1. **Definition. (Linear Combination.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$. *Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^m$.

We say $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *if the statement* $(\dagger)$ *holds:*

$(\dagger)$ *There exist some real numbers* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *such that*

$$\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n.$$

The expression $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$ *on its own is called the linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *with respect to the scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

2. **Lemma (A). ('Dictionary' between linear combinations and matrix-vector products.)**

Let A *be an* $(m \times n)$*-matrix, and* $\mathbf{t}$ *be a vector in* $\mathbb{R}^n$.

Suppose that for each $j = 1, 2, \cdots, n$, *the* j*-th column of* A *is* $\mathbf{a}_j$ *and the* j*-th entry of* $\mathbf{t}$ *is* t_j. *(So* $A = \left[\ \mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{a}_n\ \right]$ *and* $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$.*)*

Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$.

1. **Definition. (Linear Combination.)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$*. Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^m$*.*

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Then $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$*.*

This 'dictionary' has been used without our having made it explicit:
For each vector $b \in \mathbb{R}^m$, the system $\mathcal{LS}(A,b)$ may be presented as:

$$Ax = b$$

Or equivalently:

$$x_1 a_1 + x_2 a_2 + \ldots + x_n a_n = b$$

3. **Proof of Lemma (A).**

For each i, j, we denote the (i, j)-th entry of A by a_{ij}.

- The i-th entry of $A\mathbf{t}$ is given by

$$\sum_{j=1}^{n} a_{ij}t_j = t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in}.$$

- For each j, the i-th entry of $\mathbf{a}_j$ (which is the j-th column of A) is a_{ij}.

  Then the i-th entry of $t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ is $t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in}$.

The corresponding entries of

$$A\mathbf{t}, \qquad\qquad t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$$

agree with each other.

Hence $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ indeed.

Remark.

Lemma (A) looks innocent, but it will serve as a useful tool in various situations.

3. Proof of Lemma (A).

For each i, j, we denote the (i, j)-th entry of A by a_{ij}.

We are going to compare the respective entries of the column vectors At, $t_1a_1 + t_2a_2 + \ldots + t_na_n$.

- The i-th entry of $A\mathbf{t}$ is given by

$$\sum_{j=1}^{n} a_{ij}t_j = t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in}.$$

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Hence $A\mathbf{t} = t_1\mathbf{a}_1 + t_2\mathbf{a}_2 + \cdots + t_n\mathbf{a}_n$ indeed.

Remark.

Lemma (A) looks innocent, but it will serve as a useful tool in various situations.

4. **Simple concrete examples.**

(a) $$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}.$$

So $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ is the linear combination of $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ with respect to the scalars $1, 2, 3, 4, 5$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{array}\right]\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}.$$

4. **Simple concrete examples.**

(a) $$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \underset{t_1}{1} \cdot \underset{a_1}{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}} + \underset{t_2}{2} \underset{a_2}{\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}} + \underset{t_3}{3} \underset{a_3}{\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}} + \underset{t_4}{4} \underset{a_4}{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}} + \underset{t_5}{5} \underset{a_5}{\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}}.$$

So $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ is the linear combination of $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ with respect to the scalars $1, 2, 3, 4, 5$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \underset{\underbrace{[a_1|a_2|a_3|a_4|a_5]}_{\text{This is } A.}}{\left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{array}\right]} \underset{\underbrace{\begin{bmatrix}t_1\\t_2\\t_3\\t_4\\t_5\end{bmatrix}}_{\text{This is } t.}}{\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}}.$$

(b) $$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = 1\cdot\begin{bmatrix}1\\3\\5\\7\\9\end{bmatrix} + (-1)\begin{bmatrix}0\\2\\4\\6\\8\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\3\\5\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\0\\3\\6\end{bmatrix} + 0\cdot\begin{bmatrix}0\\0\\0\\0\\2\end{bmatrix}.$$

So $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ is also a linear combination of $\begin{bmatrix}1\\3\\5\\7\\9\end{bmatrix}, \begin{bmatrix}0\\2\\4\\6\\8\end{bmatrix}, \begin{bmatrix}0\\0\\1\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\\3\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\2\end{bmatrix}$ with respect to the scalars $1, -1, 2, -1, 0$.

A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\3&2&0&0&0\\5&4&1&0&0\\7&6&3&3&0\\9&8&5&6&2\end{array}\right]\begin{bmatrix}1\\-1\\2\\-1\\0\end{bmatrix}.$$

$$\text{(b)}\ \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \underset{t_1}{1}\cdot\underset{a_1}{\begin{bmatrix}1\\3\\5\\7\\9\end{bmatrix}} + \underset{t_2}{(-1)}\underset{a_2}{\begin{bmatrix}0\\2\\4\\6\\8\end{bmatrix}} + \underset{t_3}{2}\underset{a_3}{\begin{bmatrix}0\\0\\1\\3\\5\end{bmatrix}} + \underset{t_4}{(-1)}\underset{a_4}{\begin{bmatrix}0\\0\\0\\3\\6\end{bmatrix}} + \underset{t_5}{0}\cdot\underset{a_5}{\begin{bmatrix}0\\0\\0\\0\\2\end{bmatrix}}.$$

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A manifestation of the same relation is the equality below:

$$\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \underset{\underbrace{[a_1|a_2|a_3|a_4|a_5]}_{\text{This is A.}}}{\left[\begin{array}{c|c|c|c|c}1&0&0&0&0\\3&2&0&0&0\\5&4&1&0&0\\7&6&3&3&0\\9&8&5&6&2\end{array}\right]} \underset{\underbrace{\begin{bmatrix}t_1\\t_2\\t_3\\t_4\\t_5\end{bmatrix}}_{\text{This is t.}}}{\begin{bmatrix}1\\-1\\2\\-1\\0\end{bmatrix}}.$$

5. Theorem (1).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

The statements below are true:

(a) *The zero vector* $\mathbf{0}$ *in* $\mathbb{R}^m$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) *The sum of any two linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(c) *Every scalar multiple of any linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

6. Proof of Theorem (1).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

(a) [Ask: Can we name some appropriate real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which the equality

$$\mathbf{0} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$$

holds?]

We have $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_n$.

Then by definition, $\mathbf{0}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

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[This is the same as:
For any $v, w \in \mathbb{R}^m$, if v, w are linear combinations of $u_1, u_2, \ldots, u_n$ then $v+w$ is a linear combination of $u_1, u_2, \cdots, u_n$.]

(c) *Every scalar multiple of any linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

[This is the same as:
For any $v \in \mathbb{R}^m$, for any $\alpha \in \mathbb{R}$, if v is a linear combination of $u_1, u_2, \ldots, u_n$ then αv is a linear combination of $u_1, u_2, \ldots, u_n$.]

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Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$.

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We have $\mathbf{0} = 0 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2 + \cdots + 0 \cdot \mathbf{u}_n$.

Then by definition, $\mathbf{0}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(b) Suppose $\mathbf{v}, \mathbf{w}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Then, by definition, there exist some real numbers $\beta_1, \beta_2, \cdots, \beta_n$ such that

$$\mathbf{v} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n.$$

Also, there exist some real numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$ such that

$$\mathbf{w} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_n\mathbf{u}_n.$$

[Ask: Can we name some appropriate real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which the equality

$$\mathbf{v} + \mathbf{w} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$$

holds?]

Note that

$$\mathbf{v} + \mathbf{w} = ... = (\beta_1 + \gamma_1)\mathbf{u}_1 + (\beta_2 + \gamma_2)\mathbf{u}_2 + \cdots + (\beta_n + \gamma_n)\mathbf{u}_n,$$

and $\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n$ are real numbers.

Then by definition, $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(c) Exercise.

[Ask: Is $v+w$ a linear combination of $u_1, u_2, \ldots, u_n$?]

(b) Suppose $\mathbf{v}, \mathbf{w}$ are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(#)~ Then, by definition, there exist some real numbers $\beta_1, \beta_2, \cdots, \beta_n$ such that

$$\mathbf{v} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_n\mathbf{u}_n.$$

Also, there exist some real numbers $\gamma_1, \gamma_2, \cdots, \gamma_n$ such that

$$\mathbf{w} = \gamma_1\mathbf{u}_1 + \gamma_2\mathbf{u}_2 + \cdots + \gamma_n\mathbf{u}_n.$$

(♮)~ [Ask: Can we name some appropriate real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ for which the equality

$$\mathbf{v} + \mathbf{w} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n$$

holds?]

(♭)~ Note that

$$\mathbf{v} + \mathbf{w} = \ldots = (\beta_1 + \gamma_1)\mathbf{u}_1 + (\beta_2 + \gamma_2)\mathbf{u}_2 + \cdots + (\beta_n + \gamma_n)\mathbf{u}_n,$$

and $\beta_1 + \gamma_1, \beta_2 + \gamma_2, \cdots, \beta_n + \gamma_n$ are real numbers.

Then by definition, $\mathbf{v} + \mathbf{w}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

(c) Exercise. Start the argument in this way:
'Suppose v is a linear combination of $u_1, u_2, \ldots, u_n$. Suppose α is a real number.'
Then ask: Is αv a linear combination of $u_1, u_2, \ldots, u_n$?
Now imitate what is written in (#). Next ask a similar question as in (♮). Then complete the argument as in (♭).

7. **Theorem (B).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *be vectors in* $\mathbb{R}^m$.

Every linear combination of (finitely many) linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Remark.

In fact, Theorem (B) is saying the same thing as Statement (b) and Statement (c) in Theorem (1) combined.

Its conclusion part can be formulated as:

For any $\mathbf{x} \in \mathbb{R}^m$, *if*

$\mathbf{x}$ *is a linear combination of some* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \in \mathbb{R}^m$ *which are themselves linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$,

then

$\mathbf{x}$ *itself is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

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then

$\mathbf{x}$ *itself is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

If you find the argument for Theorem (B) looks scary, try to prove these statements on your own first, in the same way as proving Statement (b) and Statement (c) of Theorem (1):

(1) Let $u_1, u_2, v_1, v_2, v_3, x$ be vectors in $\mathbb{R}^m$. Suppose x is a linear combination of v_1, v_2, v_3, and each of v_1, v_2, v_3 is a linear combination of u_1, u_2. Then x is a linear combination of u_1, u_2.

(2) Let $u_1, u_2, u_3, v_1, v_2, x$ be vectors in $\mathbb{R}^m$. Suppose x is a linear combination of v_1, v_2, and each of v_1, v_2 is a linear combination of u_1, u_2, u_3. Then x is a linear combination of u_1, u_2, u_3.

8. **Proof of Theorem (B).**

[This argument carries the same essence of that for Statement (b) and Statement (c) in Theorem (1).]

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$. Pick any $\mathbf{x} \in \mathbb{R}^m$.

Suppose $\mathbf{x}$ is a linear combination of (finitely many) vectors in $\mathbb{R}^m$, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, which are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

[Reminder: We want to see why $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.]

By definition, $\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$.

Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_p \in \mathbb{R}$ such that $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p$.

[Ask: Can we link up the $\mathbf{u}_j$'s with the $\mathbf{v}_i$'s so as to see that $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$?]

By assumption, for each $j = 1, 2, \cdots, p$, there exist some $\beta_{1j}, \beta_{2j}, \cdots, \beta_{nj} \in \mathbb{R}$ such that $\mathbf{v}_j = \beta_{1j}\mathbf{u}_1 + \beta_{2j}\mathbf{u}_2 + \cdots + \beta_{nj}\mathbf{u}_n$.

Then

$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \\
&= \alpha_1(\beta_{11}\mathbf{u}_1 + \beta_{21}\mathbf{u}_2 + \cdots + \beta_{n1}\mathbf{u}_n) + \alpha_2(\beta_{12}\mathbf{u}_1 + \beta_{22}\mathbf{u}_2 + \cdots + \beta_{n2}\mathbf{u}_n) \\
&\quad + \cdots + \alpha_p(\beta_{1p}\mathbf{u}_1 + \beta_{2p}\mathbf{u}_2 + \cdots + \beta_{np}\mathbf{u}_n) \\
&= (\beta_{11}\alpha_1 + \beta_{12}\alpha_2 + \cdots + \beta_{1p}\alpha_p)\mathbf{u}_1 + (\beta_{21}\alpha_1 + \beta_{22}\alpha_2 + \cdots + \beta_{2p}\alpha_p)\mathbf{u}_2 \\
&\quad + \cdots + (\beta_{n1}\alpha_1 + \beta_{n2}\alpha_2 + \cdots + \beta_{np}\alpha_p)\mathbf{u}_n
\end{aligned}$$

Note $(\beta_{k1}\alpha_1 + \beta_{k2}\alpha_2 + \cdots + \beta_{kp}\alpha_p)$ is a real number for each $k = 1, 2, \cdots, n$.

Then $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

8. Proof of Theorem (B).

[This is done in the same spirit as that for statement (b) and statement (c) of Theorem (1).]

[This argument carries the same essence of that for Statement (b) and Statement (c) in Theorem (1).]

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be vectors in $\mathbb{R}^m$. Pick any $\mathbf{x} \in \mathbb{R}^m$.

Suppose $\mathbf{x}$ is a linear combination of (finitely many) vectors in $\mathbb{R}^m$, say, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, which are linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

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Then there exist some $\alpha_1, \alpha_2, \cdots, \alpha_p \in \mathbb{R}$ such that $\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p$.

[Ask: Can we link up the $\mathbf{u}_j$'s with the $\mathbf{v}_i$'s so as to see that $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p$?]

By assumption, for each $j = 1, 2, \cdots, p$, there exist some $\beta_{1j}, \beta_{2j}, \cdots, \beta_{nj} \in \mathbb{R}$ such that $\mathbf{v}_j = \beta_{1j}\mathbf{u}_1 + \beta_{2j}\mathbf{u}_2 + \cdots + \beta_{nj}\mathbf{u}_n$.

Then

$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_p\mathbf{v}_p \\
&= \alpha_1(\beta_{11}\mathbf{u}_1 + \beta_{21}\mathbf{u}_2 + \cdots + \beta_{n1}\mathbf{u}_n) + \alpha_2(\beta_{12}\mathbf{u}_1 + \beta_{22}\mathbf{u}_2 + \cdots + \beta_{n2}\mathbf{u}_n) \\
&\quad + \cdots + \alpha_p(\beta_{1p}\mathbf{u}_1 + \beta_{2p}\mathbf{u}_2 + \cdots + \beta_{np}\mathbf{u}_n) \\
&= (\beta_{11}\alpha_1 + \beta_{12}\alpha_2 + \cdots + \beta_{1p}\alpha_p)\mathbf{u}_1 + (\beta_{21}\alpha_1 + \beta_{22}\alpha_2 + \cdots + \beta_{2p}\alpha_p)\mathbf{u}_2 \\
&\quad + \cdots + (\beta_{n1}\alpha_1 + \beta_{n2}\alpha_2 + \cdots + \beta_{np}\alpha_p)\mathbf{u}_n
\end{aligned}$$

Note $(\beta_{k1}\alpha_1 + \beta_{k2}\alpha_2 + \cdots + \beta_{kp}\alpha_p)$ is a real number for each $k = 1, 2, \cdots, n$.

Then $\mathbf{x}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

9. **Alternative argument for Theorem (B).**

By applying mathematical induction, and by consciously applying Theorem (1), we can verify the statement

> ‘*For any positive integer* s*, if* $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_s$ *are linear combinations of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_s$ *are real numbers then* $\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v}_2+\cdots+\alpha_s\mathbf{v}_s$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$*.*’

Such an argument is also legitimate.

10. We now state a pair of results (Lemma (2), Lemma (3)) describing whether square-matrix multiplication from the left to vectors ‘preserves’ linear relations amongst vectors.

Lemma (2).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be real numbers.*

Suppose A *is an* $(m \times m)$*-square matrix, and* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Then $A\mathbf{v}$ *is a linear combination of the vectors* $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Lemma (3). (A ‘partial converse’ of Lemma (2).)

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be real numbers.*

Suppose

- A *is a non-singular* $(m \times m)$*-square matrix, and*
- $A\mathbf{v}$ *is a linear combination of the vectors*

$$A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$$

and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Then $\mathbf{v}$ *is a linear combination of the vectors*

$$\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$$

and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

11. We combine Lemma (2) and Lemma (3) to obtain Theorem (C) below:

Theorem (C).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be real numbers.*

Suppose A *is a non-singular* $(m \times m)$*-square matrix.*

Then the statements below are logically equivalent:

(a) $\mathbf{v}$ *is a linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

(b) $A\mathbf{v}$ *is a linear combination of the vectors* $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ *and the respective scalars* $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Remark.

In plain words, this result is saying that

> *linear relations amongst vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the multiplication by the same non-singular matrix from the left to the vectors.*

When we think in terms of row operations, this result is saying that

> *linear relations amongst vectors (though not necessarily the individual vectors themselves) are preserved upon*
>
> *the application of the same sequence of row operations to the vectors.*

Interpretation of the content of Theorem (C) in terms of row operations.

① linear relation given

$$\begin{tikzcd}
\alpha_1 u_1 \arrow[d, "\rho_1"'] & + \alpha_2 u_2 \arrow[d, "\rho_1"'] & + \cdots + & \alpha_n u_n \arrow[d, "\rho_1"'] & = v \arrow[d, "\rho_1"'] \\
\vdots \arrow[d, "\rho_s"'] & \vdots \arrow[d, "\rho_s"'] & \cdots & \vdots \arrow[d, "\rho_s"'] & \vdots \arrow[d, "\rho_s"'] \\
u_1' & u_2' & & u_n' & v'
\end{tikzcd}$$

same sequence of row operations applied to individual vectors

Consequence:

Resultant vectors 'linearly related' in the 'same way':

$$\alpha_1 u_1' + \alpha_2 u_2' + \dots + \alpha_n u_n' = v'$$

② linear relation given

$$\begin{tikzcd}
\alpha_1 u_1 \arrow[d, "\rho_1"'] & + \alpha_2 u_2 \arrow[d, "\rho_1"'] & + \cdots + & \alpha_n u_n \arrow[d, "\rho_1"'] & = v \\
\vdots \arrow[d, "\rho_s"'] & \vdots \arrow[d, "\rho_s"'] & \cdots & \vdots \arrow[d, "\rho_s"'] & \\
\alpha_1 u_1' & + \alpha_2 u_2' & + \cdots + & \alpha_n u_n' & = w
\end{tikzcd}$$

Same sequence of row operations applied to individual vectors in the left side.

same scalars carried through

Ask: What is w?

Answer.

w is resultant from the same sequence of row operations applied to v.

$$v \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_s} w.$$

12. Proof of Lemma (2).

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose A is an $(m \times m)$-square matrix, and $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

By assumption,

$$\mathbf{v} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n.$$

Then

$$A\mathbf{v} = A(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n) = \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n.$$

Hence $A\mathbf{v}$ is a linear combination of $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

13. **Proof of Lemma (3).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose

- A is a non-singular $(m \times m)$-square matrix, and
- $A\mathbf{v}$ is a linear combination of the vectors
$$A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$$
and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

By assumption, $A\mathbf{v} = \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n$.

[Ask: Is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$?]

Since A is non-singular, A is invertible. Therefore

$$\begin{aligned}
\mathbf{v} &= I_m\mathbf{v} = (A^{-1}A)\mathbf{v} = A^{-1}(A\mathbf{v}) \\
&= A^{-1}(\alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n) \\
&= \alpha_1 A^{-1}(A\mathbf{u}_1) + \alpha_2 A^{-1}(A\mathbf{u}_2) + \cdots + \alpha_n A^{-1}(A\mathbf{u}_n) \\
&= \alpha_1 (A^{-1}A)\mathbf{u}_1 + \alpha_2 (A^{-1}A)\mathbf{u}_2) + \cdots + \alpha_n (A^{-1}A)\mathbf{u}_n \\
&= \alpha_1 I_m\mathbf{u}_1 + \alpha_2 I_m\mathbf{u}_2 + \cdots + \alpha_n I_m\mathbf{u}_n = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n
\end{aligned}$$

Then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

13. **Proof of Lemma (3).**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ be vectors in $\mathbb{R}^m$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ be real numbers.

Suppose

- A is a non-singular $(m \times m)$-square matrix, and
- $A\mathbf{v}$ is a linear combination of the vectors

$$A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$$

  and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

By assumption, $A\mathbf{v} = \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n$.

[Ask: Is $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$?]

If we can 'erase' the symbol 'A' from the linear relation '$Av = \alpha_1 Au_1 + \ldots + \alpha_n Au_n$', then we are done. But can we? why?

Since A is non-singular, A is invertible. Therefore

$$\begin{aligned}
\mathbf{v} &= I_m\mathbf{v} = (A^{-1}A)\mathbf{v} = A^{-1}(A\mathbf{v}) \\
&= A^{-1}(\alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 + \cdots + \alpha_n A\mathbf{u}_n) \\
&= \alpha_1 A^{-1}(A\mathbf{u}_1) + \alpha_2 A^{-1}(A\mathbf{u}_2) + \cdots + \alpha_n A^{-1}(A\mathbf{u}_n) \\
&= \alpha_1(A^{-1}A)\mathbf{u}_1 + \alpha_2(A^{-1}A)\mathbf{u}_2) + \cdots + \alpha_n(A^{-1}A)\mathbf{u}_n \\
&= \alpha_1 I_m\mathbf{u}_1 + \alpha_2 I_m\mathbf{u}_2 + \cdots + \alpha_n I_m\mathbf{u}_n = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_n\mathbf{u}_n
\end{aligned}$$

Then $\mathbf{v}$ is a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ and the respective scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$.

14. **Theorem (4). (Generalization of Lemma (2) and Lemma (3).)**

Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n, \mathbf{v}$ *be vectors in* $\mathbb{R}^m$.

Let A *be a* $(p \times m)$*-matrix.*

(a) *Suppose* $\mathbf{v}$ *is a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Then $A\mathbf{v}$ *is a linear combination of the vectors* $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$.

(b) *Suppose*

- $\mathcal{N}(A) = \{\mathbf{0}\}$, *and*
- $A\mathbf{v}$ *is a linear combination of the vectors* $A\mathbf{u}_1, A\mathbf{u}_2, \cdots, A\mathbf{u}_n$.

Then $\mathbf{v}$ *is a linear combination of the vectors* $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$.

Proof of Theorem (4). Exercise.