MATH1030 Further examples on linear dependence and linear independence.

1. Consider the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots$ below. Determine whether $\mathbf{u}_1, \mathbf{u}_2, \cdots$ are linearly dependent or not. Where they are linearly dependent, also give a non-trivial linear relation for them.

   (a) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

   (b) $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}$.

   (c) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 2 \end{bmatrix}$.

   (d) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

   (e) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$.

   (f) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$.

   (g) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

   (h) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -5 \\ -4 \\ 0 \end{bmatrix}$.

   $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   (i) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   (j) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   (k) $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Answer.

1. (a) Linearly independent.
   (b) Linearly independent.
   (c) Linearly independent.
   (d) Linearly independent.
   (e) Linearly dependent.
   $-3\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$.
   (There are infinitely many non-trivial linear relations.)
   (f) Linearly independent.
   (g) Linearly independent.
   (h) Linearly dependent.
   $2\mathbf{u}_1 - \mathbf{u}_2 + \mathbf{u}_3 = \mathbf{0}$.
   (There are infinitely many non-trivial linear relations.)
   (i) Linearly independent.
   (j) Linearly independent.
   (k) Linearly dependent.
   $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}$.
   (There are infinitely many non-trivial linear relations.)
   (l) Linearly independent.