1. Recall the definition for the respective notion of *linear dependence.*

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.
We say that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent if and only if the statement* (LD) *holds:*

(LD) *There exist some* $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ *such that*

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

and $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are not all zero.*

The equality $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ *in which* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *are not all zero is called a non-trivial linear relation of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

2. The notions of linear dependence and linear combinations are linked up in the result below:

Theorem (H).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.
The statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

(b) *At least one vector amongst* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of the others.*

1. Recall the definition for the respective notion of *linear dependence.*

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

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The statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly dependent.*

There exist some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero.

(b) *At least one vector amongst* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of the others.*

Some u_j can be 'expresed' as $\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ in which u_j does not appear explicitly.

3. **Proof of Theorem (H).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

  Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero.

  Without loss of generality, assume $\alpha_1 \neq 0$.

  Then $\mathbf{u}_1 = (-\dfrac{\alpha_2}{\alpha_1})\mathbf{u}_2 + (-\dfrac{\alpha_3}{\alpha_1})\mathbf{u}_3 + \cdots + (-\dfrac{\alpha_n}{\alpha_1})\mathbf{u}_n$.

  Hence $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$.

- Suppose at least one vector amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of the others.

  Without loss of generality, assume $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$.

  Then there exist some $\beta_2, \beta_3, \cdots, \beta_n \in \mathbb{R}$ such that $\mathbf{u}_1 = \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3 + \cdots + \beta_n\mathbf{u}_n$.

  Therefore $1 \cdot \mathbf{u}_1 + (-\beta_2)\mathbf{u}_2 + (-\beta_3)\mathbf{u}_3 + \cdots + (-\beta_n)\mathbf{u}_n = \mathbf{0}$.

  This is a non-trivial linear relation of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

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Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

  Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not all zero.

  [Ask: Can some u_j be 'expressed' as $\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ in which u_j does not appear explicitly?]

  Without loss of generality, assume $\alpha_1 \neq 0$.

  Then $\mathbf{u}_1 = (-\frac{\alpha_2}{\alpha_1})\mathbf{u}_2 + (-\frac{\alpha_3}{\alpha_1})\mathbf{u}_3 + \cdots + (-\frac{\alpha_n}{\alpha_1})\mathbf{u}_n$.

  Hence $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$.

- Suppose at least one vector amongst $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ is a linear combination of the others.

  Without loss of generality, assume $\mathbf{u}_1$ is a linear combination of $\mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n$.

  Then there exist some $\beta_2, \beta_3, \cdots, \beta_n \in \mathbb{R}$ such that $\mathbf{u}_1 = \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3 + \cdots + \beta_n\mathbf{u}_n$.

  Therefore $1 \cdot \mathbf{u}_1 + (-\beta_2)\mathbf{u}_2 + (-\beta_3)\mathbf{u}_3 + \cdots + (-\beta_n)\mathbf{u}_n = \mathbf{0}$.

  This is a non-trivial linear relation of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly dependent.

  [Ask: Can we write down some equality of the form $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$ in which $\alpha_1, \ldots, \alpha_n$ are not all zero?]

4. **Illustration of the idea in Theorem (H).**

Write $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

We have

$$1 \cdot \mathbf{u}_1 + (-1)\mathbf{u}_2 + 2\mathbf{u}_3 + (-2)\mathbf{u}_4 + 3\mathbf{u}_5 = \mathbf{0}_5.$$

We also have

$$\begin{aligned}
\mathbf{u}_1 &= 1 \cdot \mathbf{u}_2 - 2\mathbf{u}_3 + 2\mathbf{u}_4 - 3\mathbf{u}_5 \\
\mathbf{u}_2 &= 1 \cdot \mathbf{u}_1 + 2\mathbf{u}_3 + (-2)\mathbf{u}_4 + 3\mathbf{u}_5 \\
\mathbf{u}_3 &= -\frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 + 1 \cdot \mathbf{u}_4 - \frac{3}{2}\mathbf{u}_5 \\
\mathbf{u}_4 &= \frac{1}{2}\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2 + 1 \cdot \mathbf{u}_3 + \frac{3}{2}\mathbf{u}_5 \\
\mathbf{u}_5 &= -\frac{1}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 - \frac{2}{3}\mathbf{u}_3 + \frac{2}{3}\mathbf{u}_4
\end{aligned}$$

Theorem (H) says that each of these six equalities is valid exactly because of the validity of each other.

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We have

$$1 \cdot \mathbf{u}_1 + (-1)\mathbf{u}_2 + 2\mathbf{u}_3 + (-2)\mathbf{u}_4 + 3\mathbf{u}_5 = \mathbf{0}_5.$$

[So u_1, u_2, u_3, u_4, u_5 are linearly dependent.]

We also have

$$\begin{aligned}
\mathbf{u}_1 &= 1 \cdot \mathbf{u}_2 - 2\mathbf{u}_3 + 2\mathbf{u}_4 - 3\mathbf{u}_5 \\
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\end{aligned}$$

[Some u_j amongst u_1, u_2, u_3, u_4, u_5 can be expressed as $\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5$ in which u_j does not appear explicitly.]

Theorem (H) says that each of these six equalities is valid exactly because of the validity of each other.

5. By logic, Corollary (1) to Theorem (H) holds as a re-formulation of Theorem (H).

The notion of linear independence can be understood through this re-formulation of Theorem (H): it corresponds to our heuristic understanding of the word *independence* in daily language.

Corollary (1) to Theorem (H).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$. *The statements below are logically equivalent:*

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

(b) *None of the vector amongst* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of the others.*

6. Corollary (2) to Theorem (H) is an immediate consequence of Theorem (H).

Corollary (2) to Theorem (H).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$.
Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *are linearly dependent.*

(b) $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

7. **Proof of Corollary (2) to Theorem (H).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$. Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.

- Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ are linearly dependent.

  Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta \in \mathbb{R}$ such that $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\cdots+\alpha_n\mathbf{u}_n+\beta\mathbf{v} = \mathbf{0}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta$ are not all zero.

  We verify that $\beta \neq 0$:

  * Suppose it were true that $\beta = 0$.
  Then $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ were not all zero.
  Therefore $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ would be linearly dependent. Contradiction arises.

  As $\beta \neq 0$, we have $\mathbf{v} = (-\frac{\alpha_1}{\beta})\mathbf{u}_1 + (-\frac{\alpha_2}{\beta})\mathbf{u}_2 + \cdots + (-\frac{\alpha_n}{\beta})\mathbf{u}_n + \beta\mathbf{v}$.

  Therefore $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

- Suppose $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

  Then there exist some $\gamma_1, \gamma_2, \cdots, \gamma_n \in \mathbb{R}$ such that $\mathbf{v} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_n\mathbf{u}_n$.

  Therefore $\gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_n\mathbf{u}_n - 1 \cdot \mathbf{v} = \mathbf{0}$.

  Hence $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ are linearly dependent.

8. By logic, Corollary (3) to Theorem (H) holds as a re-formulation of Corollary (2) to Theorem (H).

Corollary (3) to Theorem (H).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$.
Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *are linearly independent.*

(b) $\mathbf{v}$ *is not a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

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Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *are linearly independent.*

(b) $\mathbf{v}$ *is not a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Interpretation of this result in plain words:

When $u_1, u_2, \ldots, u_n$ are constituting a collection of linearly independent vectors, such a collection can be 'expanded' into a 'larger' collection of linearly independent vectors by the inclusion of an extra vector v exactly when the extra vector v is not a linear combination of $u_1, u_2, \ldots, u_n$.

9. We are going to give a useful re-formulations for the notion of *linear independence* which is useful in the study of theoretical questions.

We start by recalling the re-formulations for the notion of *linearly independence* below:

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.
Define the $(m \times n)$*-matrix* U *by* $U = \left[\, \mathbf{u}_1 \,\middle|\, \mathbf{u}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{u}_n \,\right]$.

The statements below are logically equivalent:

($\sim\diamondsuit$) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

($\sim\clubsuit$) *The only solution of homogeneous system of linear equations* $\mathcal{LS}(U, \mathbf{0}_m)$ *is the trivial solution.*

($\sim\heartsuit$) $\mathcal{N}(U) = \{\mathbf{0}_n\}$.

($\sim\spadesuit$) *For any* $\mathbf{t} \in \mathbb{R}^n$, *if* $\mathbf{t} \in \mathcal{N}(U)$ *then* $\mathbf{t} = \mathbf{0}_n$.

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This is the same as $Ut = 0_m$.

Overall, this translates (with the help of the 'dictionary' Lemma (A)) into:

For any $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$, if $\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n = 0_m$ then ($\beta_1 = 0$ and $\beta_2 = 0$ and ... and $\beta_n = 0$).

With the help of the

'dictionary' between linear combinations and matrix-vector products

applied on the statement (∼♠), we obtain another re-formulation, in the form of Lemma (I), for the notion of linear independence, which is useful for theoretical discussions (or in calculations in which the vectors involved are not given in 'concrete' terms).

Lemma (I).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

The statements below are logically equivalent to each other:

(LI_0) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

(LI) *For any* $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{R}$, *if*

$$\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n = \mathbf{0}_m$$

then

$$\beta_1 = \beta_2 = \cdots = \beta_n = 0.$$

Remark.

In fact, in many standard textbooks, people simply refers to the statement (LI) as the definition for '$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent vectors in $\mathbb{R}^m$'.

10. **Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be vectors in $\mathbb{R}^m$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.

Then for each $j = 1, 2, \cdots, k$, the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_j$ are linearly independent.

Remark.

In plain words, Theorem (1) says:

> *Any 'portion' of a collection of linearly independent vectors is itself a collection of linearly independent vectors.*

11. **Proof of Theorem (1).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be vectors in $\mathbb{R}^m$.

Suppose $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent.

Suppose $j = 1, 2, \cdots, k$. Pick any $\alpha_1, \alpha_2, \cdots, \alpha_j \in \mathbb{R}$.

Suppose $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_j\mathbf{u}_j = \mathbf{0}$.

Then $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_j\mathbf{u}_j + 0 \cdot \mathbf{u}_{j+1} + 0 \cdot \mathbf{u}_{j+2} + \cdots + 0 \cdot \mathbf{u}_k = \mathbf{0}$.

By assumption $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are linearly independent. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_j = 0$.

It follows that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_j$ are linearly independent.

12. We can re-phrase Theorem (1) at a 'cosmetic level' so as to obtain the statement ($\star$) below:

($\star$) *Let* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ *be vectors in* $\mathbb{R}^m$*.*

Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ *are linearly independent.*

Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *are linearly independent.*

The statement ($\star$) is logically equivalent to Theorem (2), which is called the

'contra-positive re-formulation' of ($\star$).

Theorem (2).
Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ *be vectors in* $\mathbb{R}^m$*.*

Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ *are linearly dependent.*

Then $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n, \mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_p$ *are linearly dependent.*

Remark.
In plain words, Theorem (2) says:

A collection of vectors is definitely linearly dependent when soem portion of it is, on its own, a collection of linearly dependent vectors.

13. We now state a pair of results (Lemma (3), Lemma (4)) describing whether square-matrix multiplication from the left to vectors 'preserves' the linear independence of a given collection of vectors.

Lemma (3).
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Suppose

- A *is an* $(m \times m)$*-square matrix, and*
- $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *are linearly independent.*

Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

Lemma (4). (A 'partial converse' of Lemma (3).)
Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Suppose

- A *is a non-singular* $(m \times m)$*-square matrix, and*
- $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

Then $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *are linearly independent.*

14. We combine Lemma (3) and Lemma (4) to obtain Theorem (J) below:

Theorem (J).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Suppose A *is a non-singular* $(m \times m)$*-square matrix.*

Then the statements below are logically equivalent:

(a) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

(b) $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *are linearly independent.*

Remark.

In plain words, Theorem (J) says that

linear independence for a collection of vectors (though not necessarily the individual vectors themselves) are preserved upon

the multiplication by the same non-singular matrix from the left to the vectors.

When we think in terms of row operations, Theorem (J) says that

linear independence for a collection of vectors (though not necessarily the individual vectors themselves) are preserved upon

the application of the same sequence of row operations to the vectors.

Interpretation of the content of Theorem (J) in terms of row operations

① Linearly independent vectors: $u_1, u_2, \ldots, u_n$

Same sequence of row operations: $\rho_1, \ldots, \rho_s$

$u_1', u_2', \ldots, u_n'$

Ask: Are they linearly independent?

→ Answer. $u_1', u_2', \ldots, u_n'$ are linearly independent.

② Linearly dependent vectors:

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$$

Non-trivial linear relation.

Same sequence of row operations: $\rho_1, \ldots, \rho_s$

$u_1', u_2', \ldots, u_n'$

Ask: Are they linearly dependent?

→ Answer. $u_1', u_2', \ldots, u_n'$ are linearly dependent. Moreover, the 'form' of the non-trivial linear relation is 'carried':

$$\alpha_1 u_1' + \alpha_2 u_2' + \ldots + \alpha_n u_n' = 0$$

15. **Proof of Lemma (3).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

Suppose A is an $(m \times m)$-square matrix, and $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ are linearly independent.

[We want to verify that $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.
This amounts to verifying the statement 'for any $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{R}$, if $\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n = \mathbf{0}_m$ then $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.']

Pick any $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{R}$.

Suppose $\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n = \mathbf{0}_m$.

[Ask: Is it true that $\beta_1 = \beta_2 = \cdots = \beta_n = 0$?]

Then

$$\begin{aligned} \beta_1 A\mathbf{u}_1 + \beta A\mathbf{u}_2 + \cdots + \beta_n A\mathbf{u}_n &= A(\beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n) \\ &= A\mathbf{0}_m = \mathbf{0}_m \end{aligned}$$

Then, by assumption, $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

Therefore $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ linearly independent.

16. **Proof of Lemma (4).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$. Suppose A is a non-singular $(m \times m)$-square matrix, and $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.

[We want to verify that $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ are linearly independent.
This amounts to verifying the statement 'for any $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{R}$, if $\beta_1 A\mathbf{u}_1 + \beta_2 A\mathbf{u}_2 + \cdots + \beta_n A\mathbf{u}_n = \mathbf{0}_m$ then $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.']

Pick any $\beta_1, \beta_2, \cdots, \beta_n \in \mathbb{R}$. Suppose $\beta_1 A\mathbf{u}_1 + \beta_2 A\mathbf{u}_2 + \cdots + \beta_n A\mathbf{u}_n = \mathbf{0}_m$.

[Ask: Is it true that $\beta_1 = \beta_2 = \cdots = \beta_n = 0$?]

Then

$$\begin{aligned}
\beta_1\mathbf{u}_1 + \beta\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n &= \beta_1 I_m\mathbf{u}_1 + \beta I_m\mathbf{u}_2 + \cdots + \beta_n I_m\mathbf{u}_n \\
&= \beta_1(A^{-1}A)\mathbf{u}_1 + \beta(A^{-1}A)\mathbf{u}_2 + \cdots + \beta_n(A^{-1}A)\mathbf{u}_n \\
&= \beta_1 A^{-1}(A\mathbf{u}_1) + \beta A^{-1}(A\mathbf{u}_2) + \cdots + \beta_n A^{-1}(A\mathbf{u}_n) \\
&= A^{-1}(\beta_1 A\mathbf{u}_1 + \beta_2 A\mathbf{u}_2 + \cdots + \beta_n A\mathbf{u}_n) = A^{-1}\mathbf{0}_m = \mathbf{0}_m
\end{aligned}$$

By assumption, $\beta_1 = \beta_2 = \cdots = \beta_n = 0$.

Therefore $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ linearly independent.

17. **Theorem (5). (Generalization of Lemma (3) and Lemma (4).)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$. *Let* A *be a* $(p \times m)$*-matrix.*

(a) *Suppose* $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$.
Then $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*

(b) *Suppose* $\mathcal{N}(A) = \{\mathbf{0}\}$, *and* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *are linearly independent.*
Then $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *are linearly independent.*

Proof of Theorem (5). Exercise.