MATH1030 Further examples on non-singularity and invertibility

1. Let α, β be real numbers. Suppose $\alpha^2 + \beta^2 = 1$.

   Define the (3×3)-square matrix A by

$$A = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 0 & \beta \\ \alpha & \beta & 0 \end{bmatrix}.$$

   (a) Compute A^2, A^3.

   (b) For each rea number x, define the (3×3)-square matrix C_x by

$$C_x = I_3 + xA + \frac{1}{2}x^2A^2.$$

   i. Show that $C_{s+t} = C_sC_t$ for any real number s, t.

   ii. Hence, or otherwise, show that C_s is invertible for each real number s.
   What is the matrix inverse of C_s?

2. Let A be a $(p \times p)$-square matrix, B be a $(p \times q)$-matrix, and C be a $(q \times q)$-square matrix, and H be the $((p + q) \times (p + q))$-square matrix given by

$$H = \left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{p\times p} & C \end{array}\right]$$

   Suppose A, C are non-singular and invertible.

   Is H non-singular and invertible? If yes, also find the matrix inverse of H in terms of A, B, C.

   Remark. Below is a hint on how to approach such a question. First ask this question:

   - Suppose it happens that the $(p \times p)$-square matrix S, the $(p \times q)$-matrix T, the $(q \times p)$-matrix U, and the $(q \times q)$-square matrix V satisfy the equality

$$\left[\begin{array}{c|c} S & T \\ \hline U & V \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{p\times p} & C \end{array}\right] = I_{p+q}.$$

     How will S, T, U, V and A, B, C relate with each other? Can each of S, T, U, V be expressed explicitly in terms of A, B, C alone?

   Then ask:

   - Suppose some matrices S, T, U, V are indeed explicitly defined in terms of A, B, C alone, like what have been obtained in the previous question. Will it happen that the equality

$$\left[\begin{array}{c|c} S & T \\ \hline U & V \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline \mathcal{O}_{p\times p} & C \end{array}\right] = I_{p+q}$$

     holds for such S, T, U, V?

3. Recall the definition for the notion of *nilpotent matrices*:

   Let A be an $(n \times n)$-square matrix. Then we say that A is nilpotent if and only if there is some positive integer p so that $A^p = \mathcal{O}_{n\times n}$.

   Prove the statements below:

   (a) Suppose A is a nilpotent matrix. Then A is singular.

   (b) Suppose A is a nilpotent $(n \times n)$-square matrix. Then, for each real number α, the matrices $I_n + \alpha A$ is nonsingular and invertible for each real number α.

Answers.

1. (a) $A^2 = \begin{bmatrix} -\beta^2 & \alpha\beta & \beta \\ \alpha\beta & -\alpha^2 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$, $A^3 = \mathcal{O}_{3\times 3}$.

   (b) i. ——

   ii. ${C_s}^{-1} = C_{-s}$ for each real number s.

2. H is invertible, and its matrix inverse is given by $H^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}BC^{-1} \\ \hline \mathcal{O}_{q\times p} & C^{-1} \end{array}\right]$.

3. ——