1. **Lemma (θ). (Invertibility as a sufficient condition for existence and uniqueness of solution for systems of linear equations.)**

   Let A be an $(n \times n)$-square matrix.

   Suppose A is invertible.

   Then, for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.

   Proof of Lemma (θ).

   Let A be an $(n \times n)$-square matrix. Suppose A is invertible.

   Then its matrix inverse A^{-1} is well-defined as an $(n \times n)$-square matrix.

   Pick any vector $\mathbf{b}$ in $\mathbb{R}^n$. Note that $A^{-1}\mathbf{b}$ is well-defined as a vector in $\mathbb{R}^n$.

   - We have $A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I_n\mathbf{b} = \mathbf{b}$.
     Hence '$\mathbf{x} = A^{-1}\mathbf{b}$' is a solution of the system $\mathcal{LS}(A, \mathbf{b})$.

   - Suppose '$\mathbf{x} = \mathbf{v}$' is a solution of the system $\mathcal{LS}(A, \mathbf{b})$.
     [Ask: Can it happen that $\mathbf{v}$ fails to be $A^{-1}\mathbf{b}$? (We hope not.)]
     Then $\mathbf{b} = A\mathbf{v}$. Therefore $A^{-1}\mathbf{b} = A^{-1}(A\mathbf{v}) = (A^{-1}A)\mathbf{v} = I_n\mathbf{v} = \mathbf{v}$.

   Hence '$\mathbf{x} = A^{-1}\mathbf{b}$' is the one and only one solution of the system $\mathcal{LS}(A, \mathbf{b})$.

1. **Lemma (θ). (Invertibility as a sufficient condition for existence and uniqueness of solution for systems of linear equations.)**

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   Then, for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.

   This is a short-hand for a pair of statements:
   - *'$x = A^{-1}b$' is a solution of $\mathcal{LS}(A,b)$.*
   - *If '$x = v$' is a solution of $\mathcal{LS}(A,b)$ then $v = A^{-1}b$.*

   Proof of Lemma (θ).

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     We are verifying: '$x = A^{-1}b$' is a solution of $\mathcal{LS}(A,b)$.

   - Suppose '$\mathbf{x} = \mathbf{v}$' is a solution of the system $\mathcal{LS}(A, \mathbf{b})$.
     [Ask: Can it happen that $\mathbf{v}$ fails to be $A^{-1}\mathbf{b}$? (We hope not.)]
     Then $\mathbf{b} = A\mathbf{v}$. Therefore $A^{-1}\mathbf{b} = A^{-1}(A\mathbf{v}) = (A^{-1}A)\mathbf{v} = I_n\mathbf{v} = \mathbf{v}$.

     We are verifying: If '$x = v$' is a solution of $\mathcal{LS}(A,b)$ then $v = A^{-1}b$.

   Hence '$\mathbf{x} = A^{-1}\mathbf{b}$' is the one and only one solution of the system $\mathcal{LS}(A, \mathbf{b})$.

2. **Question.**

Is invertibility a necessary condition for existence and uniqueness of solution for systems of linear equations?

What we are asking is whether the converse of Lemma (θ), as formulated below, is true:

'*Let A be an $(n \times n)$-square matrix.*

Suppose that, for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution.

Then A is invertible.'

Answer.

The answer to this question is *yes*.

Actually it takes 'much less' for a square matrix to be necessarily invertible.

This is the point of Lemma (ι) and Lemma (κ).

3. **Lemma (ι).**

Let A be an $(n \times n)$-square matrix.

Suppose that, for any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at most one solution.

Then A is invertible.

Lemma (κ).

Let A be an $(n \times n)$-square matrix.

Suppose that, for any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at least one solution.

Then A is invertible.

3. **Lemma (ι).**

Let A be an $(n \times n)$-square matrix.

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Lemma (κ).

Let A be an $(n \times n)$-square matrix.

Suppose that, for any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at least one solution.

Then A is invertible.

Lemma (ι) is easier to prove, as its content directly relates to the definition of null space and non-singularity.

Lemma (κ) is more difficult: the argument for Lemma (κ) involves the explicit construction of a matrix inverse of A with the information provided by the assumption.

4. **Proof of Lemma (ι).**

   Let A be an $(n \times n)$-square matrix.

   Suppose that,

   for any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at most one solution.

   [Ask: Is A invertible? Or equivalent, is A non-singular?]

   By assumption, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has at most one solution.

   Recall that $\mathcal{LS}(A, \mathbf{0})$ has at least one solution, namely, the trivial solution.

   Then $\mathcal{LS}(A, \mathbf{0})$ has exactly one solution, namely, the trivial solution.

   Therefore, by Lemma (1), A is non-singular.

   Hence, by Theorem (B), A is invertible.

5. **Proof of Lemma (κ).**

Let A be an $(n \times n)$-square matrix.

Suppose that,

for any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at least one solution.

[Ask: Can we write down a $(n \times n)$-matrix which hopefully will be a matrix inverse of A?]

Denote by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ the respective columns of I_n from left to right.
(So $I_n = \left[\, \mathbf{e}_1 \,\middle|\, \mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_n \,\right]$.)

By assumption, for each $j = 1, 2, \cdots, n$, the system $\mathcal{LS}(A, \mathbf{e}_j)$ has at least one solution, say, '$\mathbf{x} = \mathbf{g}_j$'. So by definition, $A\mathbf{g}_j = \mathbf{e}_j$.

Define $G = \left[\, \mathbf{g}_1 \,\middle|\, \mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{g}_n \,\right]$. We verify that $AG = I_n$:

- We have $AG = A\left[\, \mathbf{g}_1 \,\middle|\, \mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{g}_n \,\right] = \left[\, A\mathbf{g}_1 \,\middle|\, A\mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{g}_n \,\right] = \left[\, \mathbf{e}_1 \,\middle|\, \mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_n \,\right] = I_n$.

Now by Theorem (A), G is non-singular.

Therefore by Theorem (B), G is invertible. Furthermore, A is a matrix inverse of G. (Why?)

Hence A is also invertible (with G being a matrix inverse of A).

5. Proof of Lemma (κ).

Let A be an $(n \times n)$-square matrix.

Suppose that,

for any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at least one solution.

[Ask: Can we write down a $(n \times n)$-matrix which hopefully will be a matrix inverse of A?]

Denote by $\mathbf{e}_1, \mathbf{e}_2, \cdots , \mathbf{e}_n$ the respective columns of I_n from left to right.
(So $I_n = \left[\, \mathbf{e}_1 \,\middle|\, \mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_n \,\right]$.)

e_j is the vector in $\mathbb{R}^n$ whose j-th entry is 1 and all other entries are 0.

By assumption, for each $j = 1, 2, \cdots , n$, the system $\mathcal{LS}(A, \mathbf{e}_j)$ has at least one solution, say, '$\mathbf{x} = \mathbf{g}_j$'. So by definition, $A\mathbf{g}_j = \mathbf{e}_j$.

This is the key to the argument.

Define $G = \left[\, \mathbf{g}_1 \,\middle|\, \mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{g}_n \,\right]$. We verify that $AG = I_n$:

- We have $AG = A\left[\, \mathbf{g}_1 \,\middle|\, \mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{g}_n \,\right] = \left[\, A\mathbf{g}_1 \,\middle|\, A\mathbf{g}_2 \,\middle|\, \cdots \,\middle|\, A\mathbf{g}_n \,\right] = \left[\, \mathbf{e}_1 \,\middle|\, \mathbf{e}_2 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_n \,\right] = I_n$.

Now by Theorem (A), G is non-singular.

Therefore by Theorem (B), G is invertible. Furthermore, A is a matrix inverse of G. (Why?)

Hence A is also invertible (with G being a matrix inverse of A).

6. We can summarize our discussion above in Lemma (λ).

Lemma (λ).

Let A be an $(n \times n)$-square matrix.

The statements below are logically equivalent:

(a) A is invertible.

(b) For any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.

(c) For any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at least one solution.

(d) For any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at most one solution.

We combine Lemma (λ) and Theorem (C) to obtain an upgrade of Theorem (C) in the form of Theorem (E).

7. **Theorem (E). (Various re-formulations for the notions of non-singularity and invertibility.)**

Let A be an $(n \times n)$-matrix. The statements below are logically equivalent:

(a) A is non-singular.

(b) For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.

(c) The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.

(d) A is row-equivalent to I_n.

(e) A is invertible.

(f) There exists some $(n \times n)$-square matrix H such that $HA = I_n$.

(g) There exists some $(n \times n)$-square matrix G such that $AG = I_n$.

(h) For any vector $\mathbf{b}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{b})$ has one and only one solution, namely, '$\mathbf{x} = A^{-1}\mathbf{b}$'.

(i) For any vector $\mathbf{c}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{c})$ has at least one solution.

(j) For any vector $\mathbf{d}$ in $\mathbb{R}^n$, the system $\mathcal{LS}(A, \mathbf{d})$ has at most one solution.

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow{\rho_1} C_2 \xrightarrow{\rho_2} \cdots \cdots \xrightarrow{\rho_{p-2}} C_{p-1} \xrightarrow{\rho_{p-1}} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k. Then $[I_n | A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A | I_n]$:

$$[A|I_n] = [C_1|I_n] \xrightarrow{\rho_1} [C_2|H_1] \xrightarrow{\rho_2} [C_3|H_2 H_1] \xrightarrow{\rho_3} \cdots \cdots \xrightarrow{\rho_{p-2}} [C_{p-1}|H_{p-2} \cdots H_2 H_1] \xrightarrow{\rho_{p-1}} [C_p|H_{p-1} \cdots H_2 H_1] = [I_n|A^{-1}].$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1} \cdots H_2 H_1, \qquad A = H_1^{-1} H_2^{-1} \cdots H_{p-1}^{-1}.$$