1. **How to re-formulate the notion of non-singularity in terms of row operations and reduced row-echelon forms?**

   Suppose A is an $(n \times n)$-square matrix.

   Recall that according to the definition for the notion of *non-singularity*:

   - *A is non-singular if and only if the trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*

   Now suppose A' is the reduced row-echelon form which is row-equivalent to A, and recall that:

   - *The respective solution sets of $\mathcal{LS}(A, \mathbf{0})$ and $\mathcal{LS}(A', \mathbf{0})$ are the same as each other;*

   - *Because A' is a reduced row-echelon form, A' is non-singular if and only if $A' = I_n$.*

So we will have the mutually exclusive scenarios (†), (†′) below, dependent on whether A' is the identity matrix or not:—

(†) Suppose $A' = I_n$.
Then the trivial solution is the only solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
Therefore A is non-singular.

(†′) Suppose $A' \neq I_n$.
Then A' is singular.
Therefore there is a non-trivial solution for the homogeneous system $\mathcal{LS}(A', \mathbf{0})$, which is also a non-trivial solution for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.
Hence A is singular.

Summarizing the discussion above, we have obtained a re-formulation for the notion of non-singularity in terms of reduced row-echelon forms.

2. **Lemma (4).**

Let A be a square matrix.

A is non-singular if and only if the reduced row-echelon form which is row-equivalent to A is given by the identity matrix.

Remark.

In the light of the Lemma (3), we may state Lemma (4) in this way:

Let A be a square matrix.

The following statements are logically equivalent:

(a) *A is non-singular.*

(b) *A is row equivalent to I_n.*

Summarizing the discussion above, we have obtained a re-formulation for the notion of non-singularity in terms of reduced row-echelon forms.

2. **Lemma (4).**

Let A be a square matrix.

A is non-singular if and only if the reduced row-echelon form which is row-equivalent to A is given by the identity matrix.

Remark.

In the light of the Lemma (3), we may state Lemma (4) in this way:

Let A be a square matrix.

The following statements are logically equivalent:

(a) *A is non-singular.*

(b) *A is row equivalent to I_n.*

This provides a useful method for determining any given square matrix is non-singular or singular.

Suppose A is an $(n \times n)$-matrix.

Apply row operations to A to find the reduced row-echelon form A' which is row equivalent to A.

- If $A' = I_n$, then conclude A is non-singular.
- If $A' \neq I_n$, then conclude A is singular.

3. **Example ($\star$), as illustrations for Lemma (4).**

(a) Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A, say, through Gaussian elimination:

$$(\alpha): \quad A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots \xrightarrow[\rho_{17}]{} C_{18} \xrightarrow[\rho_{18}]{} C_{19} = A'$$

in which the ρ_k's are given by:

k	ρ_k	k	ρ_k	k	ρ_k
1	$2R_1 + R_2$	7	$1R_1 + R_5$	13	$-1R_4 + R_1$
2	$-1R_1 + R_3$	8	$1R_3 + R_4$	14	$-1R_4 + R_2$
3	$2R_1 + R_4$	9	$-2R_3 + R_5$	15	$1R_4 + R_3$
4	$1R_1 + R_5$	10	$-2R_2 + R_1$	16	$-2R_5 + R_1$
5	$1R_2 + R_3$	11	$3R_3 + R_1$	17	$1R_5 + R_2$
6	$-1R_2 + R_4$	12	$-2R_3 + R_2$	18	$-1R_5 + R_3$

It happens that $A' = I_5$. Then we may conclude that A is non-singular.

(b) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A, say, through Gaussian elimination:

$$(\alpha): \quad A = C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots \underset{\rho_{11}}{\longrightarrow} C_{12} \underset{\rho_{12}}{\longrightarrow} C_{13} = A'$$

in which the ρ_k's are given by:

k	ρ_k	k	ρ_k	k	ρ_k
1	$-1R_1 + R_2$	5	$R_3 \leftrightarrow R_4$	9	$-1R_3 + R_2$
2	$-1R_1 + R_3$	6	$1R_3 + R_4$	10	$1R_4 + R_1$
3	$R_2 \leftrightarrow R_4$	7	$(1/3)R_4$	11	$1R_4 + R_2$
4	$1R_2 + R_4$	8	$-1R_2 + R_1$	12	$-2R_4 + R_3$

It happens that $A' = I_4$. Then we may conclude that A is non-singular.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 3 & 4 & 4 & 3 \\ 2 & 2 & 1 & 1 \end{bmatrix}$.

We find the reduced row-echelon form A' which is row-equivalent to A, say, through Gaussian elimination:

$$(\alpha): \qquad A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots \xrightarrow[\rho_8]{} C_9 \xrightarrow[\rho_9]{} C_{10} = A'$$

in which the ρ_k's are given by:

k	ρ_k	k	ρ_k	k	ρ_k
1	$-1R_1 + R_2$	4	$R_2 \leftrightarrow R_3$	7	$1R_3 + R_4$
2	$-3R_1 + R_3$	5	$1R_2 + R_3$	8	$-1R_2 + R_1$
3	$-2R_1 + R_4$	6	$-1R_3$	9	$-1R_3 + R_2$

It happens that $A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then we may conclude that A is singular.

Remark. In fact, C_8 is a row-echelon form with an entire row of zeros. We can stop there and conclude that A is singular. (Why?)

4. **How to re-formulate the notion of non-singularity in terms of matrix multiplication?**

Let A be an $(n \times n)$-square matrix, and A' be the reduced row-echelon form which is row-equivalent to A. We make some observations with the help of the 'dictionary' between row operations and row-operation matrices:

(a) For such matrices A, A', we have some sequence of row operations joining A to A':

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = A'.$$

According to the 'dictionary' between row operations and row-operation matrices, for each j, there is some (unique) $(n \times n)$-square matrix H_j, namely, the row operation matrix for ρ_j, such that $C_{j+1} = H_j C_j$.

Then

$$A' = C_p = H_{p-1}C_{p-1} = H_{p-1}H_{p-2}C_{p-2} = \cdots = H_{p-1}\cdots H_2H_1A.$$

4. **How to re-formulate the notion of non-singularity in terms of matrix multiplication?**

Let A be an $(n \times n)$-square matrix, and A' be the reduced row-echelon form which is row-equivalent to A. We make some observations with the help of the 'dictionary' between row operations and row-operation matrices:

(a) For such matrices A, A', we have some sequence of row operations joining A to A':

$$A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = A'.$$

$H_1 C_1 = C_2,\ H_2 C_2 = C_3, \ldots,\ H_{p-2} C_{p-2} = C_{p-1},\ H_{p-1} C_{p-1} = C_p$

According to the 'dictionary' between row operations and row-operation matrices, for each j, there is some (unique) $(n \times n)$-square matrix H_j, namely, the row operation matrix for ρ_j, such that $C_{j+1} = H_j C_j$.

Then

$$A' = C_p = H_{p-1} C_{p-1} = H_{p-1} H_{p-2} C_{p-2} = \cdots = H_{p-1} \cdots H_2 H_1 A.$$

(b) Now further suppose A is non-singular.

Then, by Lemma (4), $A' = I_n$.

Therefore there exist some $(n \times n)$-square matrix H, namely,

$$H = H_{p-1} \cdots H_2 H_1,$$

such that $HA = I_n$.

We summarize the discovery in the above discussion in the form of Lemma (5).

5. **Lemma (5). (Converse of Lemma (2).)**

Let A be an $(n \times n)$-square matrix.

Suppose A is non-singular.

Then there exists some $(n \times n)$-square matrix H such that $HA = I_n$.

Remark. Why do we call Lemma (5) a 'converse of Lemma (2)'?

It is because Lemma (2) reads as:

Let A be an $(n \times n)$-square matrix.
Suppose there exists some $(n \times n)$-square matrix H such that $HA = I_n$.
Then A is non-singular.

Combining Lemma (1), Lemma (2), Lemma (4) and Lemma (5), we obtain Theorem (A).

6. **Theorem (A). (Re-formulation of non-singularity in terms of row operations, reduced row-echelon forms and matrix multiplication.)**

Let A be an $(n \times n)$-square matrix.

The statements below are logically equivalent:

(a) *A is non-singular.*

(b) *For any vector $\mathbf{v}$ in $\mathbb{R}^n$, if $A\mathbf{v} = \mathbf{0}$ then $\mathbf{v} = \mathbf{0}$.*

(c) *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$.*

(d) *A is row-equivalent to I_n.*

(e) *There exists some $(n \times n)$-square matrix H such that $HA = I_n$.*

7. **Towards the notion of invertibility.**

Now indeed suppose A is a non-singular $(n \times n)$-square matrix, which, according to Theorem (A), will satisfy $HA = I_n$ for some appropriate $(n \times n)$-square matrix H.

(a) **Question.**

How to write down such a matrix H explicitly?

Answer.

According to our discussion leading up to Lemma (5), such a matrix H can be obtained as the product

$$H = H_{p-1} \cdots H_2 H_1$$

in which each H_j is the row operation matrix corresponding to the row operation ρ_j in some sequence of row operations joining A to I_n:

$$(\alpha): \quad A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} C_{p-1} \xrightarrow[\rho_{p-1}]{} C_p = I_n.$$

7. **Towards the notion of invertibility.**

Now indeed suppose A is a non-singular $(n \times n)$-square matrix, which, according to Theorem (A), will satisfy $HA = I_n$ for some appropriate $(n \times n)$-square matrix H.

(a) **Question.**

How to write down such a matrix H explicitly?

Answer.

According to our discussion leading up to Lemma (5), such a matrix H can be obtained as the product

$$H = H_{p-1} \cdots H_2 H_1$$

in which each H_j is the row operation matrix corresponding to the row operation ρ_j in some sequence of row operations joining A to I_n:

$$(\alpha): \quad A = C_1 \underset{\rho_1}{\longrightarrow} C_2 \underset{\rho_2}{\longrightarrow} \cdots\cdots \underset{\rho_{p-2}}{\longrightarrow} C_{p-1} \underset{\rho_{p-1}}{\longrightarrow} C_p = I_n.$$

$$\begin{cases} C_2 = H_1 C_1 \\ C_3 = H_2 C_2 \\ C_4 = H_3 C_3 \\ \vdots \\ C_{p-1} = H_{p-2} C_{p-2} \\ C_p = H_{p-1} C_{p-1} \end{cases}$$

$$I_n = C_p = H_{p-1} H_{p-2} \cdots H_2 H_1 C_1 = \underbrace{(H_{p-1} H_{p-2} \cdots H_2 H_1)}_{H} A$$

(b) **Further question.**

But can we obtain H without multiplying so many matrices together?

Answer to further question.

Yes. How?

Write $H = H_{p-1} \cdots H_2 H_1 I_n$ (as a sleight of hand), and think of how to interpret this product in terms of row operations, according to the 'dictionary' between row operations and row operation matrices.

H_1 is the resultant of the application of the row operation ρ_1 on I_n.

$H_2 H_1$ is the resultant of the application of the row operation ρ_2 on H_1.

$H_3 H_2 H_1$ is the resultant of the application of the row operation ρ_3 on $H_2 H_1$.

So forth and so on.

H is therefore the resultant of the sequence of row operations

$$(\beta): \quad I_n \xrightarrow[\rho_1]{} H_1 \xrightarrow[\rho_2]{} H_2 H_1 \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} H_{p-2} \cdots H_2 H_1 \xrightarrow[\rho_{p-1}]{} H_{p-1} \cdots H_2 H_1 = H.$$

(b) **Further question.**

But can we obtain H without multiplying so many matrices together?

Answer to further question.

Yes. How?

Write $H = H_{p-1} \cdots H_2 H_1 I_n$ (as a sleight of hand), and think of how to interpret this product in terms of row operations, according to the 'dictionary' between row operations and row operation matrices.

H_1 is the resultant of the application of the row operation ρ_1 on I_n. $I_n \xrightarrow[\rho_1]{} H_1$

$H_2 H_1$ is the resultant of the application of the row operation ρ_2 on H_1. $H_1 \xrightarrow[\rho_2]{} H_2 H_1$

$H_3 H_2 H_1$ is the resultant of the application of the row operation ρ_3 on $H_2 H_1$. $H_1 H_2 \xrightarrow[\rho_3]{} H_3 H_2 H_1$

⋮

So forth and so on.

H is therefore the resultant of the sequence of row operations

$$(\beta): \quad I_n \xrightarrow[\rho_1]{} H_1 \xrightarrow[\rho_2]{} H_2 H_1 \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} H_{p-2} \cdots H_2 H_1 \xrightarrow[\rho_{p-1}]{} H_{p-1} \cdots H_2 H_1 = H.$$

So there is no need to perform matrix multiplication in obtaining the product $H_{p-1} H_{p-2} \cdots H_2 H_1$.

(c) **Bonus.**

Again look at the sequence

$$(\beta): \quad I_n \xrightarrow[\rho_1]{} H_1 \xrightarrow[\rho_2]{} H_2H_1 \xrightarrow[\rho_3]{} \cdots\cdots \xrightarrow[\rho_{p-2}]{} H_{p-2}\cdots H_2H_1 \xrightarrow[\rho_{p-1}]{} H_{p-1}\cdots H_2H_1 = H.$$

This tells us immediately that:

- *I_n is row-equivalent to the $(n \times n)$-square matrix H, and hence*
- *the $(n \times n)$-square matrix H is also row-equivalent to I_n.*

Then, by Theorem (A), the statements below all hold immediately and simultaneously for this matrix H:

- *H is non-singular.*
- *For any vector $\mathbf{u}$ in $\mathbb{R}^n$, if $H\mathbf{u} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$.*
- *The trivial solution is the only solution of the homogeneous system $\mathcal{LS}(H, \mathbf{0})$.*
- *There exists some $(n \times n)$-square matrix G such that $GH = I_n$.*

So now we know that $HA = I_n$, and some matrix G satisfies $GH = I_n$.

(c) **Bonus.** [Focus on H alone; forget about A for the moment.]

Again look at the sequence

$$(\beta): \quad I_n \underset{\rho_1}{\longrightarrow} H_1 \underset{\rho_2}{\longrightarrow} H_2H_1 \underset{\rho_3}{\longrightarrow} \cdots\cdots \underset{\rho_{p-2}}{\longrightarrow} H_{p-2}\cdots H_2H_1 \underset{\rho_{p-1}}{\longrightarrow} H_{p-1}\cdots H_2H_1 = H.$$

This tells us immediately that:

- I_n *is row-equivalent to the* $(n \times n)$*-square matrix* H*, and hence*
- *the* $(n \times n)$*-square matrix* H *is also row-equivalent to* I_n.

With $\tilde{\rho}_j$ being the reverse row operation of ρ_j for each j, we have the sequence $H = H_pH_{p-1}\cdots H_2H_1 \underset{\tilde{\rho}_{p-1}}{\longrightarrow} H_{p-2}\cdots H_2H_1 \longrightarrow \cdots \longrightarrow H_2H_1 \underset{\tilde{\rho}_2}{\longrightarrow} H_1 \underset{\tilde{\rho}_1}{\longrightarrow} I_n$

Then, by Theorem (A), the statements below all hold immediately and simultaneously for this matrix H:

- H *is non-singular.*
- *For any vector* $\mathbf{u}$ *in* $\mathbb{R}^n$*, if* $H\mathbf{u} = \mathbf{0}$ *then* $\mathbf{u} = \mathbf{0}$.
- *The trivial solution is the only solution of the homogeneous system* $\mathcal{LS}(H, \mathbf{0})$.
- *There exists some* $(n \times n)$*-square matrix* G *such that* $GH = I_n$.

So now we know that $HA = I_n$, and some matrix G satisfies $GH = I_n$.

(d) **Question.**

What is the matrix G?

Answer.

The matrix G can be nothing but A itself.

Justification:

For such matrices A, G, H, we have $G = GI_n = G(HA) = (GH)A = I_nA = A$.

Extra bonus. Therefore, for the same matrices A, H, it happens not only the equality $HA = I_n$ holds, but also the equality $AH = I_n$ holds.

We have obtained something unexpected discovery from the 'practical problem' of computing the matrix H which satisfies $HA = I_n$ for the non-singular matrix A. We formulate this discovery as Lemma (6).

8. **Lemma (6).**

Let A be an $(n \times n)$-square matrix.

Suppose A non-singular.

Then there exists some $(n \times n)$-square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

(d) **Question.**

What is the matrix G?

Answer.

The matrix G can be nothing but A itself.

Justification: [Reminder. Known by now: $HA = I_n$ and $GH = I_n$.]

For such matrices A, G, H, we have $G = GI_n = G(HA) = (GH)A = I_nA = A$.

Extra bonus. Therefore, for the same matrices A, H, it happens not only the equality $HA = I_n$ holds, but also the equality $AH = I_n$ holds.

We have obtained something unexpected discovery from the 'practical problem' of computing the matrix H which satisfies $HA = I_n$ for the non-singular matrix A. We formulate this discovery as Lemma (6).

8. **Lemma (6).**

Let A be an $(n \times n)$-square matrix.

Suppose A non-singular.

Then there exists some $(n \times n)$-square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.

Remark. The converse of Lemma (6) is the statement ($\sharp$):

($\sharp$) *Let A be an $(n \times n)$-square matrix.*
Suppose there exists some $(n \times n)$-square matrix H such that H is non-singular, $HA = I_n$ and $AH = I_n$.
Then A non-singular.

The statement ($\sharp$) is certainly true, by virtue of Theorem (A).

But how about statement ($\flat$) below?

($\flat$) *Let A be an $(n \times n)$-square matrix.*
Suppose there exists some $(n \times n)$-square matrix H such that $AH = I_n$.
Then A non-singular.

We are not so sure at this point, as we are assuming 'less' in Statement ($\flat$) then in Statement ($\sharp$).

It will transpire that Statement ($\flat$) is true as well, after more work is done.

9. **Example ($\star\star$), as an illustration for Lemma (6).**

Recall Example ($\star$). Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix}$.

We have the sequence of row operations

$$(\alpha): \quad A = C_1 \xrightarrow[\rho_1]{} C_2 \xrightarrow[\rho_2]{} \cdots \xrightarrow[\rho_{17}]{} C_{18} \xrightarrow[\rho_{18}]{} C_{19} = A',$$

in which the row operations ρ_k and the corresponding row-operation matrices H_k are given by:

k	ρ_k	H_k	k	ρ_k	H_k	k	ρ_k	H_k
1	$2R_1 + R_2$	$I_5 + 2E_{2,1}^{5,5}$	7	$1R_1 + R_5$	$I_5 + E_{5,2}^{5,5}$	13	$-1R_4 + R_1$	$I_5 - 1E_{1,4}^{5,5}$
2	$-1R_1 + R_3$	$I_5 - E_{3,1}^{5,5}$	8	$1R_3 + R_4$	$I_5 + E_{4,3}^{5,5}$	14	$-1R_4 + R_2$	$I_5 - E_{2,4}^{5,5}$
3	$2R_1 + R_4$	$I_5 + 2E_{4,1}^{5,5}$	9	$-2R_3 + R_5$	$I_5 - 2E_{5,3}^{5,5}$	15	$1R_4 + R_3$	$I_5 + E_{3,4}^{5,5}$
4	$1R_1 + R_5$	$I_5 + E_{5,1}^{5,5}$	10	$-2R_2 + R_1$	$I_5 - 2E_{1,2}^{5,5}$	16	$-2R_5 + R_1$	$I_5 - 2E_{1,5}^{5,5}$
5	$1R_2 + R_3$	$I_5 + E_{3,2}^{5,5}$	11	$3R_3 + R_1$	$I_5 + 3E_{1,3}^{5,5}$	17	$1R_5 + R_2$	$I_5 + E_{2,5}^{5,5}$
6	$-1R_2 + R_4$	$I_5 - E_{4,2}^{5,5}$	12	$-2R_3 + R_2$	$I_5 - 2E_{2,3}^{5,5}$	18	$-1R_5 + R_3$	$I_5 - E_{3,5}^{5,5}$

It happens that

$$I_5 = A' = C_{19} = H_{18}C_{18} = H_{18} \cdots H_2 H_1 A.$$

The matrix

$$H = H_{18}C_{18} = H_{18} \cdots H_2 H_1$$

is the resultant of the application of the row operations $\rho_1, \rho_2, \cdots, \rho_{18}$ on I_5, and is explicitly given by

$$H = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix}.$$

It so happens that H is non-singular, $HA = I_5$ and $AH = I_5$.