1. According to Lemma (γ) and Theorem (C):

(a) Each row-operation matrix is non-singular and invertible.

(b) The product of any (finitely many) row-operation matrices is non-singular and invertible.

(c) Every non-singular (and invertible) matrix is a product of row-operation matrices.

This allows us to establish a 'dictionary' between row-equivalence for general matrices and non-singularity.

2. **Lemma (μ).**

Let A, B be $(m \times n)$-matrices. Suppose A is row-equivalent to B.

Then there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

Proof of Lemma (μ).

Let A, B be $(m \times n)$-matrices. Suppose A is row-equivalent to B.

Then there is a sequence of row operations joining A to B:

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} C_2 \xrightarrow[\rho_3]{} \cdots\cdots\cdots \xrightarrow[\rho_{N-1}]{} C_N = B$$

in which the ρ_j's are the various row operations involved in this sequence.

For each k, denote by H_k the row operation matrix corresponding to ρ_k.

Then

$$C_2 = H_1 C_1 = H_1 A, C_3 = H_2 C_2, \cdots, C_{N-1} = H_{N-2} C_{N-2}, B = C_N = H_{N-1} C_{N-1}.$$

Therefore $B = HA$, in which $H = H_{N-1} H_{N-2} \cdots H_2 H_1$.

Then by Lemma (γ) and Theorem (C), the matrix H is non-singular and invertible.

2. **Lemma (μ).**

Let A, B be $(m \times n)$-matrices. Suppose A is row-equivalent to B.

$A \to \cdots\cdots \to B$ — there is some sequence of row operations joining A to B.

Then there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

B is the resultant of multiplication from the left to A by some appropriate non-singular and invertible square matrix.

Proof of Lemma (μ).

Let A, B be $(m \times n)$-matrices. Suppose A is row-equivalent to B.

Then there is a sequence of row operations joining A to B:

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} C_2 \xrightarrow[\rho_3]{} \cdots\cdots\cdots \xrightarrow[\rho_{N-1}]{} C_N = B$$

in which the ρ_j's are the various row operations involved in this sequence.

For each k, denote by H_k the row operation matrix corresponding to ρ_k.

For each k: $\begin{cases} C_k \xrightarrow[\rho_k]{} C_{k+1} \\ H_k C_k = C_{k+1} \end{cases}$

Then

$$C_2 = H_1 C_1 = H_1 A, C_3 = H_2 C_2, \cdots, C_{N-1} = H_{N-2} C_{N-2}, B = C_N = H_{N-1} C_{N-1}.$$

Therefore $B = HA$, in which $H = H_{N-1} H_{N-2} \cdots H_2 H_1$.

H can be obtained from this sequence of row operations: $I_m \xrightarrow[\rho_1]{} H_1 \xrightarrow[\rho_2]{} H_2 H_1 \to \cdots \to H_{N-1} \cdots H_1 = H$.

Then by Lemma (γ) and Theorem (C), the matrix H is non-singular and invertible.

3. **Lemma (ν).**

Let A, B be $(m \times n)$-matrices. Suppose there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

Then A is row-equivalent to B.

Proof of Lemma (ν).

Let A, B be $(m \times n)$-matrices. Suppose there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

According to Theorem (C), there exist some exist some $(m \times m)$-row-operation matrices $H_1, H_2, \cdots, H_{N-1}$ such that $H = H_{N-1}H_{N-2}\cdots H_2H_1$.

Define $C_1 = A$, $C_2 = H_1C_1$, $C_3 = H_2C_2$, ..., $C_{N-1} = H_{N-2}C_{N-2}$, $C_N = H_{N-1}C_{N-1}$.

Then, by definition,

$$B = HA = H_{N-1}H_{N-2}\cdots H_2H_1C_1 = H_{N-1}H_{N-2}\cdots H_2C_2 = \cdots = H_{N-1}C_{N-1} = C_N.$$

Denote the row operations corresponding to the respective row-operation matrices $H_1, H_2, \cdots, H_{N-1}$ by $\rho_1, \rho_2, \cdots, \rho_N$. Then, by definition, we have the sequence

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} C_2 \xrightarrow[\rho_3]{} \cdots\cdots\cdots \xrightarrow[\rho_{N-1}]{} C_N = B.$$

It follows that A is row-equivalent to B.

B = H A for some non-singular square matrix H

3. **Lemma (ν).**

Let A, B be $(m \times n)$-matrices. Suppose there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

Then A is row-equivalent to B.

A → .... → B
There is some sequence of row operations joining A to B.

Proof of Lemma (ν).

Let A, B be $(m \times n)$-matrices. Suppose there exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.

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Define $C_1 = A$, $C_2 = H_1C_1$, $C_3 = H_2C_2$, ..., $C_{N-1} = H_{N-2}C_{N-2}$, $C_N = H_{N-1}C_{N-1}$.

Then, by definition,

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Denote the row operations corresponding to the respective row-operation matrices $H_1, H_2, \cdots, H_{N-1}$ by $\rho_1, \rho_2, \cdots, \rho_N$. Then, by definition, we have the sequence

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} C_2 \xrightarrow[\rho_3]{} \cdots\cdots\cdots \xrightarrow[\rho_{N-1}]{} C_N = B.$$

It follows that A is row-equivalent to B.

4. We combine Lemma (μ) and Lemma (ν) into Theorem (F) below.

Theorem (F). (Re-formulation of row-equivalence in terms of multiplication by non-singular and invertible matrices.)

Let A, B be $(m \times n)$-matrices.

The statements below are logically equivalent:

(a) *A is row-equivalent to B.*

(b) *There exists some non-singular and invertible $(m \times m)$-square matrix H such that $B = HA$.*

Remark.

Such a re-formulation of row equivalence is useful in theoretical discussions because it brings in the equality symbol '='.

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Remark.

Such a re-formulation of row equivalence is useful in theoretical discussions because it brings in the equality symbol '='.

This will help save a lot of time and make the arguments concise:
It is usually easier to manipulate a chain of equalities for matrices than to manipulate a chain of sequences of row operations.

5. **Illustrations.**

(a) Let $A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$.

It happens that

$$A = C_1 \xrightarrow{-1R_1+R_2} C_2 \xrightarrow{-2R_1+R_3} C_3 \xrightarrow{-2R_2+R_3} C_4 \xrightarrow{-1R_3} C_5 \xrightarrow{-2R_2+R_1} C_6 \xrightarrow{-1R_3+R_2} C_7 = B$$

Then

$$B = HA,$$

in which H is the (3×3)-square matrix given by the product

$$H = H_6H_5H_4H_3H_2H_1,$$

and for each $k = 1, 2, 3, 4, 5, 6$, the matrix H_k is the row operation matrix corresponding to the row operation ρ_k joining C_k to C_{k+1}.

H is obtained as a resultant of the application of the sequence of row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5,$ on I_3:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \quad H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow{-1R_3} H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} \quad H_5H_4H_3H_2H_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{-1R_3+R_2} \quad H = H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} 0 & 1 & -2 & 1 \\ -1 & -2 & 3 & -4 \\ 2 & 7 & -12 & 11 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

It happens that

$$A = C_1 \xrightarrow{R_1 \leftrightarrow R_2} C_2 \xrightarrow{-1R_1} C_3 \xrightarrow{-2R_1+R_3} C_4 \xrightarrow{-3R_2+R_3} C_5 \xrightarrow{-2R_2+R_1} C_6 = B$$

Then

$$B = HA,$$

in which H is the (3×3)-square matrix given by the product

$$H = H_5 H_4 H_3 H_2 H_1,$$

and for each $k = 1, 2, 3, 4, 5$, the matrix H_k is the row operation matrix corresponding to the row operation ρ_k joining C_k to C_{k+1}.

H is obtained as a resultant of the application of the sequence of row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ on I_3:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_1} H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_1+R_3} \quad H_3H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{-3R_2+R_3} H_4H_3H_2H_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} \quad H = H_5H_4H_3H_2H_1 = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

(c) Let $A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ -2 & -1 & -3 & 3 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 & 0 & -8 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{bmatrix}$.

It happens that

$$A = C_1 \xrightarrow{R_1 \leftrightarrow R_2} C_2 \xrightarrow{2R_1 + R_3} C_3 \xrightarrow{-3R_2 + R_3} C_4 \xrightarrow{-2R_2 + R_1} C_5 \xrightarrow{2R_3 + R_1} C_6 \xrightarrow{-2R_3 + R_2} C_7 = B$$

Then

$$B = HA,$$

in which H is the (3×3)-square matrix given by the product

$$H = H_6 H_5 H_4 H_3 H_2 H_1$$

and for each $k = 1, 2, 3, 4, 5, 6$, the matrix H_k is the row operation matrix corresponding to the row operation ρ_k joining C_k to C_{k+1}.

H is obtained as a resultant of the application of the sequence of row operations $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5,$ on I_3:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_3} H_2 H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{-3R_2 + R_3} \quad H_3 H_2 H_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix} \xrightarrow{-2R_2 + R_1} H_4 H_3 H_2 H_1 = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_3 + R_1} \quad H_5 H_4 H_3 H_2 H_1 = \begin{bmatrix} -8 & 5 & 2 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{-2R_3 + R_2} \quad H = H_6 H_5 H_4 H_3 H_2 H_1 = \begin{bmatrix} -8 & 5 & 2 \\ 7 & -4 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

(d) Let $A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

It happens that

$$A = C_1 \xrightarrow{-1R_1+R_2} C_2 \xrightarrow{-3R_1+R_3} C_3 \xrightarrow{-1R_1+R_4} C_4 \xrightarrow{-1R_2} C_5$$
$$\xrightarrow{4R_2+R_3} C_6 \xrightarrow{3R_2+R_4} C_7 \xrightarrow{R_3 \leftrightarrow R_4} C_8 \xrightarrow{4R_3+R_4} C_8 \xrightarrow{-2R_2+R_1} C_9 \xrightarrow{-1R_3+R_1} C_{10} = B$$

Then

$$B = HA,$$

in which H is the (4×4)-square matrix given by the product

$$H = H_{10}H_9 \cdots H_3H_2H_1,$$

and for each $k = 1, 2, 3, \cdots, 9, 10$, the matrix H_k is the row operation matrix corresponding to the row operation ρ_k joining C_k to C_{k+1}.

H is obtained as a resultant of the application of the sequence of row operations $\rho_1, \rho_2, \rho_3, \cdots, \rho_9$ on I_4:

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_1+R_2} H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_1+R_3} H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-1R_1+R_4} \quad H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-1R_2} H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{4R_2+R_3} \quad H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_2+R_4} H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \quad H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 1 & -4 & 1 & 0 \end{bmatrix} \xrightarrow{4R_3+R_4} H_8H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_1} \quad H_9H_8H_7H_6H_5H_4H_3H_2H_1 = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix} \xrightarrow{-1R_3+R_1} H = H_{10}H_9 \cdots H_3H_2H_1 = \begin{bmatrix} -3 & 5 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 0 & 1 \\ 9 & -16 & 1 & 4 \end{bmatrix}$$