1. Recall the statement of Theorem (C):

Let A be an $(n \times n)$-square matrix. The statements below are logically equivalent:

(a) *A is non-singular.*

(b) *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*

(c) *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*

(d) *A is row-equivalent to I_n.*

(e) *A is invertible.*

(f) *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*

(g) *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k.
Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A|I_n]$:

$$[A|I_n] = [C_1|I_n] \xrightarrow[\rho_1]{} [C_2|H_1] \xrightarrow[\rho_2]{} [C_3|H_2H_1] \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-1}]{} [C_p|H_{p-1}\cdots H_2H_1] = [I_n|A^{-1}].$$

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1}\cdots H_2H_1, \qquad\qquad A = {H_1}^{-1}{H_2}^{-1}\cdots {H_{p-1}}^{-1}.$$

1. Recall the statement of Theorem (C):

Let A be an $(n \times n)$-square matrix. The statements below are logically equivalent:

(a) *A is non-singular.*

(b) *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*

(c) *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*

(d) *A is row-equivalent to I_n.*

(e) *A is invertible.*

(f) *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*

(g) *There exists some $(n \times n)$-square matrix G such that $AG = I_n$.*

Now suppose A is non-singular, with a sequence of row operations

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = I_n,$$

and with H_k being the row-operation matrix corresponding to ρ_k for each k.

Then $[I_n|A^{-1}]$ is the resultant of the application of the same sequence of row operations $\rho_1, \rho_2, \cdots, \rho_{p-1}$ starting from $[A|I_n]$:

C_p is the reduced row echelon form which is row-equivalent to A.

$$[A|I_n] = [C_1|I_n] \xrightarrow[\rho_1]{} [C_2|H_1] \xrightarrow[\rho_2]{} [C_3|H_2H_1] \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-1}]{} [C_p|H_{p-1}\cdots H_2H_1] = [I_n|A^{-1}].$$

This suggests how to determine whether a square matrix is invertible, and to find its matrix inverse (if it exists).

Moreover, A^{-1} and A are respectively given as products of row-operation matrices by

$$A^{-1} = H_{p-1}\cdots H_2H_1, \qquad A = H_1{}^{-1}H_2{}^{-1}\cdots H_{p-1}{}^{-1}.$$

This is the matrix inverse of A which is expected to turn up if A is invertible.

2. 'Algorithm' described by Theorem (C).

Suppose we are given an $(n \times n)$-square matrix, say, A, and want to determine whether A is invertible, and to find its matrix inverse explicitly if A is indeed invertible.

Theorem (C) provides an 'algorithm' for doing this systematically:

- Step (1).
  Form the $(2n \times n)$-matrix $\left[\, A \,\middle|\, I_n \,\right]$.

- Step (2).
  Apply row operations on $\left[\, A \,\middle|\, I_n \,\right]$ so as to result in the matrix $\left[\, A^\sharp \,\middle|\, A^\flat \,\right]$, which is row-equivalent to $\left[\, A \,\middle|\, I_n \,\right]$, and in which $A^\sharp$ is a row-echelon form row-equivalent to A.

- Step (3).
  Inspect whether $A^\sharp$ has any entire row of zeros.
  - (3a) If *yes*, conclude that A is not invertible.
  - (3b) If *no*, conclude that A is invertible. To obtain the matrix inverse of A, apply further row operations on $\left[\, A^\sharp \,\middle|\, A^\flat \,\right]$ to obtain the reduced row-echelon form which is row-equivalent to $\left[\, A \,\middle|\, I_n \,\right]$.
    The resultant reduced row-echelon form is necessarily given by $\left[\, I_n \,\middle|\, A^{-1} \,\right]$.

2. **'Algorithm' described by Theorem (C).**

Suppose we are given an $(n \times n)$-square matrix, say, A, and want to determine whether A is invertible, and to find its matrix inverse explicitly if A is indeed invertible.

Theorem (C) provides an 'algorithm' for doing this systematically:

- Step (1).
  Form the $(2n \times n)$-matrix $\left[\, A \,\middle|\, I_n \,\right]$.

- Step (2). $[A | I_n] \longrightarrow \cdots \longrightarrow [A^{\#} | A^{b}]$
  Apply row operations on $\left[\, A \,\middle|\, I_n \,\right]$ so as to result in the matrix $\left[\, A^\sharp \,\middle|\, A^\flat \,\right]$, which is row-equivalent to $\left[\, A \,\middle|\, I_n \,\right]$, and in which $A^\sharp$ is a row-echelon form row-equivalent to A.

- Step (3).
  Inspect whether $A^\sharp$ has any entire row of zeros.
  * (3a) If *yes*, conclude that A is not invertible. (In this scenario, $\mathcal{N}(A)$, which is $\mathcal{N}(A^{\#})$, contains a non-zero vector.)
  * (3b) If *no*, conclude that A is invertible. To obtain the matrix inverse of A, apply further row operations on $\left[\, A^\sharp \,\middle|\, A^\flat \,\right]$ to obtain the reduced row-echelon form which is row-equivalent to $\left[\, A \,\middle|\, I_n \,\right]$. $[A | I_n] \longrightarrow \cdots \longrightarrow [A^{\#} | A^{b}] \longrightarrow \cdots \longrightarrow [I_n | A^{-1}]$.
    The resultant reduced row-echelon form is necessarily given by $\left[\, I_n \,\middle|\, A^{-1} \,\right]$.

3. **Examples on the application of the 'algorithm' described by Theorem (C).**

For each of the square matrices below, we are going to apply row operations to determine, for each of them, whether it is invertible or not, and what its matrix inverse is when it is invertible.

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

We apply successive row operations starting from $\left[\begin{array}{c|c} A & I_2 \end{array}\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\begin{array}{c|c} A & I_2 \end{array}\right]$:

$$\begin{aligned}[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] &\xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \\ &\xrightarrow{-1R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \\ &\xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right] = [I_2|B]\end{aligned}$$

with $B = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

3. **Examples on the application of the 'algorithm' described by Theorem (C).**

For each of the square matrices below, we are going to apply row operations to determine, for each of them, whether it is invertible or not, and what its matrix inverse is when it is invertible.

(a) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

We apply successive row operations starting from $\left[\,A\,\middle|\,I_2\,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\,A\,\middle|\,I_2\,\right]$:

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] = [A^\# | A^b]$$

$A^\#$ is a row-echelon form which is row-equivalent to A.

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -1 \end{array}\right] = [I_2|B]$$

with $B = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] = [I_3|B]$$

with $B = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(b) Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \xrightarrow{-1R_3} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]} \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] = [I_3|B]$$

This is $[A^\# \,|\, A^b]$, in which $A^\#$ is a row-echelon form which is row-equivalent to A.

with $B = \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{1R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] = [I_3|B]$$

with $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{1R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-1R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right] = [I_3|B]$$

This is $[A^\# | A^b]$, in which $A^\#$ is a row-echelon form which is row-equivalent to A.

with $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$.

Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(d) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$:

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1- & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{1R_2+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{1R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{(1/3)R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{1R_4+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{1R_4+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{-2R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] = [I_4|B]$$

with $B = \begin{bmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$. Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(d) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$:

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1- & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{1R_2+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{1R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{(1/3)R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{-1R_2+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{-1R_3+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{1R_4+R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right]$$

This is $[A^\# | A^b]$, in which $A^\#$ is a row-echelon form which is row-equivalent to A.

$$\xrightarrow{1R_4+R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] \xrightarrow{-2R_4+R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & -2/3 \\ 0 & 1 & 0 & 0 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & -2/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 & 1/3 & 1/3 \end{array}\right] = [I_4|B]$$

with $B = \begin{bmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$. Hence the matrix A is invertible, and its matrix inverse is given by $A^{-1} = B$.

(e) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{array}\right]$$

We observe that A is row-equivalent to the row-echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ with an entire row of 0.

Then A is not invertible.

(e) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_3 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_3 \,\right]$:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{2R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{array}\right] = [A^{\#} | A^{b}]$$

$A^{\#}$ is a row-echelon form which is row-equivalent to A.

We observe that A is row-equivalent to the row-echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ with an entire row of 0.

Then A is not invertible.

(f) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\begin{array}{c|c} A & I_4 \end{array}\right]$:

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1R_2+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -4 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right]$$

We observe that A is row-equivalent to the row-echelon form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with an entire row of 0.

Then A is not invertible.

(f) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$.

We apply successive row operations starting from $\left[\, A \,\middle|\, I_4 \,\right]$, in such a way to obtain some row-echelon form and the reduced row-echelon form (if needed) which are row-equivalent to $\left[\, A \,\middle|\, I_4 \,\right]$:

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 3 & 4 & 4 & 3 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1R_2+R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -4 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_3} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1R_3+R_4} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] = [A^\# | A^b]$$

$A^\#$ is a row-echelon form which is row-equivalent to A.

We observe that A is row-equivalent to the row-echelon form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with an entire row of 0.

Then A is not invertible.