mathbb{R}^k$ such that $H\mathbf{x} = G\mathbf{w}$.
  Now $\mathbf{y} = AH\mathbf{x} = AG\mathbf{w}$.
  Then, by the definition of $\mathcal{C}(AG)$, we have $\mathbf{y} \in \mathcal{C}(AG)$.

- By modifying the above argument (through changing the symbols appropriately), we also deduce that for any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathcal{C}(AG)$ then $\mathbf{y} \in \mathcal{C}(AH)$.

Here we are working under the standing assumption '$\mathcal{C}(H) = \mathcal{C}(G)$'.

This reads: 'For any $y \in \mathbb{R}^m$, if $y \in \mathcal{C}(AH)$ then $y \in \mathcal{C}(AG)$.'

- [Here we verify (†).]

  Suppose $\mathbf{y} \in \mathcal{C}(AH)$. [Ask: Is it true that $y \in \mathcal{C}(AG)$? Any clue anywhere?]

  Then, by the definition of $\mathcal{C}(AH)$, there exist some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = AH\mathbf{x}$.

  [Reminder: We want to deduce $\mathbf{y} \in \mathcal{C}(AG)$. ← What does this amount to? 'There exists some $w \in \mathbb{R}^k$ such that $y = (AG)w$.'

  So we ask whether we can conceive some appropriate $\mathbf{w} \in \mathbb{R}^k$ which satisfies $\mathbf{y} = (AG)\mathbf{w}$.

  How to conceive such a $\mathbf{w}$?

  Compare the equality '$\mathbf{y} = AH\mathbf{x}$' which we have already known to be true, with the desired equality '$\mathbf{y} = AG\mathbf{w}$', which we hope to be true.

  This suggests we ask if there is indeed some $\mathbf{w} \in \mathbb{R}^k$ which satisfies $H\mathbf{x} = G\mathbf{w}$.

  It turns out that the answer is *yes*.]

  By the definition of $\mathcal{C}(H)$, $H\mathbf{x} \in \mathcal{C}(H)$. [Recall we have assumed $\mathcal{C}(H) = \mathcal{C}(G)$. What does it tell us about Hx now?]

  Then by assumption $H\mathbf{x} \in \mathcal{C}(G)$.

  Then, by the definition of $\mathcal{C}(G)$, there exists some $\mathbf{w} \in \mathbb{R}^k$ such that $H\mathbf{x} = G\mathbf{w}$.

  Now $\mathbf{y} = AH\mathbf{x} = AG\mathbf{w}$.

  Then, by the definition of $\mathcal{C}(AG)$, we have $\mathbf{y} \in \mathcal{C}(AG)$.

- By modifying the above argument (through changing the symbols appropriately), we also deduce that for any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in \mathcal{C}(AG)$ then $\mathbf{y} \in \mathcal{C}(AH)$. (‡)

12. **Proof of Lemma (5).**

[We are going to make a clever application of Lemma (4) so that we don't have to prove a set equality with direct reference to the definition of set equalities.]

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.
Suppose A is a non-singular $(m \times m)$-square matrix, and $\mathcal{C}(AH) = \mathcal{C}(AG)$.

By assumption, A has a matrix inverse, namely the $(m \times m)$-square matrix A^{-1}.

We have $H = A^{-1}(AH)$.

Then $\mathcal{C}(H) = \mathcal{C}\big(A^{-1}(AH)\big)$

We also have $G = A^{-1}(AG)$.

Then $\mathcal{C}(G) = \mathcal{C}\big(A^{-1}(AG)\big)$.

By assumption, $\mathcal{C}(AH) = \mathcal{C}(AG)$.

Then, by Lemma (4), we have $\mathcal{C}\big(A^{-1}(AH)\big) = \mathcal{C}\big(A^{-1}(AH)\big)$.

Therefore $\mathcal{C}(H) = \mathcal{C}\big(A^{-1}(AH)\big) = \mathcal{C}\big(A^{-1}(AG)\big) = \mathcal{C}(G)$.

13. Under the 'dictionary' between the notion of *column space* and *span*, Lemma (4), Lemma (5) and Theorem (L) respectively translate into Lemma (4'), Lemma (5') and Theorem (L') below.

Lemma (4').

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$*.*

Suppose

- *A is a $(m \times m)$-square matrix, and*
- $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

Then $\mathsf{Span}\,(\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\,(\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

Lemma (5'). (A 'partial converse' of Lemma (4').)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$*.*

Suppose

- *A is a non-singular $(m \times m)$-square matrix, and*
- $\mathsf{Span}\,(\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\,(\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

Then $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

Theorem (L'). (Re-formulation of Theorem (L) under the 'dictionary' between span and column space.)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

Suppose A *is a non-singular* $(m \times m)$*-square matrix.*

Then the statements below are logically equivalent:

(a) $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

(b) $\mathsf{Span}\,(\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\,(\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

Remark.

In plain words, Theorem (L') says that

> *the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the multiplication by the same non-singular matrix from the left to the vectors.*

When we think in terms of row operations, Theorem (L') says that

> *the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the application of the same sequence of row operations to the vectors.*

Theorem (L'). (Re-formulation of Theorem (L) under the 'dictionary' between span and column space.)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$.

Suppose A *is a non-singular* $(m \times m)$*-square matrix.*

Then the statements below are logically equivalent:

(a) $\mathsf{Span}\,(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\,(\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

(b) $\mathsf{Span}\,(\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\,(\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

How does Theorem (L') relate to Theorem (L)?
Write
$H = [\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]$,
$G = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_k]$.
Then:
$AH = [A\mathbf{u}_1|A\mathbf{u}_2|\cdots|A\mathbf{u}_n]$,
$AG = [A\mathbf{v}_1|A\mathbf{v}_2|\cdots|A\mathbf{v}_k]$.
Item (a) translates into: $\mathcal{C}(H) = \mathcal{C}(G)$
Item (b) translates into: $\mathcal{C}(AH) = \mathcal{C}(AG)$

Remark.

In plain words, Theorem (L') says that

> *the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the multiplication by the same non-singular matrix from the left to the vectors.*

When we think in terms of row operations, Theorem (L') says that

> *the equality between spans of vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the application of the same sequence of row operations to the vectors.*

Interpretation of the content of Theorem (L′) in terms of row operations.

equality between spans of two collections of vectors

$$\mathrm{Span}(\{u_1, u_2, \dots, u_n\}) = \mathrm{Span}(\{v_1, v_2, \dots, v_k\})$$

same sequence of row operations applied to individual vectors which give the respective spans

$$\begin{array}{ccccccc} u_1 & u_2 & \cdots & u_n & v_1 & v_2 & \cdots & v_k \\ \downarrow \rho_1 & \downarrow \rho_1 & & \downarrow \rho_1 & \downarrow \rho_1 & \downarrow \rho_1 & & \downarrow \rho_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \downarrow \rho_s & \downarrow \rho_s & & \downarrow \rho_s & \downarrow \rho_s & \downarrow \rho_s & & \downarrow \rho_s \end{array}$$

spans of the resultant vectors

$$\mathrm{Span}(\{u_1', u_2', \dots, u_n'\}) \qquad \mathrm{Span}(\{v_1', v_2', \dots, v_k'\})$$

Ask: How are these two collections of vectors related?

Answer.

$$\mathrm{Span}(\{u_1', u_2', \dots, u_n'\}) = \mathrm{Span}(\{v_1', v_2', \dots, v_k'\})$$

14. **Theorem (6). (Generalization of Lemma (4) and Lemma (5).)**

Let H be an $(m \times n)$-matrix, and G be an $(m \times k)$-matrix.

Let A be a $(p \times m)$-matrix.

(a) *Suppose* $\mathcal{C}(H) = \mathcal{C}(G)$.

Then $\mathcal{C}(AH) = \mathcal{C}(AG)$.

(b) *Suppose* $\mathcal{N}(A) = \{\mathbf{0}\}$, *and* $\mathcal{C}(AH) = \mathcal{C}(AG)$.

Then $\mathcal{C}(AH) = \mathcal{C}(AG)$.

Theorem (6'). (Generalization of Lemma (4') and Lemma (5').)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ *be vectors in* $\mathbb{R}^m$. *Let A be a $(p \times m)$-matrix.*

(a) *Suppose* $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\ (\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.

Then $\mathsf{Span}\ (\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\ (\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

(b) *Suppose* $\mathcal{N}(A) = \{\mathbf{0}\}$, *and* $\mathsf{Span}\ (\{A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n\}) = \mathsf{Span}\ (\{A\mathbf{v}_1, A\mathbf{v}_2, \cdots, A\mathbf{v}_k\})$.

Then $\mathsf{Span}\ (\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = \mathsf{Span}\ (\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\})$.